ARAB JOURNAL OF MATHEMATICAL SCIENCES

Arab J Math Sci **20**(2) (2014), 264–279

Curvelet transform for Boehmians

SUBASH MOORTHY RAJENDRAN, ROOPKUMAR RAJAKUMAR *

Department of Mathematics, Alagappa University, Karaikudi-630 004, India

Received 26 October 2012; revised 30 September 2013; accepted 7 October 2013
Available online 31 October 2013

Abstract. By proving the required auxiliary results, two Boehmian spaces are constructed for the purpose of extending the curvelet transform to the context of Boehmian spaces. A convolution theorem for curvelet transform is proved. As an application, the curvelet transform is consistently extended from one Boehmian space into the other Boehmian space and its properties like linearity, injectivity and continuity with respect to δ-convergence and Δ-convergence are obtained.

2000 MSC: 44A15; 44A35

Keywords: Boehmians; Convolution; Curvelet transform

1. NOTIONS AND NOTATIONS

We denote the set of all natural numbers, the set of all non-negative integers, the set of all real numbers, the set of all complex numbers and the set of k-tuples of real numbers, respectively by, $\mathbb{N}, \mathbb{N}_0, \mathbb{R}, \mathbb{C}$ and $\mathbb{R}^k$, where $k \in \mathbb{N}$. We also use the following notations in this article.

1. $\mathbb{S} = (0, a_0) \times \mathbb{R}^2 \times [-\pi, \pi] \subseteq \mathbb{R}^4$, for some $a_0 \in (0, \pi^2)$.
2. $\mathscr{L}^p(\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \to \mathbb{C} : \|f\|_p = \left(\int_{\mathbb{R}^2} |f(\mathbf{x})|^p \, d\mathbf{x} \right)^{1/p} < +\infty \right\}, p = 1, 2.$
3. $\hat{f}(\mathbf{t}) = \int_{\mathbb{R}^2} f(\mathbf{x}) \mathbf{e}^{-i\mathbf{x}\cdot\mathbf{t}} \, d\mathbf{x}, \forall \mathbf{t} \in \mathbb{R}^2$, for each $f \in \mathscr{L}^1(\mathbb{R}^2)$, where $\mathbf{x} \cdot \mathbf{t}$ is the usual scalar product of $\mathbf{x}$ and $\mathbf{t}$ in $\mathbb{R}^2$.
4. $\mathscr{F}(f) = \hat{f} = \mathscr{L}^2 - \lim_{n\to\infty} \hat{f}_n$, where $f_n \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2), \forall n \in \mathbb{N}$ and $f = \mathscr{L}^2 - \lim_{n\to\infty} f_n$, for each $f \in \mathscr{L}^2(\mathbb{R}^2)$.

* Corresponding author.
E-mail address: roopkumarr@rediffmail.com (R. Rajakumar)

Peer review under responsibility of King Saud University.

ELSEVIER Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2013.10.001

5. $\mathscr{L}^2_{a_0}(\mathbb{R}^2) = \{f \in \mathscr{L}^2(\mathbb{R}^2) : \hat{f}(\mathbf{t}) = 0, \forall \mathbf{t} \in E\}$, where $E \subset \mathbb{R} \times \mathbb{R}$ is the set $\left\{r\mathbf{e}^{i\omega} : 0 < r < \frac{2}{a_0} \text{ or } -\pi < \omega < \sqrt{a_0}\right\}$.
6. $\mathscr{D}(\mathbb{R}^2)$ is the Schwartz space of all infinitely differentiable complex valued functions on $\mathbb{R}^2$ with compact supports.
7. $\mathcal{Z} = \mathscr{F}(\mathscr{D}(\mathbb{R}^2))$, the image of $\mathscr{D}(\mathbb{R}^2)$ under Fourier transform.
8. $\mathscr{D}_{\mathscr{L}^2}(\mathbb{R}^2)$ is the space of all infinitely differentiable complex valued functions on $\mathbb{R}^2$ such that $\int_{\mathbb{R}^2} |f^{(k)}(\mathbf{x})|^2 \, d\mathbf{x} < +\infty, \forall k \in \mathbb{N}_0^2$. For more details about this space, we refer to [19]. The dual space of $\mathscr{D}_{\mathscr{L}^2}(\mathbb{R}^2)$ is denoted by $\mathscr{D}'_{\mathscr{L}^2}(\mathbb{R}^2)$.

Since the classical Fourier transform represents a given signal in terms of its frequency contents but with no time information, it is not efficient to process the non-stationary signals. By the introduction of wavelet transform, signals can be localized in both time and frequency. There are plenty of nice works on wavelets in the literature, to mention a few, we refer to [7–10,12–14]. As a refinement of wavelet transform, ridgelet transform is introduced by E.J. Candes [2,3], which is a hybrid integral transform *formed by using* wavelet transform, radon transform and Fourier transform. Since the ridgelet transform represents the images in recto-polar grids, the series representation of the signal by ridgelets converges faster than that by wavelets. Later, the curvelet transform has been recently introduced by E.J. Candes and D.L. Donoho [4,5], which has the ridgelet transform as a component and it is widely applied in image processing [27,28].

Now we recall the theory of continuous curvelet transform on $\mathscr{L}^2_{a_0}(\mathbb{R}^2)$ from [5]. Let $W, V \in \mathscr{D}(\mathbb{R})$ satisfy the following admissibility conditions.

(A1) $W(u) > 0, \forall u \in \left(\frac{1}{2}, 2\right)$, $W(u) = 0, \forall u \notin \left(\frac{1}{2}, 2\right)$, and $\int_{\frac{1}{2}}^{2} (W(u))^2 \frac{du}{u} = 1$.
(A2) $V(t) > 0, \forall t \in (-1, 1), V(t) = 0, \ \forall t \notin (-1, 1)$ and $\int_{-1}^{1} (V(t))^2 \, dt = 1$.

For $0 < a < a_0$, let $\gamma_{a,\mathbf{0},0}$ be the inverse Fourier transform of Λ_a, where

$$\Lambda_a(r\mathbf{e}^{i\omega}) = W(a\, r)V\big(\omega/\sqrt{a}\big)a^{3/4}, \ \forall r\mathbf{e}^{i\omega} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}, \text{ with } -\pi < \omega \leqslant \pi. \tag{1.1}$$

We observe that $\gamma_{a,\mathbf{0},0} \in \mathcal{Z}$ as $\Lambda_a \in \mathscr{D}(\mathbb{R}^2)$, for all $a \in (0, a_0)$. We also define

$$\gamma_{a,\mathbf{b},\theta}(\mathbf{x}) = \gamma_{a,\mathbf{0},0}(R_\theta(\mathbf{x} - \mathbf{b})), \ \forall \mathbf{x} \in \mathbb{R}^2, \tag{1.2}$$

where $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, which is the 2-by-2 planar rotation matrix effecting clock-wise rotation by θ radians; in other words, $R_\theta(\mathbf{x} - \mathbf{b})$ is simply the product of the two complex numbers $\mathbf{e}^{-i\theta}$ and $\mathbf{x} - \mathbf{b}$.

Definition 1 (*Curvelet transform*). For $f \in \mathscr{L}^2(\mathbb{R}^2)$ the curvelet transform is defined by

$$(\Gamma f)(a, \mathbf{b}, \theta) = \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{x})} f(\mathbf{x}) \, d\mathbf{x}, \ \forall (a, \mathbf{b}, \theta) \in \mathbb{S}. \tag{1.3}$$

Theorem 2. *Let $f \in \mathscr{L}^2(\mathbb{R}^2)$ have a Fourier transform vanishing for $|\xi| < \frac{2}{a_0}$. Then the inversion formula of the curvelet transform is given by*

$$f(\mathbf{x}) = \int_{\mathbb{S}} (\Gamma f)(a, \mathbf{b}, \theta)\gamma_{a,\mathbf{b},\theta}(\mathbf{x}) \frac{da}{a^3} d\mathbf{b} d\theta, \ \forall \mathbf{x} \in \mathbb{R}^2. \tag{1.4}$$

Under the same assumption, we also get the Parseval's identity, $\|f\|_2 = |||\Gamma f|||_2$.

While we probe into the proof of the above theorem, to know the reason for introducing the condition that $\hat{f}$ is vanishing on the open ball $B\left(\mathbf{0}, \frac{2}{a_0}\right)$ in $\mathbb{R}^2$, we could observe that the authors of the paper [5] introduced this condition to achieve the identity $\int_0^{a_0} \int_0^{2\pi} W(ar)^2 V\left(\frac{\omega-\theta}{\sqrt{a}}\right)^2 a^{3/2} d\theta \frac{da}{a^3} = 1, \forall r\mathbf{e}^{i\omega}$ on the support of $\hat{f}$, which is required to prove both conclusions of this theorem. Though, the above condition is useful to get $a^{1/2} \int_0^{a_0} W(ar)^2 a^{3/2} \frac{da}{a^3} = 1$, for $r > \frac{2}{a_0}$, we note that $\int_0^{2\pi} V\left(\frac{\omega-\theta}{\sqrt{a}}\right)^2 d\theta = a^{1/2}$ could not be obtained for all $\omega \in (-\pi, \pi)$ or for all $\omega \in (0, 2\pi)$. Indeed, $\int_0^{2\pi} V\left(\frac{\omega-\theta}{\sqrt{a}}\right) d\theta = a^{1/2} \int_{\frac{\omega-2\pi}{\sqrt{a}}}^{\frac{\omega}{\sqrt{a}}} V(t)^2 dt$, which is equal to $a^{1/2}$ iff $(-1,1) \subseteq \left(\frac{\omega-2\pi}{\sqrt{a}}, \frac{\omega}{\sqrt{a}}\right)$, since V is supported on $(-1,1)$ and $\int_{-1}^{1} V(t)^2 dt = 1$. This is possible only for $\omega \in (\sqrt{a}, 2\pi - \sqrt{a})$. Hence it is necessary to restrict ω such that $(-1,1) \subseteq \left(\frac{\omega-2\pi}{\sqrt{a}}, \frac{\omega}{\sqrt{a}}\right)$ and the bounds of ω should be independent of a. The one possible better restriction on ω so that $(-1,1) \subseteq \left(\frac{\omega-2\pi}{\sqrt{a}}, \frac{\omega}{\sqrt{a}}\right)$ is $\sqrt{a_0} < \omega < \pi$. Thus the above theorem is valid if the condition "$\hat{f}(\xi) = 0$, for $|\xi| < \frac{2}{a_0}$." is modified as "$\hat{f}(r\mathbf{e}^{i\omega}) = 0$, for $0 < r < \frac{2}{a_0}$ or $-\pi < \omega < \sqrt{a_0}$."

This is the motivation for introducing the space $\mathscr{L}^2_{a_0}(\mathbb{R}^2)$ at the beginning of this section.

On the other hand, motivated from the Boehme's regular operators [1], the concept of Boehmians was first introduced by J. Mikusiński and P. Mikusiński [15] and two notions of convergence called δ-convergence and Δ-convergence on a Boehmian space are introduced and discussed in [16]. From these remarkable works, a new avenue was opened in the area of generalized functions and lot of integral transforms have been extended on different Boehmian spaces. For a complete bibliography on Boehmians, we refer the reader to the website http://mikusinski.cos.ucf.edu/boehmians.pdf. In particular, the wavelet transform and ridgelet transform are also extended to the context of Boehmians. See [20–26].

In this article, we prove a suitable convolution theorem for the curvelet transform. Further, very first time the curvelet transform is extended to the context of Bohmians by constructing two Boehmian spaces, one is for the domain and the another is for the codomain of the extended curvelet transform. Then we prove that the extended curvelet transform is well-defined, consistent with the classical curvelet transform, linear, one-to-one and continuous with respect to the two notions of convergences in the context of Boehmians. Finally, we justify that the domain of the extended curvelet transform is properly larger than $\mathscr{D}'_{\mathscr{L}^2}(\mathbb{R}^2)$.

2. Boehmian spaces

From [18], we briefly recall the construction of an abstract Boehmian space $\mathscr{B} = \mathscr{B}(G, (S, \cdot), \odot, \Delta)$, where G is a topological vector space over $\mathbb{C}$, $(S, \cdot)$ is a commutative semi-group, $\odot : G \times S \to G$ satisfies the following conditions:

1. $(g_1 + g_2) \odot s = g_1 \odot s + g_2 \odot s, \forall g_1, g_2 \in G$ and $\forall s \in S$,
2. $(cg) \odot s = c(g \odot s), \forall c \in \mathbb{C}, \forall g \in G$ and $\forall s \in S$,
3. $g \odot (s \cdot t) = (g \odot s) \odot t, \forall g \in G$ and $\forall s, t \in S$,
4. If $g_n \to g$ as $n \to \infty$ in G and $s \in S$, then $g_n \odot s \to g \odot s$ as $n \to \infty$;

and Δ is a collection of sequences from S with the following properties:

1. If $(s_n), (t_n) \in \Delta$, then $(s_n \cdot t_n) \in \Delta$,
2. If $g_n \to g$ in G as $n \to \infty$ and $(s_n) \in \Delta$, then $g_n \odot s_n \to g$ as $n \to \infty$ in G.

A pair of sequences $((g_n), (s_n))$ with $g_n \in G, \forall n \in \mathbb{N}$ and $(s_n) \in \Delta$ is called a quotient if $g_n \odot s_m = g_m \odot s_n, \ \forall m, n \in \mathbb{N}$ and is denoted by $\frac{(g_n)}{(s_n)}$. The equivalence class $\left[\frac{(g_n)}{(s_n)}\right]$ containing $\frac{(g_n)}{(s_n)}$ induced by the equivalence relation $\sim$ defined on the collection of all quotients by

$$\frac{(g_n)}{(s_n)} \sim \frac{(h_n)}{(t_n)} \text{ if } g_n \odot t_m = h_m \odot s_n, \ \forall m, n \in \mathbb{N}$$

is called a Boehmian and the collection of all Boehmians $\mathscr{B}$ is a vector space with respect to the addition and scalar multiplication defined as follows.

$$\left[\frac{(g_n)}{(s_n)}\right] + \left[\frac{(h_n)}{(t_n)}\right] = \left[\frac{(g_n \odot t_n + h_n \odot s_n)}{(s_n \cdot t_n)}\right], \ c\left[\frac{(g_n)}{(s_n)}\right] = \left[\frac{(cg_n)}{(s_n)}\right].$$

Every member $g \in G$ can be uniquely identified as a member of $\mathscr{B}$ by $\left[\frac{(g \odot t_n)}{(t_n)}\right]$, where $(t_n) \in \Delta$ is arbitrary and the operation $\odot$ is also extended to $\mathscr{B} \times S$ by $\left[\frac{(g_n)}{(s_n)}\right] \odot t = \left[\frac{(g_n \odot t)}{(s_n)}\right]$. There are two notions of convergence on $\mathscr{B}$ namely δ-convergence and Δ-convergence which are defined as follows.

Definition 3 [16, δ-convergence]. We say that $X_m \xrightarrow{\delta} X$ as $m \to \infty$ in $\mathscr{B}$, if there exist $g_{m,n}, g_n \in G, m, n \in \mathbb{N}$ and $(s_n) \in \Delta$ such that $X_m = \left[\frac{(g_{m,n})}{(s_n)}\right], X = \left[\frac{(g_n)}{(s_n)}\right]$ and for each $n \in \mathbb{N}, g_{m,n} \to g_n$ as $m \to \infty$ in G.

Definition 4 [16, Δ-convergence]. We say that $X_m \xrightarrow{\Delta} X$ as $m \to \infty$ in $\mathscr{B}$, if there exist $g_m \in G, \forall m \in \mathbb{N}$ and $(s_n) \in \Delta$ such that $(X_m - X) \odot s_m = \left[\frac{(g_m \odot s_n)}{(s_n)}\right]$ and $g_m \to 0$ as $m \to \infty$ in G.

We construct the Boehmian space $\mathscr{B}^2_{a_0}(\mathbb{R}^2) = \mathscr{B}(\mathscr{L}^2_{a_0}(\mathbb{R}^2), (\mathscr{L}^1(\mathbb{R}^2), *), *, \Delta_T)$, where $*$ denotes the convolution that is defined by

$$(f * \phi)(\mathbf{x}) = \int_{\mathbb{R}^2} f(\mathbf{x} - \mathbf{y})\phi(\mathbf{y}) \ d\mathbf{y}, \ \forall \mathbf{x} \in \mathbb{R}^2 \tag{2.1}$$

and Δ_T is a collection of all sequences (ϕ_n) from $\mathscr{L}^1(\mathbb{R}^2)$ satisfying the following conditions.

(Δ_1) $\int_{\mathbb{R}^2} \phi_n(\mathbf{x}) \ d\mathbf{x} = 1, \forall n \in \mathbb{N}$.
(Δ_2) $\int_{\mathbb{R}^2} |\phi_n(\mathbf{x})| \ d\mathbf{x} \leqslant M, \forall n \in \mathbb{N}$, for some $M > 0$.
(Δ_3) For each $\delta > 0, \int_{|\mathbf{x}| \geqslant \delta} |\mathbf{x}||\phi_n(\mathbf{x})| \ d\mathbf{x} \to 0$ as $n \to \infty$.

This collection of sequences is similar to the one introduced by P. Mikusiński in [17].

Lemma 5. *If* $f \in \mathscr{L}^2_{a_0}(\mathbb{R}^2)$ *and* $\phi \in \mathscr{L}^1(\mathbb{R}^2)$, *then* $f * \phi \in \mathscr{L}^2_{a_0}(\mathbb{R}^2)$.

Proof. First we prove that

$$\|f * \phi\|_2 \leqslant \|f\|_2 \|\phi\|_1. \tag{2.2}$$

If $\|\phi\|_1 = 0$, then the inequality is obvious. Otherwise, $|\phi(\mathbf{y})| \frac{d\mathbf{y}}{\|\phi\|_1}$ is a probability measure on $\mathbb{R}^2$. Thus, we get

$$\begin{aligned}
\|f * \phi\|_2^2 &= \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} f(\mathbf{x} - \mathbf{y}) \phi(\mathbf{y})\, d\mathbf{y} \right|^2 d\mathbf{x} \\
&\leqslant \|\phi\|_1^2 \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |f(\mathbf{x} - \mathbf{y})|\, |\phi(\mathbf{y})| \frac{d\mathbf{y}}{\|\phi\|_1} \right)^2 d\mathbf{x} \\
&\leqslant \|\phi\|_1^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(\mathbf{x} - \mathbf{y})|^2 |\phi(\mathbf{y})| \frac{d\mathbf{y}}{\|\phi\|_1}\, d\mathbf{x} \\
&\quad \text{(since } t \mapsto t^2 \text{ is a convex function on } [0, \infty) \\
&\quad \text{and by Jensen's inequality)} \\
&\leqslant \|\phi\|_1^2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(\mathbf{x} - \mathbf{y})|^2 |\phi(\mathbf{y})|\, d\mathbf{x} \frac{d\mathbf{y}}{\|\phi\|_1} \\
&\quad \text{(by Fubini's theorem)} \\
&= \|\phi\|_1 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |f(\mathbf{z})|^2\, d\mathbf{z}\, |\phi(\mathbf{y})|\, d\mathbf{y} \\
&\quad \text{(Applying the change of variable } \mathbf{z} = \mathbf{x} - \mathbf{y}) \\
&= \|f\|_2^2 \|\phi\|_1^2.
\end{aligned}$$

For $f_n \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2(\mathbb{R}^2), \forall n \in \mathbb{N}$ such that $f_n \to f$ in $\mathscr{L}^2(\mathbb{R}^2)$ as $n \to \infty$, we have $f_n * \phi \to f * \phi$ in $\mathscr{L}^2(\mathbb{R}^2)$ as $n \to \infty$ (by the estimate (2.2)) and hence

$$(f * \hat{\phi}) = \mathscr{L}^2 - \lim_{n \to \infty} (f_n * \hat{\phi}) = \mathscr{L}^2 - \lim_{n \to \infty} \hat{f}_n \hat{\phi} = \hat{f} \hat{\phi}.$$

Thus $(f * \hat{\phi})$ vanishes at all points at which $\hat{f}$ vanishes. This completes the proof of this lemma. □

Lemma 6. *If* $f, g \in \mathscr{L}^2_{a_0}(\mathbb{R}^2), \phi, \psi \in \mathscr{L}^1(\mathbb{R}^2)$ *and* $\alpha \in \mathbb{C}$, *then*

1. $(f + g) * \phi = f * \phi + g * \phi$.
2. $(\alpha f) * \phi = \alpha (f * \phi)$.
3. $\phi * \psi = \psi * \phi$.
4. $f * (\phi * \psi) = (f * \phi) * \psi$.

The proof of the lemma is well known.

Lemma 7. *If* $f_n \to f$ *in* $\mathscr{L}^2_{a_0}(\mathbb{R}^2)$ *and* $\phi \in \mathscr{L}^1(\mathbb{R}^2)$, *then* $f_n * \phi \to f * \phi$ *as* $n \to \infty$ *in* $\mathscr{L}^2_{a_0}(\mathbb{R}^2)$.

The proof of this lemma follows immediately from the estimate (2.2).

Lemma 8. *If* $(\phi_n), (\psi_n) \in \Delta_T$, *then* $(\phi_n * \psi_n) \in \Delta_T$.

Proof. Since verifying the first two properties (Δ_1) and (Δ_2) of $(\phi_n * \psi_n)$ is straightforward, we prove the property (Δ_3) for $(\phi_n * \psi_n)$.

Let $\epsilon > 0$ be given.

$$\begin{aligned}
\int_{|\mathbf{x}|\geqslant\epsilon} |\mathbf{x}||(\phi_n * \psi_n)(\mathbf{x})|d\mathbf{x} &= \int_{|\mathbf{x}|\geqslant\epsilon} |\mathbf{x}|\left|\int_{\mathbb{R}^2} \phi_n(\mathbf{x}-\mathbf{y})\psi_n(\mathbf{y})d\mathbf{y}\right|d\mathbf{x} \\
&\leqslant \int_{|\mathbf{x}|\geqslant\epsilon}\int_{\mathbb{R}^2} |\mathbf{x}||\phi_n(\mathbf{x}-\mathbf{y})||\psi_n(\mathbf{y})|d\mathbf{y}d\mathbf{x} \\
&\leqslant \int_{\mathbb{R}^2}\int_{|\mathbf{x}|\geqslant\epsilon} |\mathbf{x}||\phi_n(\mathbf{x}-\mathbf{y})||\psi_n(\mathbf{y})|d\mathbf{x}d\mathbf{y} \\
&\quad \text{(by Fubini's theorem)} \\
&\leqslant \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{x}||\phi_n(\mathbf{x}-\mathbf{y})||\psi_n(\mathbf{y})|d\mathbf{x}d\mathbf{y} \\
&\leqslant \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{y}+\mathbf{z}||\phi_n(\mathbf{z})||\psi_n(\mathbf{y})|d\mathbf{z}d\mathbf{y} \\
&\leqslant \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} (|\mathbf{y}|+|\mathbf{z}|)|\phi_n(\mathbf{z})||\psi_n(\mathbf{y})|d\mathbf{z}d\mathbf{y} \\
&\leqslant \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{y}||\phi_n(\mathbf{z})||\psi_n(\mathbf{y})|d\mathbf{z}d\mathbf{y} \\
&\quad + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{z}||\phi_n(\mathbf{z})||\psi_n(\mathbf{y})|d\mathbf{z}d\mathbf{y} \\
&= I_n + J_n \text{ (say)}
\end{aligned}$$

Using $\int_{|\mathbf{y}|\geqslant\epsilon}|\mathbf{y}||\psi_n(\mathbf{y})|d\mathbf{y} \to 0$ as $n \to \infty$, we choose $N_1 \in \mathbb{N}$ such that $\int_{|\mathbf{y}|\geqslant\epsilon}|\mathbf{y}||\psi_n(\mathbf{y})|d\mathbf{y} < \epsilon, \forall n \geqslant N_1$.

Now for $n \geqslant N_1$, we have

$$\begin{aligned}
I_n &= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\mathbf{y}||\phi_n(\mathbf{z})||\psi_n(\mathbf{y})|d\mathbf{z}d\mathbf{y} \\
&= \int_{\mathbb{R}^2} |\mathbf{y}||\psi_n(\mathbf{y})|\int_{\mathbb{R}^2}|\phi_n(\mathbf{z})|d\mathbf{z}\ d\mathbf{y} \\
&\leqslant M_1\int_{\mathbb{R}^2}|\mathbf{y}||\psi_n(\mathbf{y})|d\mathbf{y} \\
&= M_1\left(\int_{|\mathbf{y}|<\epsilon}|\mathbf{y}||\psi_n(\mathbf{y})|d\mathbf{y} + \int_{|\mathbf{y}|\geqslant\epsilon}|\mathbf{y}||\psi_n(\mathbf{y})|d\mathbf{y}\right) \\
&< M_1\left(\epsilon\int_{|\mathbf{y}|<\epsilon}|\psi_n(\mathbf{y})|d\mathbf{y} + \int_{|\mathbf{y}|\geqslant\epsilon}|\mathbf{y}||\psi_n(\mathbf{y})|d\mathbf{y}\right) \\
&< M_1(M_2+1)\epsilon.
\end{aligned}$$

Therefore, $I_n \to 0$ as $n \to \infty$. By a similar argument, we get that $J_n \to 0$ as $n \to \infty$ and hence $(\phi_n * \psi_n) \in \Delta_T$. $\square$

Lemma 9. *If $f \in \mathscr{L}^2_{a_0}(\mathbb{R}^2)$ and $(\phi_n) \in \Delta_T$, then $f * \phi_n \to f$ as $n \to \infty$ in $\mathscr{L}^2_{a_0}(\mathbb{R}^2)$.*

Proof. For a given $\epsilon > 0$, using the denseness of $C_c(\mathbb{R}^2)$ in $\mathscr{L}^2(\mathbb{R}^2)$, we choose $g \in C_c(\mathbb{R}^2)$ such that $\|f - g\|_2 < \epsilon$. Since g is uniformly continuous on $\mathbb{R}^2$, there exists $\delta > 0$ such that

$$|g(\mathbf{x}) - g(\mathbf{y})| < \epsilon, \text{whenever } \mathbf{x}, \mathbf{y} \in \mathbb{R}^2 \text{ with } |\mathbf{x} - \mathbf{y}| < \delta.$$

Now, for each $n \in \mathbb{N}$, we get

$$\begin{aligned}
\|g * \phi_n - g\|_2^2 &= \int_{\mathbb{R}^2} |(g * \phi_n)(\mathbf{x}) - g(\mathbf{x})|^2 d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} g(\mathbf{x} - \mathbf{y})\phi_n(\mathbf{y})d\mathbf{y} - g(\mathbf{x}) \int_{\mathbb{R}^2} \phi_n(\mathbf{y})d\mathbf{y} \right|^2 d\mathbf{x} \\
&\quad \text{(by using property } (\Delta_1) \text{ of } (\phi_n)) \\
&\leqslant \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})||\phi_n(\mathbf{y})|d\mathbf{y} \right)^2 d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^2} \|\phi_n\|_1 |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})| \frac{|\phi_n(\mathbf{y})|}{\|\phi_n\|_1} d\mathbf{y} \right)^2 d\mathbf{x} \\
&\leqslant \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \|\phi_n\|_1^2 |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})|^2 \frac{|\phi_n(\mathbf{y})|}{\|\phi_n\|_1} d\mathbf{y}d\mathbf{x} \\
&\quad \text{(Since } t \mapsto t^2 \text{ is a convex function on } [0, \infty) \\
&\quad \text{and by Jensen's inequality.)} \\
&= \|\phi_n\|_1 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})|^2 |\phi_n(\mathbf{y})| d\mathbf{y}d\mathbf{x} \\
&= \|\phi_n\|_1 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})|^2 |\phi_n(\mathbf{y})| d\mathbf{x}d\mathbf{y} \\
&\quad \text{(by Fubini's theorem)} \\
&\leqslant M \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})|^2 |\phi_n(\mathbf{y})| d\mathbf{x}d\mathbf{y} \\
&\quad \text{(where } M > 0 \text{ is as in property } (\Delta_2) \text{ of } (\phi_n)) \\
&\leqslant M \int_{|\mathbf{y}|<\delta} \int_{\mathbb{R}^2} |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})|^2 |\phi_n(\mathbf{y})| d\mathbf{x}d\mathbf{y} \\
&\quad + M \int_{|\mathbf{y}|\geqslant\delta} \int_{\mathbb{R}^2} |g(\mathbf{x} - \mathbf{y}) - g(\mathbf{x})|^2 |\phi_n(\mathbf{y})| d\mathbf{x}d\mathbf{y} \\
&= T_1(n) + T_2(n) \text{ (Say)}
\end{aligned}$$

Now

$$
\begin{aligned}
T_1(n) &\leqslant M\int_{|\mathbf{y}|<\delta}\int_K \epsilon^2|\phi_n(\mathbf{y})|d\mathbf{x}d\mathbf{y}, \\
&\qquad \text{where } K \text{ is a compact set containing } (B_\delta(0)+\operatorname{supp} g). \\
&\leqslant M\epsilon^2 m(K)\int_{|\mathbf{y}|<\delta}|\phi_n(\mathbf{y})|d\mathbf{y} \\
&\qquad \text{where } m(K) \text{ is the Lebesgue measure of } K. \\
&\leqslant M^2\epsilon^2 m(K).
\end{aligned}
$$

Next by using the property (Δ_3) of $(\phi_n)\in\Delta_T$, we choose $N_0\in\mathbb{N}$ such that

$$
\int_{|\mathbf{y}|\geqslant\delta}|\mathbf{y}||\phi_n(\mathbf{y})|d\mathbf{y}<\epsilon,\ \forall n\geqslant N_0.
$$

Therefore, for $n\geqslant N_0$, we have

$$
\begin{aligned}
T_2(n) &\leqslant M\int_{|\mathbf{y}|\geqslant\delta}\left(\int_{\mathbb{R}^2}(|g(\mathbf{x}-\mathbf{y})|+|g(\mathbf{x})|)^2 d\mathbf{x}\right)|\phi_n(\mathbf{y})|d\mathbf{y} \\
&\leqslant M\int_{|\mathbf{y}|\geqslant\delta}\left(\int_{\mathbb{R}^2}2(|g(\mathbf{x}-\mathbf{y})|^2+|g(\mathbf{x})|^2)d\mathbf{x}\right)|\phi_n(\mathbf{y})|d\mathbf{y} \\
&\leqslant 4M\|g\|_2^2\int_{|\mathbf{y}|\geqslant\delta}|\mathbf{y}|^{-1}|\mathbf{y}||\phi_n(\mathbf{y})|d\mathbf{y} \\
&\leqslant 4M\|g\|_2^2\delta^{-1}\epsilon.
\end{aligned}
$$

Therefore,

$$
\|g*\phi_n-g\|_2^2 < M\epsilon\Big(M\epsilon m(K)+4\|g\|_2^2\delta^{-1}\Big). \tag{2.3}
$$

Now, for $n\geqslant N_0$, using the estimate (2.2), we obtain that

$$
\begin{aligned}
\|f*\phi_n-f\|_2 &= \|f*\phi_n-g*\phi_n+g*\phi_n-g+g-f\|_2 \\
&\leqslant \|f*\phi_n-g*\phi_n\|_2+\|g*\phi_n-g\|_2+\|f-g\|_2 \\
&\leqslant \|(f-g)*\phi_n\|_2+\sqrt{M\epsilon\Big(M\epsilon m(K)+4\|g\|_2^2\delta^{-1}\Big)}+\epsilon \\
&\leqslant \|f-g\|_2\|\phi_n\|_1+\sqrt{M\epsilon\Big(M\epsilon m(K)+4\|g\|_2^2\delta^{-1}\Big)}+\epsilon \\
&\leqslant M\epsilon+\sqrt{M\epsilon\Big(M\epsilon m(K)+4\|g\|_2^2\delta^{-1}\Big)}+\epsilon.
\end{aligned}
$$

Therefore $f*\phi_n\to f$ in $\mathscr{L}^2_{a_0}(\mathbb{R}^2)$ as $n\to\infty$. □

Lemma 10. *If $f_n\to f$ as $n\to\infty$ in $\mathscr{L}^2_{a_0}(\mathbb{R}^2)$ and $(\phi_n)\in\Delta_T$, then $f_n*\phi_n\to f$ as $n\to\infty$ in $\mathscr{L}^2_{a_0}(\mathbb{R}^2)$.*

Proof. For all $n\in\mathbb{N}$,

$$\begin{aligned}\|f_n * \phi_n - f\|_2 &\leqslant \|(f_n - f) * \phi_n\|_2 + \|f * \phi_n - f\|_2 \\ &\leqslant \|(f_n - f)\|_2 \|\phi_n\|_1 + \|f * \phi_n - f\|_2 \\ &\leqslant M\|f_n - f\|_2 + \|f * \phi_n - f\|_2,\end{aligned}$$

where $M > 0$ is as in the property (Δ_3) of (ϕ_n). On the right hand side, the first term tends to zero by hypothesis, and the second term tends to zero by applying the previous lemma, as $n \to \infty$. Hence the lemma follows. □

Next we prove the auxiliary results required to construct another Boehmian space which will contain the extended curvelet transforms of square integrable Boehmians. In the following sequel, we shall use the notation $\mathfrak{L}^2(\mathbb{S})$ to denote $\{F : \mathbb{S} \to \mathbb{C} : |||F|||_2 < \infty\}$, where $|||F|||_2 = \left(\int_{\mathbb{S}} |F(a, \mathbf{b}, \theta)|^2 \frac{da}{a^3} d\mathbf{b} d\theta\right)^{1/2}$.

Definition 11. Let $F \in \mathfrak{L}^2(\mathbb{S})$ and let $\phi \in \mathscr{L}^1(\mathbb{R}^2)$. Define

$$(F \times \phi)(a, \mathbf{b}, \theta) = \int_{\mathbb{R}^2} F(a, \mathbf{b} - \mathbf{y}, \theta)\phi(\mathbf{y})\ d\mathbf{y},\ \forall (a, \mathbf{b}, \theta) \in \mathbb{S}.$$

Lemma 12. *If* $F \in \mathfrak{L}^2(\mathbb{S})$ *and* $\phi \in \mathscr{L}^1(\mathbb{R}^2)$, *then* $|||F \times \phi|||_2 \leqslant |||F|||_2 \|\phi\|_1$ *and* $F \times \phi \in \mathfrak{L}^2(\mathbb{S})$.

Proof. If ϕ is identically zero, then the lemma follows obviously. So, we assume that ϕ is not identically zero. Therefore, $\|\phi\|_1 \neq 0$ and hence $|\phi(\mathbf{y})| \frac{d\mathbf{y}}{\|\phi\|_1}$ is a probability measure on $\mathbb{R}^2$. Using Jensen's inequality and Fubini's theorem, we get that

$$\begin{aligned}|||F \times \phi|||_2^2 &= \int_{\mathbb{S}} \left| \int_{\mathbb{R}^2} F(a, \mathbf{b} - \mathbf{y}, \theta)\phi(\mathbf{y})\ d\mathbf{y} \right|^2 \frac{da}{a^3} d\mathbf{b} d\theta \\ &\leqslant \|\phi\|_1^2 \int_{\mathbb{S}} \left(\int_{\mathbb{R}^2} |F(a, \mathbf{b} - \mathbf{y}, \theta)|\ |\phi(\mathbf{y})| \frac{d\mathbf{y}}{\|\phi\|_1} \right)^2 \frac{da}{a^3} d\mathbf{b} d\theta \\ &\leqslant \|\phi\|_1^2 \int_{\mathbb{S}} \int_{\mathbb{R}^2} |F(a, \mathbf{b} - \mathbf{y}, \theta)|^2 |\phi(\mathbf{y})| \frac{d\mathbf{y}}{\|\phi\|_1} \frac{da}{a^3} d\mathbf{b} d\theta \\ &\text{(since } t \mapsto t^2 \text{ is a convex function on } [0, \infty)) \\ &= \|\phi\|_1 \int_{\mathbb{R}^2} \int_{\mathbb{S}} |F(a, \mathbf{b} - \mathbf{y}, \theta)|^2 \frac{da}{a^3} d\mathbf{b} d\theta |\phi(\mathbf{y})|\ d\mathbf{y} \\ &= \|\phi\|_1 \int_{\mathbb{R}^2} \int_{\mathbb{S}} |F(a, \mathbf{c}, \theta)|^2 \frac{da}{a^3} d\mathbf{c} d\theta |\phi(\mathbf{y})|\ d\mathbf{y} \\ &\text{(Applying the change of variable } \mathbf{c} = \mathbf{b} - \mathbf{y}) \\ &= |||F|||_2^2 \|\phi\|_1^2.\end{aligned}$$

This completes the proof of this lemma. □

Lemma 13. *If* $F_n \to F$ *in* $\mathfrak{L}^2(\mathbb{S})$ *as* $n \to \infty$ *and* $\phi \in \mathscr{L}^1(\mathbb{R}^2)$, *then* $F_n \times \phi \to F \times \phi$ *in* $\mathfrak{L}^2(\mathbb{S})$ *as* $n \to \infty$.

Proof. The lemma follows from the inequality, $|||F \times \phi|||_2 \leqslant |||F|||_2 \|\phi\|_1$, which is obtained in the proof of the previous lemma. □

Lemma 14. *Let* $F, F_1, F_2 \in \mathfrak{L}^2(\mathbb{S}), c_1, c_2 \in \mathbb{C}$ *and* $\phi, \phi_1, \phi_2 \in \mathscr{L}^1(\mathbb{R}^2)$. *Then*

1. $(c_1F_1 + c_2F_2) \times \phi = c_1(F_1 \times \phi) + c_2(F_2 \times \phi)$.
2. $F \times (\phi_1 * \phi_2) = (F \times \phi_1) \times \phi_2$.

Proof. The proof of the first statement is straightforward and hence we prove only the second statement. Let $(a, \mathbf{b}, \theta) \in \mathbf{S}$ be arbitrary. Using Fubini's theorem, we get

$$\begin{aligned}(F \times (\phi_1 * \phi_2))(a, \mathbf{b}, \theta) &= \int_{\mathbb{R}^2} F(a, \mathbf{b} - \mathbf{x}, \theta) \int_{\mathbb{R}^2} \phi_1(\mathbf{x} - \mathbf{y})\phi_2(\mathbf{y})\ d\mathbf{y}\ d\mathbf{x} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} F(a, \mathbf{b} - \mathbf{x}, \theta)\phi_1(\mathbf{x} - \mathbf{y})\ d\mathbf{x}\phi_2(\mathbf{y})\ d\mathbf{y} \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} F(a, \mathbf{b} - (\mathbf{z} + \mathbf{y}), \theta)\phi_1(\mathbf{z})\ d\mathbf{z}\phi_2(\mathbf{y})\ d\mathbf{y} \\ &\qquad \text{(replacing } \mathbf{x} \text{ by } \mathbf{z} \text{ through } \mathbf{z} = \mathbf{x} - \mathbf{y}\text{)} \\ &= \int_{\mathbb{R}^2} (F \times \phi_1)(a, \mathbf{b} - \mathbf{y}, \theta)\phi_2(\mathbf{y})\ d\mathbf{y} \\ &= ((F \times \phi_1) \times \phi_2)(a, \mathbf{b}, \theta).\end{aligned}$$

This completes the proof of this lemma. □

We can prove the following two lemmas by slightly modifying the proofs of Lemmas 9 and 10.

Lemma 15. *If* $F \in \mathfrak{L}^2(\mathbb{S})$ *and* $(\phi_n) \in \Delta_T$, *then* $F \times \phi_n \to F \times \phi$ *in* $\mathfrak{L}^2(\mathbb{S})$ *as* $n \to \infty$.

Lemma 16. *If* $F_n \to F$ *in* $\mathfrak{L}^2(\mathbb{S})$ *as* $n \to \infty$ *and* $(\phi_n) \in \Delta_T$, *then* $F_n \times \phi_n \to F$ *in* $\mathfrak{L}^2(\mathbb{S})$ *as* $n \to \infty$.

Thus the Bohmian space $\mathscr{B}^2(\mathbb{S}) = \mathscr{B}(\mathfrak{L}^2(\mathbb{S}), (\mathscr{L}^1(\mathbb{R}^2), *), \times, \Delta_T)$ has been constructed. Next, we prove the convolution theorems which are applied to extend the curvelet transform to the Boehmian space $\mathscr{B}^2_{a_0}(\mathbb{R}^2)$ in the next section.

Lemma 17. *For* $0 < a < a_0$ *and* $\theta \in [-\pi, \pi]$,

$$\hat{\gamma}_{a,\mathbf{0},\theta}(r\mathbf{e}^{i\omega}) = W(a\ r)V((\omega - \theta)/\sqrt{a})a^{3/4},\ \forall r\mathbf{e}^{i\omega} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}.$$

Proof. First we observe that if $\mathbf{x} \cdot \mathbf{y}$ is the usual scalar product of two vectors in $\mathbb{R}^2$, then

$$\mathbf{e}^{i\omega} \cdot \mathbf{e}^{i(\eta+\theta)} = \mathbf{e}^{i(\omega-\theta)} \cdot \mathbf{e}^{i\eta}. \tag{2.4}$$

For $0 < a < a_0$ and $\theta \in [-\pi, \pi]$,

$$\begin{aligned}
\hat{\gamma}_{a,\mathbf{0},\theta}(r\mathbf{e}^{i\omega}) &= \int_{\mathbb{R}^2} \gamma_{a,\mathbf{0},0}(R_\theta(\rho\mathbf{e}^{i\zeta}))\mathbf{e}^{-ir\mathbf{e}^{i\omega}\cdot\rho\mathbf{e}^{i\zeta}}\ d(\rho\mathbf{e}^{i\zeta}) \\
&= \int_{\mathbb{R}^2} \gamma_{a,\mathbf{0},0}(\rho\mathbf{e}^{i(\zeta-\theta)})\mathbf{e}^{-ir\mathbf{e}^{i\omega}\cdot\rho\mathbf{e}^{i\zeta}}\ d(\rho\mathbf{e}^{i\zeta}) \\
&= \int_{\mathbb{R}^2} \gamma_{a,\mathbf{0},0}(\rho\mathbf{e}^{i\eta})\mathbf{e}^{-ir\mathbf{e}^{i\omega}\cdot\rho\mathbf{e}^{i(\eta+\theta)}}\ d(\rho\mathbf{e}^{i\eta}) \\
&\quad \text{(using the change of variable } \eta = \zeta - \theta) \\
&= \int_{\mathbb{R}^2} \gamma_{a,\mathbf{0},0}(\rho\mathbf{e}^{i\eta})\mathbf{e}^{-ir\mathbf{e}^{i(\omega-\theta)}\cdot\rho\mathbf{e}^{i\eta}}\ d(\rho\mathbf{e}^{i\eta}) \quad \text{(using 2.4)} \\
&= \hat{\gamma}_{a,\mathbf{0},0}(r\mathbf{e}^{i(\omega-\theta)}) \\
&= W(a\ r)V((\omega-\theta)/\sqrt{a})a^{3/4} \quad \text{(using 1.1)}
\end{aligned}$$

Hence the lemma follows. □

Theorem 18 (*Convolution theorem*). *If* $f \in \mathscr{L}^2_{a_0}(\mathbb{R}^2)$ *and* $\phi \in \mathscr{L}^1(\mathbb{R}^2)$, *then* $\Gamma(f * \phi) = (\Gamma f) \times \phi$.

Proof. Let $(a, \mathbf{b}, \theta) \in \mathbb{S}$ be arbitrary. Applying Fubini's theorem, we get that

$$\begin{aligned}
(\Gamma(f*\phi))(a,\mathbf{b},\theta)z &= \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{x})}(f*\phi)(\mathbf{x})\ d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{x})} \int_{\mathbb{R}^2} f(\mathbf{x}-\mathbf{y})\phi(\mathbf{y})\ d\mathbf{y}\ d\mathbf{x} \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{x})} f(\mathbf{x}-\mathbf{y})\ d\mathbf{x}\phi(\mathbf{y})\ d\mathbf{y} \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{b},\theta}(\mathbf{z}+\mathbf{y})} f(\mathbf{z})\ d\mathbf{z}\phi(\mathbf{y})\ d\mathbf{y} \\
&\quad \text{(applying the change of variable } \mathbf{z} = \mathbf{x}-\mathbf{y}) \\
&= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \overline{\gamma_{a,\mathbf{b}-\mathbf{y},\theta}(\mathbf{z})} f(\mathbf{z})\ d\mathbf{z}\ \phi(\mathbf{y})\ d\mathbf{y} \quad \text{(Using (1.2))} \\
&= \int_{\mathbb{R}^2} (\Gamma f)(\mathbf{b}-\mathbf{y})\phi(\mathbf{y})\ d\mathbf{y} \\
&= ((\Gamma f) \times \phi)(a,\mathbf{b},\theta).
\end{aligned}$$

Thus the proof is completed. □

3. Extended curvelet transform

Definition 19. The extended curvelet transform $\Gamma : \mathscr{B}^2_{a_0}(\mathbb{R}^2) \to \mathscr{B}^2(\mathbb{S})$ is defined by

$$\Gamma\left(\left[\frac{(f_n)}{(\phi_n)}\right]\right) = [(\Gamma f_n)/(\phi_n)].$$

Lemma 20. *The above definition is well defined.*

Proof. Let $\left[\frac{(f_n)}{(\phi_n)}\right] \in \mathscr{B}^2_{a_0}(\mathbb{R}^2)$. Then $f_n \in \mathscr{L}^2_{a_0}(\mathbb{R}^2), \forall n \in \mathbb{N}$ and $(\phi_n) \in \Delta_T$.

This implies that $\Gamma f_n \in \mathfrak{L}^2(\mathbb{S}), \forall n \in \mathbb{N}$ and

$$f_n * \phi_m = f_m * \phi_n, \ \forall m, n \in \mathbb{N}.$$

Applying Theorem 18, we get

$$(\Gamma f_n) \times \phi_m = (\Gamma f_m) \times \phi_n, \ \forall m, n \in \mathbb{N}.$$

Therefore, $[(\Gamma f_n)/(\phi_n)]$ is a Boehmian in $\mathscr{B}^2_{a_0}(\mathbb{R}^2)$. Next we show that the definition of Γ is independent of the choice of the representatives of the Boehmians. If $\left[\frac{(f_n)}{(\phi_n)}\right] = \left[\frac{(g_n)}{(\psi_n)}\right]$ in $\mathscr{B}^2_{a_0}(\mathbb{R}^2)$. Then we have

$$f_n * \psi_m = g_m * \phi_n, \ \forall m, n \in \mathbb{N}.$$

By applying curvelet transform and by using Theorem 18, we obtain that

$$\Gamma f_n \times \psi_m = \Gamma g_m \times \phi_n, \ \forall m, n \in \mathbb{N}.$$

Thus, $\Gamma\left(\left[\frac{(f_n)}{(\phi_n)}\right]\right) = \Gamma\left(\left[\frac{(g_n)}{(\psi_n)}\right]\right)$ in $\mathscr{B}^2(\mathbb{S})$ and hence $\Gamma : \mathscr{B}^2_{a_0}(\mathbb{R}^2) \to \mathscr{B}^2(\mathbb{S})$ is a well defined function. □

We note that the extended curvelet transform $\Gamma : \mathscr{B}^2_{a_0}(\mathbb{R}^2) \to \mathscr{B}^2(\mathbb{S})$ is consistent with the classical curvelet transform $\Gamma : \mathscr{L}^2_{a_0}(\mathbb{R}^2) \to \mathfrak{L}^2(\mathbb{S})$. More explicitely, if $\mathscr{I}_1 : \mathscr{L}^2_{a_0}(\mathbb{R}^2) \to \mathscr{B}^2_{a_0}(\mathbb{R}^2)$ and $\mathscr{I}_2 : \mathfrak{L}^2(\mathbb{S}) \to \mathscr{B}^2(\mathbb{S})$ are the identification mappings defined by

$$\mathscr{I}_1(f) = \left[\frac{(f * \phi_n)}{(\phi_n)}\right] \text{ and } \mathscr{I}_2(F) = [(F \times \phi_n)/(\phi_n)],$$

where $(\phi_n) \in \Delta_T$ is arbitrary, then $(\Gamma \circ \mathscr{I}_1)(f) = (\mathscr{I}_2 \circ \Gamma)(f), \ \forall f \in \mathscr{L}^2_{a_0}(\mathbb{R}^2)$.

Indeed, if $f \in \mathscr{L}^2_{a_0}(\mathbb{R}^2)$, then

$$(\Gamma \circ \mathscr{I}_1)(f) = \Gamma\left[\frac{(f * \phi_n)}{(\phi_n)}\right] = [(\Gamma(f * \phi_n))/(\phi_n)] = [(\Gamma f \times \phi_n)/(\phi_n)] = (\mathscr{I}_2 \circ \Gamma)(f).$$

Theorem 21. *The extended curvelet transform* $\Gamma : \mathscr{B}^2_{a_0}(\mathbb{R}^2) \to \mathscr{B}^2(\mathbb{S})$ *is linear.*

Proof. Proof of this theorem is straightforward by using Theorem 18 and the linearity of the curvelet transform $\Gamma : \mathscr{L}^2_{a_0}(\mathbb{R}^2) \to \mathfrak{L}^2(\mathbb{S})$. □

Theorem 22. *The extended curvelet transform* $\Gamma : \mathscr{B}^2_{a_0}(\mathbb{R}^2) \to \mathscr{B}^2(\mathbb{S})$ *is one-to-one.*

Proof. Let $X \in \mathscr{B}^2_{a_0}(\mathbb{R}^2)$ be such that $\Gamma X = 0$. If $X = \left[\frac{(f_n)}{(\phi_n)}\right]$, then by assumption, we have $[(\Gamma f_n)/(\phi_n)] = 0$, and hence $\Gamma f_n \times \psi_m = 0, \ \forall m, n \in \mathbb{N}$. Then, applying Theorem 18, we get $\Gamma(f_n * \psi_m) = 0, \ \forall m, n \in \mathbb{N}$. Using the injectivity of $\Gamma : \mathscr{L}^2_{a_0}(\mathbb{R}^2) \to \mathfrak{L}^2(\mathbb{S})$, we obtain that $f_n * \psi_m = 0,\ \forall m, n \in \mathbb{N}$. So, by Lemma 9, we have $f_n = \lim_{m\to\infty} f_n * \delta_m = 0, \forall n \in \mathbb{N}$. Therefore, $X = 0$ and hence Γ is one-to-one, since it is a linear map. □

Theorem 23. *The range of extended curvelet transform $\Gamma : \mathscr{B}^2_{a_0}(\mathbb{R}^2) \to \mathscr{B}^2(\mathbb{S})$ is the subspace of $\mathscr{B}^2(\mathbb{S})$ consisting of all Y having a representation $[(F_n)/(\psi_n)]$ with $F_n \in \Gamma(\mathscr{L}^2_{a_0}(\mathbb{R}^2)), \forall n \in \mathbb{N}$.*

Proof. By definition, if $Y \in \Gamma(\mathscr{B}^2_{a_0}(\mathbb{R}^2))$, then there exists $\left[\frac{(f_n)}{(\phi_n)}\right] \in \mathscr{B}^2_{a_0}(\mathbb{R}^2)$ such that $\Gamma\left(\left[\frac{(f_n)}{(\phi_n)}\right]\right) = Y$. Obviously, $[(\Gamma f_n)/(\phi_n)]$ itself is a required representation of Y. Conversely, let $Y \in \mathscr{B}^2(\mathbb{S})$ be such that Y has a representation $[(F_n)/(\psi_n)]$ such that $F_n \in \Gamma(\mathscr{L}^2_{a_0}(\mathbb{R}^2)),\ \forall n \in \mathbb{N}$. Then, there exists $f_n \in \mathscr{L}^2_{a_0}(\mathbb{R}^2)$ such that $\Gamma f_n = F_n, \forall n \in \mathbb{N}$. We claim that $\left[\frac{(f_n)}{(\psi_n)}\right]$ is a Boehmian in $\mathscr{B}^2_{a_0}(\mathbb{R}^2)$. From $[(F_n)/(\psi_n)] \in \mathscr{B}^2(\mathbb{S})$, we have

$$(\Gamma f_n) \times \psi_m = F_n \times \psi_m = F_m \times \psi_n = (\Gamma f_m) \times \psi_n, \forall m, n \in \mathbb{N}.$$

Then Theorem 18 implies that

$$\Gamma(f_n * \psi_m) = \Gamma(f_m * \psi_n),\ \forall m, n \in \mathbb{N}.$$

By invoking the injectivity of $\Gamma : \mathscr{L}^2_{a_0}(\mathbb{R}^2) \to \mathfrak{L}^2(\mathbb{S})$, we obtain that $f_n * \psi_m = f_m * \psi_n,\ \forall m, n \in \mathbb{N}$ and hence our claim holds. Then, $\left[\frac{(f_n)}{(\psi_n)}\right] \in \mathscr{B}^2_{a_0}(\mathbb{R}^2)$ and $\Gamma\left(\left[\frac{(f_n)}{(\psi_n)}\right]\right) = [(\Gamma f_n)/(\psi_n)] = [(F_n)/(\psi_n)]$. □

As an immediate consequence of the previous theorem, we can describe the range $\Gamma(\mathscr{B}^2_{a_0}(\mathbb{R}^2))$ as the Boehmian space $\mathscr{B}\left(\Gamma\left(\mathscr{L}^2_{a_0}(\mathbb{R}^2)\right), (\mathscr{L}^1(\mathbb{R}^2), *), \times, \Delta_T\right)$. At this juncture, we point out that characterizing $\Gamma\left(\mathscr{L}^2_{a_0}(\mathbb{R}^2)\right)$ is an interesting open problem.

Definition 24. The extended inverse curvelet transform $\Gamma(\mathscr{B}^2_{a_0}(\mathbb{R}^2)) \to \mathscr{B}^2_{a_0}(\mathbb{R}^2)$ is defined by $\Gamma^{-1}[(F_n)/(\phi_n)] = \left[\frac{(\Gamma^{-1}F_n)}{(\phi_n)}\right],\ \forall [(F_n)/(\phi_n)] \in \Gamma(\mathscr{B}^2_{a_0}(\mathbb{R}^2))$.

Theorem 25. *If $X \in \mathscr{B}^2_{a_0}(\mathbb{R}^2)$ and $\phi \in \mathscr{L}^1(\mathbb{R}^2)$, then $\Gamma(X * \phi) = \Gamma(X) \times \phi$.*

Proof. Let $X = \left[\frac{(f_n)}{(\phi_n)}\right]$. Then, applying Theorem 18, we obtain that

$$\begin{aligned}
\Gamma(X * \phi) &= \Gamma\left(\left[\frac{(f_n * \phi)}{(\phi_n)}\right]\right) \\
&= [(\Gamma(f_n * \phi))/(\phi_n)] \\
&= [(\Gamma f_n \times \phi)/(\phi_n)] \\
&= [(\Gamma f_n)/(\phi_n)] \times \phi \\
&= \Gamma(X) \times \phi.
\end{aligned}$$

Thus the theorem follows. $\square$

Theorem 26. *The extended curvelet transform* $\Gamma : \mathscr{B}^2_{a_0}(\mathbb{R}^2) \to \mathscr{B}^2(\mathbb{S})$ *is continuous with respect to* δ*-convergence as well as* Δ*-convergence.*

Proof. Let $X_m \xrightarrow{\delta} X$ as $m \to \infty$ in $\mathscr{B}^2_{a_0}(\mathbb{R}^2)$. Then by [16, Lemma 2.4], there exists $f_{m,n}, f_n \in \mathscr{L}^2_{a_0}(\mathbb{R}^2)$ and $(\phi_n) \in \Delta_T$ such that $X_m = \left[\frac{(f_{m,n})}{(\phi_n)}\right], X = \left[\frac{(f_n)}{(\phi_n)}\right]$ and

$$\text{for each } n \in \mathbb{N},\ f_{m,n} \to f_n \text{ in } \mathscr{L}^2_{a_0}(\mathbb{R}^2) \text{ as } m \to \infty.$$

Since $\Gamma : \mathscr{L}^2_{a_0}(\mathbb{R}^2) \to \mathfrak{L}^2(\mathbb{S})$ is continuous we have

$$\Gamma f_{m,n} \to \Gamma f_n \text{ as } m \to \infty \text{ in } \mathfrak{L}^2(\mathbb{S}).$$

Since $\Gamma X_m = [(\Gamma f_{m,n})/(\phi_n)]$, $\forall m \in \mathbb{N}$ and $\Gamma X = [(\Gamma f_n)/(\phi_n)]$, we get $\Gamma X_n \xrightarrow{\delta} \Gamma X$ as $n \to \infty$ in $\mathscr{B}^2(\mathbb{S})$.

Let $X_n \xrightarrow{\Delta} X$ as $n \to \infty$ in $\mathscr{B}^2_{a_0}(\mathbb{R}^2)$. Then by definition, there exists $h_n \in \mathscr{L}^2_{a_0}(\mathbb{R}^2), \forall n \in \mathbb{N}$ and $(\phi_n) \in \Delta_T$ such that

$$(X_n - X) * \phi_n = \left[\frac{h_n * \phi_k}{\phi_k}\right], \forall n \in \mathbb{N} \text{ and } h_n \to 0 \text{ in } \mathscr{L}^2_{a_0}(\mathbb{R}^2) \text{ as } n \to \infty.$$

Since the curvelet transform $\Gamma : \mathscr{L}^2_{a_0}(\mathbb{R}^2) \to \mathfrak{L}^2(\mathbb{S})$ is continuous, $\Gamma h_n \to 0$ as $n \to \infty$ in $\mathfrak{L}^2(\mathbb{S})$. Applying Theorems 21 and 25, for each $n \in \mathbb{N}$, we obtain that

$$\begin{aligned}
(\Gamma X_n - \Gamma X) \times \phi_n &= \Gamma(X_n - X) \times \phi_n \\
&= \Gamma((X_n - X) * \phi_n) \\
&= \Gamma\left[\frac{(h_n * \phi_k)}{(\phi_k)}\right] \\
&= [(\Gamma(h_n * \phi_k))/(\phi_k)] \\
&= [(\Gamma h_n \times \phi_k)/(\phi_k)].
\end{aligned}$$

Therefore, it follows that $\Gamma X_n \xrightarrow{\Delta} \Gamma X$ as $n \to \infty$ in $\mathscr{B}^2(\mathbb{S})$. Hence, Γ is continuous with respect to Δ-convergence. $\square$

By a similar set of arguments used for extended curvelet transform, one can prove that extended inverse curvelet transform also is consistent with the inverse curvelet transform on $\Gamma\left(\mathscr{L}^2_{a_0}(\mathbb{R}^2)\right)$ and continuous with respect to δ-convergence as well as Δ-convergence.

4. Conclusion

We slightly modify the example of a Boehmian not representing any distribution given in [16], so that it belongs to $\mathscr{B}^2_{a_0}(\mathbb{R}^2)$. Since $\sum \frac{(2(n-1))!}{(2n)!} = \sum \frac{1}{4n^2-2n} < \infty$, by Denjoy-Carleman theorem [11,6], we conclude that $C\{(2n)!\}$ is not a quasi-analytic class. Therefore there exists $\phi \in \mathscr{D}(\mathbb{R}^2)$ such that $\phi(0) \neq 0$ and $\phi^{(\mathbf{k})}(0) = 0$, $\forall \mathbf{k} \in \mathbb{N}_0^2$ with

$\sup_{\mathbf{x}\in\mathbb{R}^2}|\phi^{(\mathbf{k})}(\mathbf{x})| \leqslant \beta B^{|\mathbf{k}|}(2\mathbf{k})!$, for some $\beta, B \in (0,\infty)$, where $\mathbf{k} = (k_1, k_2), |\mathbf{k}| = k_1 + k_2$, and $(2\mathbf{k})! = (2k_1)!(2k_2)!$. For each $n \in \mathbb{N}$, define

$$f_n(\mathbf{x}) = \sum_{|\mathbf{k}|=0}^{\infty} \frac{\phi_n^{(\mathbf{k})}(\mathbf{x})}{(3\mathbf{k})!}, \ \forall \mathbf{x} \in \mathbb{R}^2, \text{where } \phi_n(\mathbf{x}) = n^2\phi(n\mathbf{x}), \ \forall \mathbf{x} \in \mathbb{R}.$$

If $X = \left[\frac{(f_n)}{(\phi_n)}\right]$, then X is a C^∞-Boehmian and it does not represent any distribution. To prove that this Boehmian X belongs to $\mathscr{B}^2_{a_0}(\mathbb{R}^2)$, we first note that if $\psi \in \mathscr{L}^1(\mathbb{R}^2) \cap \mathscr{L}^2_{a_0}(\mathbb{R}^2)$ is such that $\int_{\mathbb{R}^2} \psi(\mathbf{x})\, d\mathbf{x} = 1$ and $\int_{\mathbb{R}^2} |\mathbf{x}|\ |\psi(\mathbf{x})|\, d\mathbf{x} < \infty$, then $(\psi_n) \in \Delta_T$, where $\psi_n(\mathbf{x}) = n^2\psi(n\mathbf{x}), \ \forall \mathbf{x} \in \mathbb{R}^2, \forall n \in \mathbb{N}$.

If we choose, $(\psi_n) \in \Delta_T$ as described above, by using Lemmas 5 and 8, we get $f_n * \psi_n \in \mathscr{L}^2_{a_0}(\mathbb{R}^2), \forall n \in \mathbb{N}$ (since $f_n \in \mathscr{D}(\mathbb{R}^2) \subset \mathscr{L}^1(\mathbb{R}^2)$ and $\psi_n \in \mathscr{L}^2_{a_0}(\mathbb{R}^2)$) and $(\phi_n * \psi_n) \in \Delta_T$. Therefore, $X = \left[\frac{(f_n)}{(\phi_n)}\right] = \left[\frac{(f_n*\psi_n)}{(\phi_n*\psi_n)}\right] \in \mathscr{B}^2_{a_0}(\mathbb{R}^2)$. In fact, the space $\mathscr{D}'_{\mathscr{L}^2}(\mathbb{R}^2)$ is identified as a proper subspace of $\mathscr{B}_{\mathbb{R}^2}$ by the map $u \mapsto \left[\frac{(u*\psi_n)}{(\psi_n)}\right]$, where (ψ_n) is as described above and $(u * \psi_n)(x) = \langle u(\mathbf{t}), \psi_n(\mathbf{x} - \mathbf{t})\rangle, \ \forall \mathbf{x} \in \mathbb{R}^2$.

Acknowledgement

The authors express their sincere thanks to the referee for valuable comments and suggestions.

References

[1] T.K. Boehme, The support of Mikusiński operators, Trans. Amer. Math. Soc. 176 (1973) 319–334.
[2] E.J. Candès, Ridgelets: Theory and applications, Ph.D. Thesis, Department of Statistics, Stanford University, (1998).
[3] E.J. Candès, Harmonic analysis of nueral networks, Appl. Comput. Harmon. Anal. 6 (1999) 197–218.
[4] E.J. Candès, D.L. Donoho, Continuous curvelet transform: I. resolution of the wavefront set, Appl. Comput. Harmon. Anal. 19 (2005) 162–197.
[5] E.J. Candès, D.L. Donoho, Continuous curvelet transform: II. discretization and frames, Appl. Comput. Harmon. Anal. 19 (2005) 198–222.
[6] T. Carleman, Les fonctions quasi-analytiquesCollection Borel, Gauthier-Villars, Paris, 1926.
[7] C.K. Chui, An introduction to wavelets, Academic Press, Boston, 1992.
[8] I. Daubechies, Orthonormal bases of compactly supported wavelets, Comm. Pure Appl. Math. 41 (1998) 909–996.
[9] I. Daubechies, Wavelet transform, time - frequency localizations and signal analysis, IEEE Trans. Inform. Theory 36 (1990) 961–1005.
[10] I. Daubechies, Ten lectures on wavelets, SIAM, Philadelphia, 1992.
[11] A. Denjoy, Sur les fonctions quasi-analytiques de variable réelle, C.R. Acad. Sci. Paris 173 (1921) 1320–1322.
[12] S.G. Mallat, Multiresolution approximations and wavelet orthonormal bases of $L^2(R)$, Trans. Amer. Math. Soc. 315 (1989) 69–87.
[13] Y. Meyer, Wavelets and operators, Cambridge University Press, Cambridge, 1993.
[14] Y. Meyer, R.D. Rya, Wavelets: Algorithms and applications, SIAM, Philadelphia, 1994.
[15] J. Mikusiński, P. Mikusiński, Quotients de suites et leurs applications dans l'anlyse fonctionnelle, C.R. Acad. Sci. Paris 293 (1981) 463–464.

[16] P. Mikusiński, Convergence of Boehmians, Japan J. Math. 9 (1983) 159–179.
[17] P. Mikusiński, Tempered Boehmians and ultra distributions, Proc. Amer. Math. Soc. 123 (1995) 813–817.
[18] P. Mikusiński, On flexibility of Boehmians, Integral Transforms Spec. Funct. 4 (1996) 141–146.
[19] R.S. Pathak, Integral transform for generalized functions and their applications, Gordon and Breach Science Publishers, Amsterdam, 1997.
[20] R. Roopkumar, Wavelet analysis on a Boehmian space, Int. J. Math. Math. Sci. 15 (2003) 917–926.
[21] R. Roopkumar, An extension of distributional wavelet transform, Colloq. Math. 115 (2009) 195–206.
[22] R. Roopkumar, Convolution theorems for wavelet transform on tempered distributions and their extensions to tempered Boehmians, Asian-Eur. J. Math. 2 (2009) 117–127.
[23] R. Roopkumar, Ridgelet transform on square integrable Boehmians, Bull. Korean Math. Soc. 46 (2009) 835–844.
[24] R. Roopkumar, Ridgelet transform on tempered distributions, Comment. Math. Univ. Carolin. 51 (2010) 431–439.
[25] R. Roopkumar, Extended ridgelet transform on distributions and Boehmians, Asian-Eur. J. Math. 4 (2011) 207–521.
[26] R. Roopkumar, Extension of ridgelet transform to tempered Boehmians, Novi Sad J. Math. 42 (2012) 19–32.
[27] J.L. Starck, E.J. Candès, D.L. Donoho, The curvelet transform for image denoising, IEEE Trans. Image Process. 11 (2002) 670–684.
[28] J.L. Starck, D.L. Donoho, E.J. Candès, Astronomical image representation by the curvelet transform, Astronomy and Astrophysics 398 (2003) 785–800.