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Distribution in the sense of eigenvalues of g-Toeplitz sequences: Clustering and attraction

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Abstract. This paper considers the spectral distribution and the concept of clustering and attraction in the sense of eigenvalues sequence of g-Toeplitz structures $\{T_{n,g}(f)\}$ defined by $T_{n,g}(f) = \left[\hat{f}_{r-gs}\right]_{r,s=0}^{n-1}$, where g is a given nonnegative parameter, $\{\hat{f}_k\}$ is the sequence of Fourier coefficients of the function $f \in L^1(\mathbb{T}^d)$ with $\mathbb{T} = (-\pi, \pi)$, d is a positive integer, and where f is real-valued and essentially bounded. A detailed treatment of the unilevel case is given, that is, $d = 1$ and $g \in \mathbb{N}$. The generalizations to the blocks and multilevel case are also presented for the case where g is a vector with nonnegative integer entries.

Keywords: Toeplitz; g-Toeplitz; Eigenvalues; Distribution; Clustering; Attraction; Multilevel blocs

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1. INTRODUCTION

Let f be a Lebesgue function defined over the domain $\mathbb{T} = (-\pi, \pi)$. We recall that for a given nonnegative integer, an $n \times n$ matrix A_n is called g-Toeplitz if $A_n = \left[\hat{f}_{r-gs}\right]_{r,s=0}^{n-1}$. In that case, a g-Toeplitz matrix is denoted by $T_{n,g}(f)$ and the sequence $\{\hat{f}_k\}_k$ of entries of $T_{n,g}(f)$ is the sequence of Fourier coefficients of the symbol f. For the algebraic properties of such matrices we refer to Section 5.1 of the classical book by Davis [7]. The first motivation

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of this study is due to the variety of fields where such matrices can be encountered, such as wavelet analysis [6] and in the refinement equations associated with the subdivision algorithms (see [8] and references therein). In addition, Gilbert Strang [24] has found interesting relationships between dilation equations in the wavelet context and multigrid methods [5,10], for the restriction/prolongation operators [11,1] with boundary conditions. More especially, the analysis of boundary conditions naturally arises when dealing with signal/image restoration problems or differential equations (see [22,20]).

Let $d \in \mathbb{N}$ and $f \in L^1(\mathbb{T}^d)$ be a real-valued function. For a fixed nonnegative parameter $g = (g_1, \ldots, g_d)$ we define the sequence $\{T_{n,g}(f)\}_n$ of g-Toeplitz matrices, where $n = (n_1, \ldots, n_d)$ with $n_j \neq 0$ for every $j = 1, \ldots, d$, and we denote by $\{\Lambda_{n,g}\}_n$ the sequence of its spectra, where $\Lambda_{n,g} = \{\lambda_j : j = 0, 1, \ldots, n_1 n_2 \ldots n_d - 1\}$. An interesting question is to know how the spectrum $\Lambda_{n,g}$ can be related to a symbol $\theta_f^{(g)} \in L^\infty(\mathbb{T}^d)$ when the generating function f of g-Toeplitz sequences is real-valued or, even if $f \in L^1(\mathbb{T}^d)$, to study the convergence of the sequence of sets $\{\Lambda_{n,g}\}_n$. When $g = 1$, an essential result concerning the sequence of spectra is the famous Szegö theorem which says that, if f is real-valued and essentially bounded then

$$\lim_{n\to\infty} \sum_{\lambda \in \Lambda_n} F(\lambda) = \frac{1}{mes(\mathbb{T})} \int_{\mathbb{T}} F(f(\exp(\hat{i}t)))dt, \quad (\hat{i}^2 = -1) \tag{1}$$

for every continuous function F with compact support (see, for example, [13]). Here, $mes(\mathbb{T})$ denotes the Lebesgue measure, that is, $mes(\mathbb{T}) = 2\pi$. Furthermore, Tilli and Tyrtyshnikov/Zamarashkin, independently, showed that relation (1) holds for any integrable function f which is just real-valued, see [25,26]. Parter is the first researcher who has obtained the corresponding result for a complex-valued function f and the sequence of sets of its singular values when replacing f by $|f|$ under the hypothesis of continuous times uni-modular symbols, see [17], Avram (essentially bounded symbols [2]), and Tyrtyshnikov/Zamarashkin [25,26], independently, when the symbol f is just integrable. A large class of test functions F in [25,21,4] satisfies the Eq. (1) and the case of functions f of several variables (multilevel case) and matrix-valued functions was studied in [25,18] in the context of preconditioning (other related results were established by Linnik, Widom, Doktorski, see Section 6.9 in [5]).

In some recent works [15,14,9] we studied the spectral features and asymptotic properties for g-circulants and g-Toeplitz sequences and we addressed the problem of regularizing preconditioning of g-Toeplitz sequences via g-circulants, in the case where the sequence of entries $\{\hat{f}_k\}_k$ is the sequence of Fourier coefficients of a function $f \in L^1(\mathbb{T})$. Such results were plainly generalized to the block, multilevel case, amounting to choose the symbol f multivariate, i.e., defined on the set $\mathbb{T}^d$ for some $d > 1$, and matrix-valued, i.e., such that $f(x)$ is a matrix of given size $p \times q$. Here we treat the notion of spectral distribution and the concept of clustering and attraction in the eigenvalues sequence of g-Toeplitz structures $\{T_{n,g}(f)\}_n$. In particular, we consider the case where the sequence of values $\{\hat{f}_k\}_k$ is the Fourier coefficients of a real-valued function and essentially bounded and the interesting result is that the distribution function is more sparsely vanishing than the distribution function obtained in [15] (case where the parameter $g \notin \{0; 1\}$, otherwise the two results are the same). From this analysis we observe that the g-Toeplitz sequences $\{T_{n,g}(f)\}_n$ are sparsely vanishing (for the notion of sparsely vanishing matrix sequences, one can refer to [9]). We generalize the results obtained to one dimension, i.e., $d = p = q = 1$ and g a positive integer to the block and multilevel case, amounting to choose $f \in L^\infty(\mathbb{T}^d)$ to be "real-valued" for some $d > 1$, that is, $f(x)$ is a matrix of given size p.

We proceed as follows. In Section 2, we recall definitions and main tools. The problem of eigenvalue distribution of g-Toeplitz sequences is analyzed and discussed in Section 3. Section 4 generalizes the results obtained in Section 3 to block and multilevel case. We end the paper by drawing the general conclusion in Section 5.

2. Definitions and main tools

In this section, we give some basic definitions and we introduce some general tools for the spectral distribution (in both cases: eigenvalue and singular value) of matrix sequences $\{A_n\}_n$. As already mentioned in the previous section, if we denote by $T_n(f)$ the standard Toeplitz matrix generated by $f \in L^1(\mathbb{T})$, that is, $T_n(f) = [a_{r-c}]_{r,c=0}^{n-1}$, and by $T_{n,g}(f)$ the g-Toeplitz matrix generated by the same symbol, it is proven in [15, page 12] that for n and g generic,

$$T_{n,g}(f) = [\widehat{T}_{n,g}|\widetilde{T}_{n,g}] = [T_n(f)\widehat{Z}_{n,g}|\widetilde{T}_{n,g}] \tag{2}$$

where $\widehat{T}_{n,g} = T_n(f)\widehat{Z}_{n,g} \in \mathbb{C}^{n\times\mu_g}$, ($\mu_g = \lceil \frac{n}{g} \rceil$), is the matrix obtained from $T_{n,g}(f)$ by considering only its μ_g first columns, $\widetilde{T}_{n,g} \in \mathbb{C}^{n\times(n-\mu_g)}$ is the matrix obtained from $T_{n,g}(f)$ by considering only its $n-\mu_g$ last columns, and $\widehat{Z}_{n,g}$ is the matrix defined in (3) by considering only the μ_g first columns.

$$Z_{n,g} = [\delta_{r-gc}]_{r,c}^{n-1} \quad \text{where} \quad \delta_k = \begin{cases} 1, & \text{if } k \equiv 0 \ (mod\, n), \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

As stated in formula (2), the matrix $T_{n,g}(f)$ can be written as

$$T_{n,g}(f) = [T_n(f)\widehat{Z}_{n,g}|\widetilde{T}_{n,g}] = T_n(f)[\widehat{Z}_{n,g}|0] + [0|\widetilde{T}_{n,g}]. \tag{4}$$

To study the spectral distribution and the concept of clustering and attraction in the sense of eigenvalues sequence of g-Toeplitz structures, the idea is to solve the following problems. For $f \in L^\infty(\mathbf{T})$ real-valued:

- **(p1)** show that the matrix sequence $\{T_{n,g}(f)\}$ is uniformly bounded by a positive constant $\widehat{C}$ independent of n,
- **(p2)** show that $\|[0|\widetilde{T}_{n,g}]\|_1 = o(n)$, $n \to \infty$,
- **(p3)** show that the sequence $\{T_n(f)[\widehat{Z}_{n,g}|0]\}$ distributes in the sense of eigenvalues as a real-valued function $\theta_f^{(g)} \in L^\infty(\mathbb{T})$.

Now, denoting by $\sigma_1(A_n)$, $\sigma_2(A_n)$, …,$\sigma_n(A_n)$, the singular values of an $n \times n$ matrix A_n, for $p \in [1, \infty)$ we define $\|A_n\|_p$ the Schatten p-norm of A_n (see [3]) to be the l^p norm of the singular values vector

$$\|A_n\|_p = \left[\sum_{j=1}^{n} (\sigma_j(A_n))^p\right]^{\frac{1}{p}}.$$

In the following, we are especially interested in the norm $\|\cdot\|_1$ which is known as the Trace norm and the norm $\|\cdot\|$ which is the usual operator norm $\|A_n\| = \sup_{x\in\mathbb{C}^n,\|x\|_2=1}\|A_n x\|_2$. If $\lambda_j(A_n)$, $j = 1,\ldots,n$, are the eigenvalues of A_n then the spectrum of A_n is defined by $\Lambda_n = \{\lambda_j(A_n), 1 \le j \le n\}$. So, for any function F defined on $\mathbb{C}$, the symbol $\Sigma_\lambda(F, A_n)$ stands for the mean

$$\Sigma_\lambda(F, A_n) := \frac{1}{n}\sum_{j=1}^{n} F\left(\lambda_j(A_n)\right) = \frac{1}{n}\sum_{\lambda\in\Lambda_n} F(\lambda). \tag{5}$$

The corresponding symbol $\Sigma_\sigma(F, A_n)$ denotes the expression with the singular values obtained by replacing the eigenvalues.

In this work, we are particularly interested in explicit formulae of the distribution results for g-Toeplitz sequences. Following what is known in the standard case of $g = 1$ (or $g = e$ in the multilevel setting), we need to link the coefficients of the g-Toeplitz sequence to a function $\theta \in L^\infty(\mathbb{T})$. We define the matrix sequences as sequences $\{A_n\}$ where A_n is an $n\times n$ matrix and, Toeplitz or g-Toeplitz sequences (where g is a d-dimensional vector of nonnegative integers) as matrix sequences of the form $\{A_n\}$ or $\{A_{n,g}\}$ with $A_n = T_n(f) = \left[\hat{f}_{j-r}\right]_{j,r=\underline{0}}^{n-e}$ and $A_{n,g} = T_{n,g}(f) = \left[\hat{f}_{j-g\circ r}\right]_{j,r=\underline{0}}^{n-e}$. Here: $d \in \mathbb{N}^*$, $g = (g_1,\ldots,g_d)$, $n = (n_1,\ldots,n_d)$, $j = (j_1,\ldots,j_d)$, $r = (r_1,\ldots,r_d)$, $e = (1,\ldots,1)$, $\underline{0} = (0,\ldots,0)$, and where f is an integrable function defined over $\mathbb{T}^d = (-\pi,\pi)^d$ the d-fold cartesian product of the unit circle in the complex plane and $\{\hat{f}_k\}$ is the sequence of Fourier coefficients of f defined by

$$\begin{aligned}\hat{f}_j = \hat{f}_{(j_1,\ldots,j_d)}(f) &= \frac{1}{(2\pi)^d}\int_{[-\pi,\pi]^d} f(t_1,\ldots,t_d)\\ &\quad\times \exp\left(-\hat{i}(j_1t_1+\cdots+j_dt_d)\right)dt_1\ldots dt_d \quad (\hat{i}^2=-1)\end{aligned} \tag{6}$$

for integers j_l such that $-\infty < j_l < \infty$, with $1 \le l \le d$. If f is a matrix-valued function of d variables whose component functions are all integrable, then the $(j_1,\ldots,j_d)$th Fourier coefficient is considered to be the matrix whose (r,s)th entry is the $(j_1,\ldots,j_d)$th Fourier coefficient of the function $[f(t_1,\ldots,t_d)]_{r,s}$. Of course, the "$\circ$" operation is the componentwise Hadamard product between vectors or matrices of the same size.

The following definition is motivated by the Szegö and Tilli theorems characterizing the spectral approximation of a Toeplitz operator (in certain cases) by the spectra of the elements of the natural approximating matrix sequences $\{A_n\}$, where A_n is formed by the first n rows and columns of the matrix representation of the operator.

Definition 2.1. Let $\mathcal{C}_0(\mathbb{C})$ be the set of continuous functions with bounded support defined over the complex field, d a positive integer and θ a complex-valued measurable function defined on a set $G \subset \mathbb{C}^d$ of finite and positive Lebesgue measure $m(G)$. Here G will be equal to $\mathbb{T}^d$. A matrix sequence $\{A_n\}$ is said to be distributed (in the sense of eigenvalues) as the pair (θ, G), or to have the distribution function θ if, $\forall F \in \mathcal{C}_0(\mathbb{C})$, the following limit relation holds

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} F\left(\lambda_j(A_n)\right) = \frac{1}{m(G)}\int_G F\left(\theta(t)\right)dt, \quad \forall F \in \mathcal{C}_0(\mathbb{C}) \tag{7}$$

where $\lambda_j(A_n)$, $j = 1, 2, \ldots, n$, are the eigenvalues of A_n. We denote this by $\{A_n\} \sim_\lambda (\theta, G)$.

If (7) holds for every $F \in \mathcal{C}_0(\mathbb{R}_0^+)$ in place of $F \in \mathcal{C}_0(\mathbb{C})$, with the singular values σ_j, $j = 1, \ldots, n$, in place of the eigenvalues, and with $|\theta(t)|$ in place of $\theta(t)$, we say that $\{A_n\} \sim_\sigma (\theta, G)$ or that the matrix sequence $\{A_n\}$ is distributed (in the sense of singular values) as the pair (θ, G). More specifically, for every $F \in \mathcal{C}_0(\mathbb{R}_0^+)$ we have

$$\lim_{n\to\infty} \sum_\sigma (F, A_n) = \frac{1}{m(G)} \int_G F\left(|\theta(t)|\right) dt \tag{8}$$

where we have in view of (5)

$$\sum_\sigma (F, A_n) = \frac{1}{n} \sum_{j=1}^{n} F(\sigma_j(A_n)).$$

Furthermore, in order to treat block Toeplitz matrices, we consider measurable functions $\theta : G \to \mathcal{M}_{MN}$, where $\mathcal{M}_{MN}$ is the space of $M \times N$ matrices with complex entries and a function is considered to be measurable if and only if the component functions are. In that case $\{A_n\} \sim_\lambda (\theta, G)$ means that $M = N$ and

$$\lim_{n\to\infty} \sum_\lambda (F, A_n) = \frac{1}{m(G)} \int_G \frac{\sum\limits_{j=1}^{N} F(\lambda_j(\theta(t)))}{N} dt \tag{9}$$

$\forall F \in \mathcal{C}_0(\mathbb{C})$, where $\lambda_j(\theta(t))$ in relation (9) are the eigenvalues of the matrix $\theta(t)$.

When $N \neq M$, θ takes values in $\mathcal{M}_{NM}$, in that case, we say that $\{A_n\} \sim_\sigma (\theta, G)$ when for every $F \in \mathcal{C}_0(\mathbb{R}_0^+)$ we have

$$\lim_{n\to\infty} \sum_\sigma (F, A_n) = \frac{1}{m(G)} \int_G \frac{\sum\limits_{j=1}^{\min\{N,M\}} F(\lambda_j(\sqrt{\theta^*(t)\theta(t)}))}{\min\{N, M\}} dt. \tag{10}$$

Finally, two matrix sequences $\{A_n\}$ and $\{B_n\}$ are equally distributed in the sense of eigenvalues and/or singular values if $\forall F \in \mathcal{C}_0(\mathbb{C})$, we have

$$\lim_{n\to\infty} \left[\sum_\nu (F, A_n) - \sum_\nu (F, B_n) \right] = 0 \tag{11}$$

with $\nu = \lambda$ or $\nu = \sigma$.

It is important to recall that two matrix sequences having the same distribution function are equally distributed. The reverse of this result is not true, moreover, two equally distributed matrix sequences may be not associated with a distribution function at all. For example, when considering any diagonal matrix sequence $\{D_n\}$ and for a matrix sequence $\{A_n\}$ where $A_n = D_n - \epsilon_n I_n$, $\epsilon_n \to 0$ when $n \to \infty$. Then if the matrix sequence $\{D_n\}$ is not associated with a distribution function (in the sense of eigenvalues) (example: $D_n = (-1)^n I_n$), we will have $\{D_n\}$ and $\{A_n\}$ equally distributed even though it is not possible to associate a distribution function with either of them. On the other hand, if one of them distributes as a function, then the other necessary has the same one. This is easy to prove using the definitions (see also [19], Remark 6.1]).

Now, notice that a matrix sequence $\{A_n\}$ is distributed as a pair (θ, G) if and only if the sequence of linear functionals $\{\phi_n\}$ defined by $\phi_n(F) = \sum_\lambda(F, A_n)$ converges weakly to the functional $\phi(F) = \frac{1}{m(G)}\int_G F(\theta(t))dt$ as in (7).

In this paper, we also introduce the concept of clustering and attraction of sequences $\{\Lambda_n\}$ to describe what the distribution result (in the sense of eigenvalues) really means about the asymptotic qualities of the spectrum. Here Λ_n is the set of eigenvalues of A_n.

Definition 2.2. A matrix sequence $\{A_n\}$ is strongly clustered at $s \in \mathbb{C}$ (in the sense of eigenvalues), if for any $\epsilon > 0$, the number of the eigenvalues of A_n off the disc

$$D(s, \epsilon) := \{z \in \mathbb{C},\ |z| < \epsilon\} \tag{12}$$

can be bounded by a constant q_ϵ possibly depending of ϵ, but independent of n. In order words

$$q_\epsilon(n, s) := \#\{j : \lambda_j(A_n) \notin D(s, \epsilon)\} = O(1) \quad n \to \infty$$

where $\#A$ means the cardinality of the set A.

If every A_n has only real eigenvalues (at least for large n) then we may assume that s is real and that the disc $D(s, \epsilon)$ is the interval $(s - \epsilon, s + \epsilon)$.

Definition 2.3. A matrix sequence $\{A_n\}$ is strongly clustered as a nonempty closed set $S \subset \mathbb{C}$ (in the sense of eigenvalues), if for any $\epsilon > 0$ the number of the eigenvalues of A_n off the disc

$$D(S, \epsilon) := \bigcup_{s \in S} D(s, \epsilon)$$

can be bounded by a constant $q_\epsilon(n, S)$ possibly depending of ϵ, but independent of n. Moreover

$$q_\epsilon(n, S) := \#\{j : \lambda_j(A_n) \notin D(S, \epsilon)\} = O(1) \quad n \to \infty.$$

Here $\bigcup_{s \in S} D(s, \epsilon)$ is called the ϵ-neighborhood of S. If every A_n has only real eigenvalues (at least for large n) then S is a nonempty closed subset of $\mathbb{R}$.

Remark. When replacing the term "strongly" by "weakly" in the Definitions 2.2–2.3, one has

$$q_\epsilon(n, s) = o(n), \qquad q_\epsilon(n, S) = o(n), \quad n \to \infty$$

in the case of a point s or a closed subset S. Finally, if we replace eigenvalues with singular values we obtain all the corresponding definitions for singular values.

Remark. It is clear that $\{A_n\} \sim_\lambda (\theta, G)$, with $\theta \equiv s$ (s being a constant function) if and only if $\{A_n\}$ is weakly clustered at $s \in \mathbb{C}$ (for more details and relations between the notions of equal distribution, equal localization, spectral distribution, spectral clustering etc., see [19, section 4]). We introduce another interesting notion concerning the eigenvalues of a matrix sequence.

Definition 2.4. Let $\{A_n\}$ be a matrix sequence and let Λ_n be the spectrum of A_n. We say that $\{A_n\}$ is strongly attracted by $s \in \mathbb{C}$ (in the sense of eigenvalues), if

$$\lim_{n\to\infty} dist(s, \Lambda_n) = 0 \tag{13}$$

where dist(X, Y) is the usual euclidian distance between two subsets X and Y of the complex plane. Furthermore, if we order the eigenvalues according to their distance from s, i.e.,

$$|\lambda_1(A_n) - s| \leq |\lambda_2(A_n) - s| \leq \cdots \leq |\lambda_n(A_n) - s|$$

then we say that the attraction to s is of order $r(s) \in \mathbb{N}^*$, $r(s)$ is a fixed number if

$$\lim_{n\to\infty} |\lambda_{r(s)}(A_n) - s| = 0, \qquad \lim_{n\to\infty} \inf |\lambda_{r(s)+1}(A_n) - s| > 0$$

and that the attraction is of order $r(s) = \infty$ if

$$\lim_{n\to\infty} |\lambda_j(A_n) - s| = 0$$

for every fixed j. Finally, one defines weak attraction by replacing "lim" with "lim inf" in (13).

Remark. If $\{A_n\}$ is at least weakly clustered at a point s, then s strongly attracted $\{A_n\}$ with infinite order. Indeed, if there is an attraction of finite order $r(s)$ then

$$\lim_{n\to\infty} \frac{\#\{\lambda \in \Lambda_n : \lambda \notin D(s, \delta)\}}{n} = 1$$

for some $\delta > 0$ and this is impossible if $\{A_n\}$ is weakly clustered at s. On the other hand, there are sequences which are strongly attracted by s with infinite order, but not even weakly clustered at s. Indeed, the notion of weak clustering does not say anything concerning weak attraction or attraction of finite order.

Example. Let $\{A_n\}$ be a sequence of matrices with $A_n = \frac{1}{n+1} I_n$, where I_n is the identity matrix of order n. Then A_n has one eigenvalue $\lambda_n = \frac{1}{n+1}$ of multiplicity n. In addition, the sequence $\{A_n\}$ is strongly attracted by zero with infinite order, but not weakly clustered at zero. Indeed, setting $\Lambda_n = \{\frac{1}{n+1}\}$, then $\lim_{n\to\infty} \frac{1}{n+1} = 0$. But, there exists $\epsilon_n = \frac{1}{2^n}$ such that $q_{\epsilon_n}(n, 0) = \#\{\lambda \in \Lambda_n : |\lambda| \geq \epsilon_n\} = n$.

Remark. It is obvious that any of the notions introduced in this section for eigenvalues has a natural analogy for singular values, as explicitly described for the concept of distribution in relations (7) and (8).

Now, let us recall the definition of the essential range which plays an important role in the study of asymptotic properties of the spectrum.

Definition 2.5. Given a measurable complex-valued function θ defined on a Lebesgue measurable set G, the essential range of θ is the set $S(\theta)$ of points $s \in \mathbb{C}$ such that, for every $\epsilon > 0$, the Lebesgue measure of the set $\theta^{-1}(D(s,\epsilon)) := \{t \in G : \theta(t) \in D(s,\epsilon)\}$ is positive, with $D(s,\epsilon)$ as in (12). The function θ is essentially bounded if its essential range is bounded. Furthermore, if θ is real-valued, then the essential supremum (infimum) is defined as the supremum (infimum) of its essential range. Finally, if the function θ is $N \times N$ matrix-valued and measurable, then the essential range of θ is the union of the essential ranges of the complex-valued eigenvalues $\lambda_{j(\theta)}, j = 1, \ldots, N$.

Remark. $S(\theta)$ is a closed set because its complement in $\mathbb{C}$ is open.

Theorem 2.1 (*[23]*). *Let θ be a measurable function defined on G with finite and positive Lebesgue measure, and $S(\theta)$ be the essential range of θ. Let $\{A_n\}$ be a matrix sequence distributed as θ in the sense of eigenvalues, in that case, defining Λ_n to be the set of eigenvalues of A_n, the following facts are true:*

(a) *$S(\theta)$ is a weak cluster for $\{A_n\}$,*
(b) *each point $s \in S(\theta)$ strongly attracts $\{A_n\}$ with infinite order $r(s) = \infty$,*
(c) *there exists a sequence $\{\lambda^{(n)}\}$, where $\lambda^{(n)}$ is an eigenvalue of A_n such that $\liminf_{n\to\infty} |\lambda^{(n)}| \geq \|\theta\|$.*

Theorem 2.2 (*[12, Theorem 3.4]*). *Let $\{B_n\}$ and $\{C_n\}$ be two matrix sequences, where B_n is Hermitian and $A_n = B_n + C_n$. Assume further that $\{B_n\}$ is distributed as (θ, G) in the sense of the eigenvalues, where G is of finite and positive Lebesgue measure, both $\{B_n\}$ and $\{C_n\}$ are uniformly bounded by a positive constant $\widehat{C}$ independent of n, and $\|C_n\|_1 = o(n)$, $n \to \infty$. Then θ is real-valued and $\{A_n\}$ is distributed as (θ, G) in the sense of the eigenvalues. In particular, if $S(\theta)$ is the essential range of θ, then $\{A_n\}$ is weakly clustered at $S(\theta)$, and $S(\theta)$ strongly attracts the spectra of $\{A_n\}$ with an infinite order of attraction for any of its points.*

Theorem 2.3 (*[23]*). *Let $f, g \in L^\infty(\mathbb{T})$ be such that $h = fg$ is real-valued. Then $\{B_n\} \sim_\lambda (h, \mathbb{T})$ with $B_n = T_n(f)T_n(g)$, $S(h)$ is a weak cluster for $\{B_n\}$, and any $s \in S(h)$ strongly attracts the spectra of $\{B_n\}$ with infinite order.*

Theorem 2.4 (*[23]*). *Let $d \in \mathbb{N}^+$ and let $f, g \in L^\infty(\mathbb{T}^d)$ be such that $h = fg$ is real-valued. Setting $B_n = T_n(f)T_n(g)$, then $\{B_n\} \sim_\lambda (h, \mathbb{T}^d)$, $S(h)$ is a weak cluster for $\{B_n\}$, and any $s \in S(h)$ strongly attracts the spectra of $\{B_n\}$ with infinite order.*

With the above results we begin the study of the spectral distribution and the concept of clustering and attraction in the sense of eigenvalues sequence of g-Toeplitz structures $\{T_{n,g}(f)\}$, where $f \in L^\infty(\mathbb{T}^d)$ is real-valued. As mentioned above, we recall that the aim of this work is to give the general picture for any nonnegative vector g. Since the notations can be quite heavy, for the sake of readability, we start with the case $d = p = q = 1$. Several generalizations, including also the degenerate case in which g has some zero entries, are given in Section 4, which imply that the general analysis can be reduced to the case where all the entries of g are positive, that is, $g_j > 0$, $j = 1, \ldots, d$.

Let $f \in L^\infty(\mathbb{T}^d)$ (with $d \in \mathbb{N}$) be real-valued, it is known by the famous Szegö theorem that the Toeplitz sequence $\{T_n(f)\}$ distributes (in the sense of eigenvalues) as the symbol f. From relation (4), we can observe that $T_{n,g}(f) = T_n(f)[\widehat{Z}_{n,g}|0] + [0|\widetilde{\mathcal{T}}_{n,g}]$. Therefore, we study separately the distribution of the two sequences $\{T_n(f)[\widehat{Z}_{n,g}|0]\}$ and $\{[0|\widetilde{\mathcal{T}}_{n,g}]\}$, and then apply Theorems 2.2–2.3 to obtain the distribution of the g-Toeplitz sequences. However, in that case we have additional difficulties with respect to the Toeplitz sequences (for example, see [15,16] in the case of singular value distribution). Of course, for $g = 1$ (unilevel case) the g-Toeplitz matrix becomes the classical Toeplitz matrix, so the assumption made on the symbol f guarantees the use of the famous Szegö theorem. In [15, page 22] it was shown that the sequence $\{T_{n,g}(f)\}$ where $f \in L^1(\mathbb{T}^d)$ be complex-valued distributes (in the sense of singular values) as a symbol θ_f given by

$$\theta_f(x,t) = \begin{cases} \sqrt{\dfrac{1}{g}\displaystyle\sum_{j=1}^{g-1} |f|^2\left(\dfrac{x+2j\pi}{g}\right)}, & \text{if } t \in [0, 1/g), \\ 0, & \text{if } t \in [1/g, 1]. \end{cases} \tag{14}$$

In the following, we prove that the sequence $\{T_{n,g}(f)\}$ distributes in the sense of eigenvalues as a symbol $\theta_f^{(g)}$ which is sparsely vanishing (in the sense introduced by Tyrtyshnikov: the functions whose set of zeros has zero Lebesgue measure) if f is sparsely vanishing, and the symbol $\theta_f^{(g)}$ equals zero whenever the parameter g is strictly greater than 1.

The following remark plays a fundamental role in the study of problem (p1) stated above.

Remark. For $f \in L^\infty(\mathbb{T})$ real-valued and $p, q \in L^1(\mathbb{T})$ also real-valued, setting $h = p + \hat{\imath} q$ (with $\hat{\imath}^2 = -1$) the following inequality holds

$$\left| \int_{-\pi}^{\pi} f(x)h(x)dx \right| \le \|f\|_\infty \left| \int_{-\pi}^{\pi} h(x)dx \right| \tag{15}$$

where $|\cdot|$ denotes the norm in $\mathbb{C}$.

3. Eigenvalue distribution of g-Toeplitz sequences $\{T_{n,g}(f)\}$

In this section, we state main results namely Lemmas 3.1–3.4, and Theorem 3.1 which are solutions of problems (p1)–(p3) stated at the beginning of Section 2. We deduce the eigenvalue distribution of the sequence $\{T_{n,g}(f)\}$ according to Theorems 2.1–2.2. Moreover, we establish the relation between the asymptotic properties (clustering and attraction) of g-Toeplitz sequences and the Toeplitz structures.

Lemma 3.1. *The matrix sequence $\{[\widetilde{Z}_{n,0}|0]\}$ is distributed (in the sense of eigenvalues) as a real-valued function which is uniformly bounded by a positive constant independent of n.*

Proof. If $g = 1$, the matrix $[\widehat{Z}_{n,g}|0] = I_n$, otherwise $[\widehat{Z}_{n,g}|0]$ can be written as

$$[\widehat{Z}_{n,g}|0] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & * & \ddots & \vdots \\ * & * & \dots & 0 \end{bmatrix}, \quad \text{(case where } g > 1)$$

then the spectrum of $[\widehat{Z}_{n,g}|0]$ (for $g > 1$) is $\Lambda_n = \{1, 0\}$, where 1 is an eigenvalue of multiplicity 1 and 0 the other one of multiplicity $n - 1$. In addition, $[\widehat{Z}_{n,g}|0]$ is a lower triangular matrix, so for every $F \in \mathcal{C}_0(\mathbb{C})$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} F(\lambda_j) = \begin{cases} F(1), & \text{if } g = 1, \\ F(0), & \text{if } g > 1. \end{cases}$$

Then

$$\{[\widehat{Z}_{n,g}|0]\} \sim_\lambda (\theta^{(g)}, \mathbb{T}) \tag{16}$$

where $\theta^{(g)} = \begin{cases} 1, & \text{if } g = 1, \\ 0, & \text{otherwise.} \end{cases}$ is a real-valued function. $\square$

Lemma 3.2. *For any $f \in L^\infty(\mathbb{T})$ real-valued, the matrix sequence $\{T_{n,g}(f)\}$ is uniformly bounded by a positive constant $\widehat{C}$ independent of n.*

Proof. Let us recall that,

$$\|T_{n,g}(f)\|^2 = \sup_{x\in\mathbb{C}^n, x\neq 0} \frac{x^* T_{n,g}(f)^* T_{n,g}(f)x}{x^* x}.$$

Furthermore,

$$\begin{aligned} x^* T_{n,g}(f)^* T_{n,g}(f)x &= \left[\left(\sum_{k=0}^{n-1} \bar{x}_k \bar{\hat{f}}_{k-gj} \right)_{j=0}^{n-1} \right]' \left(\sum_{l=0}^{n-1} x_l \hat{f}_{l-gp} \right)_{p=0}^{n-1} \\ &= \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \bar{x}_k \bar{\hat{f}}_{k-gj} \sum_{l=0}^{n-1} x_l \hat{f}_{l-gj} \\ &= \frac{1}{4\pi^2} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} \bar{x}_k \sum_{l=0}^{n-1} x_l \int_{-\pi}^{\pi} f(x) \exp(\hat{i}(k - gj)x) dx \\ &\quad \times \int_{-\pi}^{\pi} f(x) \exp(-\hat{i}(l - gj)x) dx \\ &= \frac{1}{4\pi^2} K_n \end{aligned}$$

where

$$K_n = \sum_{j=0}^{n-1} \overline{\left[\int_{-\pi}^{\pi} f(x) \sum_{k=0}^{n-1} x_k \exp(-\hat{i}(k-gj)x)dx\right]} \times \left[\int_{-\pi}^{\pi} f(x) \sum_{k=0}^{n-1} x_k \exp(-\hat{i}(k-gj)x)dx\right] = \sum_{j=0}^{n-1} \left|\int_{-\pi}^{\pi} f(x) \sum_{k=0}^{n-1} x_k \exp(-\hat{i}(k-gj)x)dx\right|^2 .$$

According to relation (15), we have that

$$\sum_{j=0}^{n-1} \left|\int_{-\pi}^{\pi} f(x) \sum_{k=0}^{n-1} x_k \exp(-\hat{i}(k-gj)x)dx\right|^2 \leq \|f\|_\infty^2 \sum_{j=0}^{n-1} \left|\int_{-\pi}^{\pi} \sum_{k=0}^{n-1} x_k \exp(-\hat{i}(k-gj)x)dx\right|^2 .$$

So,

$$K_n \leq \|f\|_\infty^2 \sum_{j=0}^{n-1} \left|\int_{-\pi}^{\pi} \sum_{k=0}^{n-1} x_k \exp(-\hat{i}(k-gj)x)dx\right|^2 = \|f\|_\infty^2 \sum_{gj=\leq n-1} \left|\sum_{k=0}^{n-1} x_k \int_{-\pi}^{\pi} \exp(-\hat{i}(k-gj)x)dx\right|^2$$

and because $\int_{-\pi}^{\pi} \exp(-\hat{i}(k-gj)x)dx = \begin{cases} 2\pi & \text{if } k = gj, \\ 0 & \text{otherwise,} \end{cases}$ it follows that

$$K_n \leq 4\pi^2 \|f\|_\infty^2 \sum_{gj=\leq n-1} |x_{gj}|^2 \leq 4\pi^2 \|f\|_\infty^2 \|x\|_2^2. \quad \square$$

Proposition 3.1 *([15, page 21]). The sequence $\{[0|\widetilde{\mathcal{T}}_{n,g}]\}$ is spectrally distributed (in the sense of singular values) as the null function.*

One deduces from Proposition 3.1 the following Lemma.

Lemma 3.3. *For every $f \in L^\infty(\mathbb{T})$, the following inequality holds*

$$\|[0|\widetilde{\mathcal{T}}_{n,g}]\|_1 = o(n), \quad n \to \infty.$$

Proof. Let $p \in [1, \infty]$ and A and B be two matrices of size n. We have $\|A + B\|_p \leq \|A\|_p + \|A\|_p$ and $\|AB\|_p \leq \|A\| \times \|B\|_p$. It follows from these inequalities and relation (4) that

$$\begin{aligned}\|[0|\widetilde{\mathcal{T}}_{n,g}]\| &= \|T_{n,g}(f) - T_n(f)[\widehat{Z}_{n,g}|0]\| \\ &\leq \|T_{n,g}(f)\| + \|T_n(f)[\widehat{Z}_{n,g}|0]\| \\ &\leq \|f\|_\infty + \|T_n(f)\| \times \|[\widehat{Z}_{n,g}|0]\| \\ &\leq 2\|f\|_\infty \end{aligned} \tag{17}$$

where the second formula follows from Lemma 3.2 and the last one holds because $\|[\widehat{Z}_{n,g}|0]\| = 1$.

In addition, we know by Proposition 3.1 that the sequence $\{[0|\widetilde{\mathcal{T}}_{n,g}]\}$ is spectrally distributed (in the sense of singular values) as the null function, whence for every $F \in \mathcal{C}_0(\mathbb{R}_0^+)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} F(\sigma_j) = 0 \tag{18}$$

where σ_j: $j = 0, 1, \ldots, n-1$, are the singular values of $[0|\widetilde{\mathcal{T}}_{n,g}]$. Relation (18) holds with $F(x) = x$ (which has a unbounded support), since by inequality (17), $\|[0|\widetilde{\mathcal{T}}_{n,g}]\| \leq 2\|f\|_\infty$, and so the spectra of $\{[0|\widetilde{\mathcal{T}}_{n,g}]\}$ are contained in the interval $[0, 2\|f\|_\infty]$. Hence,

$$\|[0|\widetilde{\mathcal{T}}_{n,g}]\|_1 = \sum_{j=0}^{n-1} \sigma_j = o(n), \quad n \to \infty. \quad \square \tag{19}$$

Remark. The proof of Lemma 3.3 was concluded by taking into account the case where the parameter g is strictly greater than 1. In the case where $g = 1$, we have $T_{n,g}(f) = T_n(f)$ and so $[\widehat{Z}_{n,g}|0] = I_n$, the matrix $[0|\widetilde{\mathcal{T}}_{n,g}]$ becomes the identically null matrix.

Lemma 3.4. *Let $f \in L^\infty(\mathbb{T})$ be real-valued. Then the matrix sequence $\{T_n(f)[\widehat{Z}_{n,g}|0]\}$ is distributed (in the sense of eigenvalues) as a real-valued function $\theta_f^{(g)} \in L^\infty(\mathbb{T})$ given by (see (24))*

$$\theta_f^{(g)} = \begin{cases} f, & \text{if } g = 1, \\ 0, & \text{if } g > 1. \end{cases}$$

Proof. Since $f \in L^\infty(\mathbb{T})$, then the sequences $\{T_n(f)\}$ and $\{[\widehat{Z}_{n,g}|0]\}$ are uniformly bounded by a positive constant independent of n. According to relation (16), the sequence $\{[\widehat{Z}_{n,g}|0]\}$ is distributed (in the sense of eigenvalues) as the real-valued function $\theta^{(g)}$ by Lemma 3.1. f being real-valued, it follows from the famous Szegö theorem that the sequence $\{T_n(f)\}$ is distributed (in the sense of eigenvalues) as the symbol f.

Setting $\theta_f^{(g)} = f \times \theta^{(g)}$, let us show that the matrix sequences $\{T_n(f)[\widehat{Z}_{n,g}|0]\}$ and $\{T_n(\theta_f^{(g)})\}$ are equally distributed (in the sense of eigenvalues). Here, it suffices to prove that the two sequences have the same distribution function. Indeed: since $\{[\widehat{Z}_{n,g}|0]\}$ is distributed (in the sense of eigenvalues) as the real-valued function $\theta^{(g)}$, then the sequences $\{[\widehat{Z}_{n,g}|0]\}$ and $\{T_n(\theta^{(g)})\}$ are equally distributed (in the sense of eigenvalues). Whence

$$[\widehat{Z}_{n,g}|0] = T_n(\theta^{(g)}) + R_{n,g}, \quad \text{with } \|R_{n,g}\|_1 = o(n),\ n \to \infty. \tag{20}$$

Multiplying equality (20) member by member by the matrix $T_n(f)$, we get

$$\begin{aligned} &T_n(f)[\widehat{Z}_{n,g}|0] = T_n(f)T_n(\theta^{(g)}) + T_n(f)R_{n,g}, \quad \text{with} \\ &\quad \|T_n(f)R_{n,g}\|_1 = o(n),\ n \to \infty. \end{aligned} \tag{21}$$

In fact, $\|T_n(f)R_{n,g}\|_1 \le \|T_n(f)\| \times \|R_{n,g}\|_1 \le \|f\|_\infty \times \|R_{n,g}\|_1 = o(n)$, $n \to \infty$. Since $f, \theta^{(g)} \in L^\infty(\mathbb{T})$ are real-valued then $f \times \theta^{(g)}$ is real-valued, so it follows from Theorem 2.3 that the sequence $\{T_n(f)T_n(\theta^{(g)})\}$ is distributed as the symbol $\theta_f^{(g)} = f \times \theta^{(g)}$, so the sequences $\{T_n(f)T_n(\theta^{(g)})\}$ and $\{T_n(\theta_f^{(g)})\}$ are equally distributed (in the sense of eigenvalues). One can write

$$T_n(f)T_n(\theta^{(g)}) = T_n(\theta_f^{(g)}) + \widetilde{R}_{n,g}, \quad \text{with } \|\widetilde{R}_{n,g}\|_1 = o(n),\ n \to \infty. \tag{22}$$

It follows from relations (21) and (22) that

$$T_n(f)[\widehat{Z}_{n,g}|0] = T_n(\theta_f^{(g)}) + \widetilde{\widetilde{R}}_{n,g}, \quad \text{with } \|\widetilde{\widetilde{R}}_{n,g}\|_1 = o(n),\ n \to \infty \tag{23}$$

where

$$\theta_f^{(g)} = \begin{cases} f, & \text{if } g = 1, \\ 0, & \text{if } g > 1. \end{cases} \tag{24}$$

From the assumption that $f \in L^\infty(\mathbb{T})$ is real-valued, it follows that $\theta_f^{(g)} \in L^\infty(\mathbb{T})$ is also real-valued. According to Szegö theorem, the sequence $\{T_n(\theta_f^{(g)})\}$ is distributed (in the sense of eigenvalues) as the symbol $\theta_f^{(g)}$. More precisely, $\{T_n(\theta_f^{(g)})\}$ is a sequence of Hermitian matrices which are uniformly bounded by a positive constant independent of n. It is also obvious that the sequence $\{\widetilde{\widetilde{R}}_{n,g}\}$ is uniformly bounded by a positive constant independent of n. So, we deduce from Theorem 2.2 that the sequence $\{T_n(f)[\widehat{Z}_{n,g}|0]\}$ is distributed (in the sense of eigenvalues) as $\theta_f^{(g)}$. $\square$

Remark. Setting $S(\theta_f^{(g)})$ and $S(f)$ the essential range of $\theta_f^{(g)}$ and f, respectively, then

$$S(\theta_f^{(g)}) = \begin{cases} S(f), & \text{if } g = 1, \\ \{0\}, & \text{for } g > 1. \end{cases} \tag{25}$$

It follows from Theorems 2.1–2.2 that $\{T_n(f)[\widehat{Z}_{n,g}|0]\}$ is weakly clustered at $S(\theta_f^{(g)})$, and $S(\theta_f^{(g)})$ strongly attracts the spectra of $\{T_n(f)[\widehat{Z}_{n,g}|0]\}$ with an infinite order of attraction for any of its points.

Theorem 3.1. *Let $f \in L^\infty(\mathbb{T})$ be real-valued and $g \in \mathbb{N}^*$, where $g < n$ is a fixed parameter independent of n. Then the matrix sequence $\{T_{n,g}(f)\}$ is distributed (in the* *sense of eigenvalues) as the real-valued function $\theta_f^{(g)} \in L^\infty(\mathbb{T})$ given by (24). In addition, $\{T_{n,g}(f)\}$ is weakly clustered at $S(\theta_f^{(g)})$ (in the sense of Definitions 2.2–2.3) and $S(\theta_f^{(g)})$ strongly attracts the spectra of $\{T_{n,g}(f)\}$ with an infinite order of attraction for any of its points (in the sense of Definition 2.4).*

Proof. One deduces from (4) and Lemmas 3.3 and 3.4 that

$$T_{n,g}(f) = T_n(\theta_f^{(g)}) + Q_{n,g}, \quad \text{with } \|Q_{n,g}\|_1 = o(n),\ n \to \infty \tag{26}$$

where $Q_{n,g} = \tilde{\tilde{R}}_{n,g} + [0|\widetilde{\mathcal{T}}_{n,g}]$. The rest of proof follows from Theorems 2.1–2.2 and the Szegö theorem. $\square$

We end this section by a simple example which confirms our distribution result. For the sake of simplicity, we consider the case where the symbol $f \in L^\infty(\mathbb{T})$ is a constant function and the parameter g is strictly greater than 1. For $g = 1$, $T_{n,g}(f)$ is the classical Toeplitz matrix $T_n(f)$ and so the distribution formula (26) holds thanks to the famous Szegö theorem.

Example. Let $f = \alpha$ be a non null constant function. The entries of $T_{n,g}(f)$ are given by

$$\hat{f}_{r-gs} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \alpha(x)\exp(-\hat{i}(r-gs)x)dx = \begin{cases} \alpha, & \text{if } r = gs, \\ 0, & \text{otherwise,} \end{cases}$$

for $r, s = 0, 1, \ldots, n-1$. So, $T_{n,g}(f)$ is a lower triangular matrix having as eigenvalues α of multiplicity 1 and 0 of multiplicity $n-1$. So, for every $F \in \mathcal{C}_0(\mathbb{C})$ continuous with bounded support, we have

$$\Sigma_\lambda(F, T_{n,g}(f)) = \frac{1}{n}F(\alpha) + \left(1 - \frac{1}{n}\right)F(0)$$

then

$$\lim_{n\to\infty} \Sigma_\lambda(F, T_{n,g}(f)) = F(0) = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(0)dt.$$

So

$$\{T_{n,g}(f)\} \sim_\lambda (0, \mathbb{T}).$$

Remark. Another useful argument of our distribution result (given by Theorem 3.1) is justified by the fact that the distribution function $\theta_f^{(g)}$ given by (24) is more sparsely vanishing than the distribution function $\theta_f(\cdot,\cdot)$ given by (14) (case of singular values). For more notions on the sparsely vanishing functions and also the sparsely vanishing matrix sequences (for example, see [9]). The reason follows from the fact that the number of non zero eigenvalues of a matrix is less or equal than the number of its non zero singular values. There is equality if the matrix is Hermitian (or at least normal). To see this, let us consider the matrix $[\widehat{Z}_{n,g}|0]$, with $g > 1$ defined in relation (4). According to the proof of Lemma 3.1, the eigenvalues of this matrix are 1 (with multiplicity 1) and 0 (with multiplicity $n-1$). Furthermore, it is easy to prove that $[\widehat{Z}_{n,g}|0]^*[\widehat{Z}_{n,g}|0] = \left[\begin{array}{c|c} I_{\mu_g} & 0 \\ \hline 0 & 0 \end{array}\right]$ where I_{μ_g} is the identity matrix of size $\mu_g = \lceil \frac{n}{g} \rceil$. Then its singular values are 1 (of multiplicity μ_g) and 0 (of multiplicity $n - \mu_g$).

Remark. The fact that the distribution function is not unique should not be a surprise. In this work, we have assumed that the symbol f of Toeplitz (or g-Toeplitz) sequences is real-valued and essentially bounded in view to use the Szegö theorem (for eigenvalue distribution of Toeplitz sequences). Now, the question is to know if something can be said on the eigenvalue distribution of g-Toeplitz sequences when the generating function is just real-valued. In that precise case the idea is to use the result of Tilli, Tyrtyrshnikov/Zamarashkin ([25,26], 1990) which says that a Toeplitz sequence generated by a real-valued integrable function is distributed in the sense of eigenvalues as the symbol.

4. Generalization to block and multilevel setting

We start this section by recalling that it is proven in [23] that the sequence $\{T_n(f)T_n(g)\}$ is distributed (in the sense of eigenvalues) as the symbol $h = fg$ with $f, g \in L^\infty(\mathbb{T}^d)$ ($d \in \mathbb{N}$, $d > 1$, $\mathbb{T} = (-\pi, \pi)$) such that $h = fg$ is real-valued. If $S(h)$ denotes the essential range of h, then $S(h)$ is a weak cluster for $\{T_n(f)T_n(g)\}$, and any $s \in S(h)$ strongly attracts the spectra of $\{T_n(f)T_n(g)\}$ with infinite order (see also Theorem 2.4). This fact is sufficient for extending the proof of the relation $\{T_{n,g}(f)\} \sim_\lambda (\theta_f^{(g)}, \mathbb{T})$ to the case where $\theta_f^{(g)}$ is defined as in (24) with the function $f \in L^\infty(\mathbb{T}^d)$ is real-valued.

Let us consider the general multilevel case, where $f \in L^\infty(\mathbb{T}^d)$ is real-valued and matrix-valued. When g is a positive vector, we have

$$\{T_{n,g}(f)\} \sim_\lambda (\theta_f^{(g)}, \mathbb{T}^d) \tag{27}$$

where

$$\theta_f^{(g)} = \begin{cases} f, & \text{if } g = e, \\ 0, & \text{for } g > e, \end{cases} \qquad S(\theta_f^{(g)}) = \begin{cases} S(f), & \text{if } g = e, \\ \{\underline{0}\}, & \text{for } g > e, \end{cases} \tag{28}$$

and all the arguments are extended componentwise, that is, $g = e$ and $g > e$, respectively, means that $g_r = 1$ and $g_r > 1$ for $r = 1, \dots, d$. In addition, $S(\theta_f^{(g)})$ is a weak cluster for $\{T_{n,g}(f)\}$ (in the sense of Definitions 2.2–2.3) and any $s \in S(\theta_f^{(g)})$ strongly attracts the spectra of $\{T_{n,g}(f)\}$ with an infinite order (in the sense of Definition 2.4).

5. General conclusions

In this paper, we have studied the spectral distribution in the eigenvalues sequence of g-Toeplitz structures and then we have provided an analysis of the concept of clustering and attraction of these sequences. The generalization to the block and multilevel setting (case where the parameter g is a vector with positive integer entries) has been considered. We have also worked under the hypotheses that the generating function f of g-Toeplitz sequences is both real-valued and essentially bounded so that the famous Szegö can be used for the Toeplitz case. Since Tilli, Tyrtyshnikov/Zamarashkin [25,26], independently, have proven the same distribution formula under the only assumption that the symbol $f \in L^1(\mathbb{T})$ is real-valued, this latter point will be the subject of our future investigation for the g-Toeplitz structures. Other interesting problems will be the study of eigenvector behaviors both for g-Toeplitz and g-circulant matrices.

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