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Crosscap of the ideal based zero-divisor graph

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Abstract. Let R be a commutative ring and I be an ideal of R. The ideal based zero-divisor graph, denoted by $\Gamma_I(R)$, is the graph with the vertex set $\{x \in R - I : xy \in I \text{ for some } y \in R - I\}$ and two distinct vertices x and y are adjacent if and only if $xy \in I$. In this paper, we classify all finite quotient rings R/I and ideals I of R for which the crosscap of $\Gamma_I(R)$ is at most one. Moreover, we investigate certain properties on the crosscap of $\Gamma_I(R)$ in the general case also.

Keywords: Zero-divisor graph; Ideal based zero-divisor graph; Local ring; Crosscap of a graph

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1. INTRODUCTION

Through out this paper R denotes a commutative ring with $1 \neq 0$. Let I be an ideal of R and $Z(R)$ be the set of all zero-divisors of R. The study of zero-divisor graphs has become an exciting research topic in the last twenty years, leading to many fascinating results and developments. The concept of the zero-divisor graph of a commutative ring was due to Anderson and Livingston in [3]. For a commutative ring R, the zero-divisor graph of R, denoted as $\Gamma(R)$, is the graph whose vertices are the non-zero zero-divisors of R and two distinct vertices x and y are adjacent if and only if $xy = 0$. Several authors extensively studied about the zero-divisor graph of a commutative ring R, for instance see [2,3,10]. Recently, Redmond [11] generalized the concept of zero-divisor graph and introduced the ideal based zero-divisor graph of R. For an ideal I of R, the ideal based

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zero-divisor graph $\Gamma_I(R)$ is a graph with vertex set $\{x \in R - I : xy \in I$ for some $y \in R - I\}$ and in which distinct vertices x and y are adjacent if and only if $xy \in I$. In the case of $I = \{0\}$, $\Gamma_I(R) = \Gamma(R)$. Also note that if I is a prime ideal of R, then the graph $\Gamma_I(R)$ is empty. The ideal based zero-divisor graph $\Gamma_I(R)$ provides an excellent setting for studying some aspects of algebraic properties of commutative rings. Especially, the embeddings of ideal based zero-divisor graph $\Gamma_I(R)$ help us to explore some interesting results related to algebraic structures of rings. Certain domination properties of the ideal based zero-divisor graph of a near-ring were studied by authors in [15].

The main objective of topological graph theory is to embed a graph into a surface. There are many studies [1,4,5,9,12–14,16] concerning embeddings of the zero-divisor graph and other graphs. In [17], similar embeddings were discussed for ideal based zero-divisor graphs. Recently, Hsieh [8] investigated the crosscap (i.e., genus of non-orientable surface) of the zero-divisor graph $\Gamma(R)$ and illustrated all finite commutative rings R (up to isomorphism) for which $\Gamma(R)$ has crosscap one.

In this connection, we establish a goal for an embedding of $\Gamma_I(R)$ in a non-orientable surface. In Section 2, first we state some known results and easy observations in order to obtain our main results. We give some necessary and sufficient conditions for the crosscap to be of at most one. Also we give some embeddings to show that there exist ideal based zero-divisor graphs with crosscap one. In Section 3, we investigate the crosscap of $\Gamma_I(R)$ in general case. Finally we give a sufficient condition, for a general commutative ring R and a nonzero ideal I of R such that crosscap upper bound of $\Gamma_I(R)$ is one.

Let G be a simple graph with vertex set $V(G)$. A graph G is said to be complete bipartite graph if $V(G)$ can be partitioned into two disjoint sets V_1, V_2 such that no two vertices of V_1 or V_2 are adjacent and every vertex of V_1 is adjacent to every vertex of V_2. $K_{m,n}$ denotes the complete bipartite graph where $|V_1| = m$ and $|V_2| = n$. A graph G is complete if each pair of distinct vertices in G is adjacent and K_n denotes the complete graph with n vertices. For notations and terminology and basic results on graph theory, we refer [7].

Let $\bar{S}_k$ denote the sphere with k crosscaps, where k is a non-negative integer, that is, $\bar{S}_k$ is a non-oriented surface with k crosscaps. The crosscap of a graph G, denoted as $\bar{\gamma}(G)$, is the minimal integer n such that the graph G can be embedded in $\bar{S}_n$. Intuitively, G is embedded in a surface if it can be drawn in the surface so that its edges intersect only at their common vertices. We say a graph G is planar if $\bar{\gamma}(G) = 0$, and projective if $\bar{\gamma}(G) = 1$. It is easy to see that $\bar{\gamma}(H) \le \bar{\gamma}(G)$ for all subgraphs H of G. The following results are useful in the subsequent sections.

Lemma 1.1 ([6]). *Let m, n be integers and for a real number x, $\lceil x \rceil$ is the least integer that is greater than or equal to x. Then*

(i) $$\bar{\gamma}(K_n) = \begin{cases} \left\lceil \frac{1}{6}(n-3)(n-4) \right\rceil & \text{if } n \ge 3 \text{ and } n \ne 7; \\ 3 & \text{if } n = 7. \end{cases}$$

(ii) $\bar{\gamma}(K_{m,n}) = \left\lceil \frac{1}{2}(m-2)(n-2) \right\rceil$, *where* $m, n \ge 2$.

Lemma 1.2 ([8]). *Suppose that H and H' are two subgraphs of a graph G such that H and H' are isomorphic to $K_{3,3}$ or K_5. If $H \cap H' = \{v\}$, where v is a vertex of G, then $\bar{\gamma}(G) > 1$.*

2. Crosscap one in the finite case

In this section, we determine all finite quotient rings R/I and ideals I of R for which the crosscap of $\Gamma_I(R)$ is at most one. To attain this, we need the following results.

Theorem 2.1 *([11], Theorem 7.2).* *Let R be a finite ring and I be a nonzero ideal of R. Then $\Gamma_I(R)$ is planar if and only if $\Gamma(\frac{R}{I})$ contains no cycles and either (a) $|I| = 2$ or (b) $|\Gamma(\frac{R}{I})| = 1$ with $|I| \leq 4$.*

Theorem 2.2 *([9], Theorem 3.5.1).* *Let (R, M) be a finite local ring which is not a field. Then $\Gamma(R)$ is planar if and only if R is isomorphic to one of the following 29 rings: $\mathbb{Z}_4$, $\mathbb{Z}_8$, $\mathbb{Z}_9$, $\mathbb{Z}_{16}$, $\mathbb{Z}_{25}$, $\mathbb{Z}_{27}$, $\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_2[x]}{(x^4)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,y^2)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^3,xy,y^2-x^2)}$, $\frac{\mathbb{F}_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_3[x]}{(x^2)}$, $\frac{\mathbb{Z}_3[x]}{(x^3)}$, $\frac{\mathbb{Z}_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2-2,x^4)}$, $\frac{\mathbb{Z}_4[x]}{(x^3-2,x^4)}$, $\frac{\mathbb{Z}_4[x]}{(x^2-2,x^3)}$, $\frac{\mathbb{Z}_4[x]}{(x^3,x^2-2x)}$, $\frac{\mathbb{Z}_4[x]}{(x^3+x^2-2,x^4)}$, $\frac{\mathbb{Z}_4[x,y]}{(x^2,y^2,xy-2)}$, $\frac{\mathbb{Z}_4[x,y]}{(x^3,x^2-2,xy,y^2-2)}$, $\frac{\mathbb{Z}_5[x]}{(x^2)}$, $\frac{\mathbb{Z}_8[x]}{(x^2-4,2x)}$, $\frac{\mathbb{Z}_9[x]}{(x^2-3,x^3)}$, $\frac{\mathbb{Z}_9[x]}{(x^2+3,x^3)}$.*

Theorem 2.3 *([8], Theorem 2.8).* *Let (R, M) be a finite local ring which is not a field. Then $\bar{\gamma}(\Gamma(R)) = 1$ if and only if R is isomorphic to one of the following 13 rings. $\mathbb{Z}_{32}$, $\mathbb{Z}_{49}$, $\frac{\mathbb{Z}_2[x]}{(x^5)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^3,xy,y^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^3-2,x^5)}$, $\frac{\mathbb{Z}_4[x]}{(x^4-2,x^5)}$, $\frac{\mathbb{Z}_4[x]}{(x^3,2x)}$, $\frac{\mathbb{Z}_4[x,y]}{(x^3,x^2-2,xy,y^2)}$, $\frac{\mathbb{Z}_7[x]}{(x^2)}$, $\frac{\mathbb{Z}_8[x]}{(x^2,2x)}$, $\frac{\mathbb{Z}_8[x]}{(x^2-2,x^5)}$, $\frac{\mathbb{Z}_8[x]}{(x^2-2x+2,x^5)}$, $\frac{\mathbb{Z}_8[x]}{(x^2+2x-2,x^5)}$.*

Lemma 2.4 *([17], Lemma 4.5).* *If (R, M) is a finite local ring, $t = |\frac{R}{M}|$ and k is the smallest integer for which $M^k = 0$, then $|M^i| = t^{n_i}|M^{i+1}|$ for $i = 0, \ldots, k-1$. In particular, $|R| = t^n$ for some n.*

For later use, we list out some properties of $\Gamma_I(R)$ from [11,17].

Remark 2.5. Let I be an ideal of R.

(i) If $\{x_\lambda + I : \lambda \in \Lambda\}$ is the set of non-zero zero-divisors in $\frac{R}{I}$, then by the definition of $\Gamma_I(R)$, the vertex set of $\Gamma(\frac{R}{I})$ is $\{x_\lambda + a : \lambda \in \Lambda, a \in I\}$, and hence $|V(\Gamma_I(R))| = |I|.|V(\Gamma(\frac{R}{I}))|$.

(ii) Let $I = \{a_1, \ldots, a_t\}$ and $\{x_1 + I, \ldots, x_m + I\}$ be the set of non-zero zero-divisors of $\frac{R}{I}$. Then the vertex set of $\Gamma_I(R)$ is $\{x_j + a_i : 1 \leq i \leq t,\ 1 \leq j \leq m\}$ and the edge set of $\Gamma_I(R)$ is $\{(x_j + a_i)(x_\ell + a_k) : x_j x_\ell \in I\}$. From this, if $\Gamma(\frac{R}{I}) \cong K_{m,n}$, then $K_{mt,nt}$ is a subgraph of $\Gamma_I(R)$. Also, if $\Gamma(\frac{R}{I}) \cong K_n$, then $K_{t,t,\ldots,t_{(n\ times)}}$ is a subgraph of $\Gamma_I(R)$.

(iii) If $\frac{R}{I}$ has no nilpotent elements, then $\Gamma(\frac{R}{I}) \cong K_{m,n}$ implies $\Gamma_I(R) \cong K_{mt,nt}$. Further if $\Gamma(\frac{R}{I}) \cong K_n$, then $\Gamma_I(R) \cong K_{t,t,\ldots,t_{(n\ times)}}$. If every element in $Z(\frac{R}{I})$ is nilpotent, then $\Gamma(\frac{R}{I}) \cong K_n$ implies $\Gamma_I(R) \cong K_{nt}$.

(iv) By (i), (ii) and (iii), $\Gamma_I(R)$ depends on $\frac{R}{I}$ and $|I|$. From this, the graphs $\Gamma_I(R)$ and $\Gamma_{(0)\times A}(\frac{R}{I} \times A)$ are isomorphic, where A is any ring with $|A| = |I|$. For example, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $I = (0) \times (0) \times \mathbb{Z}_2$, then $\Gamma_I(R) \cong K_{2,2}$. Here $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore $\Gamma_{(0)\times(0)\times A}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times A)$, $|A| = 2$ and so $\Gamma_{(0)\times(0)\times\mathbb{Z}_2}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \cong K_{2,2}$. Hence $\Gamma_I(R) \cong \Gamma_{(0)\times(0)\times A}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times A)$.

The following results provide effective criterion for discussing crosscap one in the finite case.

Lemma 2.6. *Let R be a commutative ring and I be a nonzero ideal of R such that $(\frac{R}{I}, \frac{M}{I})$ is a finite local ring with $(\frac{M}{I})^2 \neq (0)$ and $|\frac{R}{M}| \geq 3$. Then $\bar{\gamma}(\Gamma_I(R)) > 1$.*

Proof. Let k be the index of $\frac{M}{I}$ and $k \geq 3$. By Lemma 2.4, $|(\frac{M}{I})^{k-1} - (0)| \geq |\frac{R}{M}| - 1 \geq 2$ and $|(\frac{M}{I})^{k-2} - (\frac{M}{I})^{k-1}| \geq (|\frac{R}{M}| - 1)|(\frac{M}{I})^{k-1}| \geq 4$. Therefore there are distinct elements $u_1, u_2 \in (\frac{M}{I})^{k-1} - \{0\}$ and $v_1, v_2 \in (\frac{M}{I})^{k-2} - (\frac{M}{I})^{k-1}$. Since $k \geq 3$, $u_i v_j = 0$ for all i, j. Thus $K_{2,2}$ is a subgraph of $\Gamma(\frac{R}{I})$. In view of $|I| \geq 2$ and Remark 2.5, $K_{4,4}$ is a subgraph of $\Gamma_I(R)$, so that Lemma 1.1 gives $\bar{\gamma}(\Gamma_I(R)) > 1$. □

Lemma 2.7. *Let $R \cong \mathbb{Z}_2 \times S$, where S is a finite local ring. Then the following holds:*

(i) *If $|V(\Gamma(S))| \leq 1$, then $\Gamma(R)$ is planar and contains no cycles;*
(ii) *If $|V(\Gamma(S))| > 1$, then K_3 and $K_{2,2}$ are subgraphs of $\Gamma(R)$.*

Proof. (i) If $|V(\Gamma(S))| = 0$, then S is a finite field and so $\Gamma(R) \cong K_{1,|S|}$. Hence $\Gamma(R)$ is planar and contains no cycles. If $|V(\Gamma(S))| = 1$, then by Theorem 2.2, $S \cong \mathbb{Z}_4$ or $S \cong \frac{\mathbb{Z}_2[x]}{(x^2)}$, so that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$. By Fig. 1 [10], $\Gamma(R)$ are trees.

(ii) Suppose that $|V(\Gamma(S))| = 2$. Then by Theorem 2.2, $S \cong \mathbb{Z}_9$ or $S \cong \frac{\mathbb{Z}_3[x]}{(x^2)}$, so that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_9$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_3[x]}{(x^2)}$. Again by Fig. 12 [10], K_3 and $K_{2,2}$ are subgraphs of $\Gamma(R)$. If $|V(\Gamma(S))| > 2$, then there exist three distinct elements $a, b, c \in V(\Gamma(S))$ such that $ab = ac = 0$. Let $u_1 = (0, b), u_2 = (0, c), v_1 = (1, 0)$ and $v_2 = (1, a)$, then $u_i v_j = 0$ for all i, j, so $K_{2,2}$ is a subgraph of $\Gamma(R)$. Also $v_1 - (0, a) - u_1 - v_1$ is a triangle in $\Gamma(R)$ and hence K_3 is a subgraph of $\Gamma(R)$. □

Lemma 2.8. *Let I be a nonzero ideal of R and $\frac{R}{I} \cong R_1 \times R_2 \times \cdots \times R_k$, where R_i is a finite local ring for every i and $k \geq 3$. If $\bar{\gamma}(\Gamma_I(R)) \leq 1$, then $k = 3$ and $R_i \cong \mathbb{Z}_2$ for every i.*

Proof. Suppose that $k = 4$. Let $u_1 = (1, 0, 0, 0), u_2 = (0, 1, 0, 0), v_1 = (0, 0, 1, 0)$ and $v_2 = (0, 0, 0, 1)$. Then $u_i v_j = 0$ for all i, j, so that $K_{2,2}$ is a subgraph of $\Gamma(\frac{R}{I})$. By Remark 2.5 and $|I| \geq 2$, we get that $K_{4,4}$ is a subgraph of $\Gamma_I(R)$. Therefore by Lemma 1.1, $\bar{\gamma}(\Gamma_I(R)) > 1$. Hence $\frac{R}{I} \cong R_1 \times R_2 \times R_3$. If there exists at least one i such that $|R_i| \geq 3$, without loss of generality, say $|R_3| \geq 3$. Let $u_1 = (1, 0, 0), u_2 = (0, 1, 0), v_1 = (0, 0, 1)$ and $v_2 = (0, 0, 2)$. Now $u_i v_j = 0$ for all i, j, so that $K_{2,2}$ is a subgraph of $\Gamma(\frac{R}{I})$ which implies that $K_{4,4}$ is a subgraph of $\Gamma_I(R)$. Thus $\bar{\gamma}(\Gamma_I(R)) > 1$, a contradiction. □

Lemma 2.9. *Let R be a finite ring and I be a non-zero ideal of R. Suppose the following holds:*

(i) *$\frac{R}{I}$ is local with unique maximal ideal $\frac{M}{I}$;*
(ii) *$|\frac{R}{M}| = 2$;*
(iii) *$\bar{\gamma}(\Gamma(\frac{R}{I})) = 1$.*

Then $\bar{\gamma}(\Gamma_I(R)) > 1$.

Proof. By (i) and (iii), $\frac{R}{I}$ is isomorphic to one of the rings in the statement of Theorem 2.3. In view of (ii) and Theorem 2.3, $K_{2,2}$ is a subgraph of corresponding zero-divisor graph of $\Gamma(\frac{R}{I})$. Thus by Lemma 1.1, $\bar{\gamma}(\Gamma_I(R)) > 1$. □

The following theorem plays an important role in this paper.

Theorem 2.10. *Let R be a finite ring and I be a non-zero ideal of R. Suppose that $\omega(\Gamma(\frac{R}{I})) \geq 4$ or $\bar{\gamma}(\Gamma(\frac{R}{I})) \geq 1$. Then $\bar{\gamma}(\Gamma_I(R)) > 1$.*

Proof. Let $\frac{R}{I} \cong R_1 \times R_2 \times \cdots \times R_k$, where R_i is a finite local ring for every i. If $k \geq 4$, then by Lemma 2.8, $\bar{\gamma}(\Gamma_I(R)) > 1$. Therefore, we assume that $k \leq 2$ or $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. However, $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ implies that $\omega(\Gamma(\frac{R}{I})) = 3$ and $\Gamma(\frac{R}{I})$ is planar, a contradiction. Hence $k \leq 2$. Now suppose that $\frac{R}{I}$ is not local and $\frac{R}{I} \cong R_1 \times R_2$. If $|R_1| \geq 3$ and $|R_2| \geq 3$, then $K_{2,2}$ is a subgraph of $\Gamma(\frac{R}{I})$, so that $\bar{\gamma}(\Gamma_I(R)) > 1$. Hence we assume that $R_1 \cong \mathbb{Z}_2$. By Lemma 2.7, $\bar{\gamma}(\Gamma_I(R)) > 1$.

Finally, let $\frac{R}{I}$ be a local ring with unique maximal ideal $\frac{M}{I}$.

Case 1: $(\frac{M}{I})^2 = (0)$. Since $\omega(\Gamma(\frac{R}{I})) \geq 4$ or $\bar{\gamma}(\Gamma(\frac{R}{I})) \geq 1$, there exist distinct non-zero zero-divisors $u_1, u_2, u_3, u_4 \in \frac{R}{I}$ such that $u_i u_j = 0$ for all i, j, so that $K_{2,2}$ is a subgraph of $\Gamma(\frac{R}{I})$ implies $K_{4,4}$ is a subgraph of $\Gamma_I(R)$. Thus by Lemma 1.1, $\bar{\gamma}(\Gamma_I(R)) > 1$.

Case 2: $(\frac{M}{I})^2 \neq (0)$. If $|\frac{R}{M}| \geq 3$, then Lemma 2.6 gives that $\bar{\gamma}(\Gamma_I(R)) > 1$. Hence we assume that $|\frac{R}{M}| = 2$. As mentioned Lemma 2.4, $|V(\Gamma(\frac{R}{I}))| = 2^n - 1$ for some positive integer n. Since $\omega(\Gamma(\frac{R}{I})) \geq 4$ or $\bar{\gamma}(\Gamma(\frac{R}{I})) \geq 1$, $n \geq 3$. By Lemma 2.9, the assumption $\bar{\gamma}(\Gamma(\frac{R}{I})) \geq 1$ yields that $\bar{\gamma}(\Gamma_I(R)) > 1$. Suppose $\omega(\Gamma(\frac{R}{I})) = 4$ and $\Gamma(\frac{R}{I})$ is planar. As $\Gamma(\frac{R}{I})$ is planar and $|V(\Gamma(\frac{R}{I}))| = 2^{n-1}$, Theorem 2.2 gives that $|V(\Gamma(\frac{R}{I}))| = 7$. Again by Theorem 2.2, $\frac{R}{I}$ is isomorphic to one of the following rings $\mathbb{Z}_{16}$, $\frac{\mathbb{Z}_4[x,y]}{(x^2,y^2,xy-2)}$, $\frac{\mathbb{Z}_4[x,y]}{(x^3,x^2-2,xy,y^2-2)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,y^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^3,xy,y^2-x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^3,x^2-2x)}$ and $\frac{\mathbb{Z}_8[x]}{(x^2-4,2x)}$. One can check that corresponding zero-divisor graphs $\Gamma(\frac{R}{I})$ have $\omega(\Gamma(\frac{R}{I})) = 3$, a contradiction. □

Theorem 2.11. *Let R be a finite ring and I be a non-zero ideal of R. Suppose that $\omega(\Gamma(\frac{R}{I})) \leq 2$. Then $\bar{\gamma}(\Gamma_I(R)) \leq 1$ if and only if one of the following holds:*

(i) $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *and* $|I| \leq 3$*;*
(ii) $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ *and* $|I| = 2$*;*
(iii) $\frac{R}{I} \cong \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$ *and* $|I| = 2$*;*
(iv) $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{F}_q$*,* $q \geq 3$ *and* $|I| = 2$*;*
(v) $\frac{R}{I} \cong \mathbb{Z}_4$*, and* $|I| \leq 6$*;*
(vi) $\frac{R}{I} \cong \frac{\mathbb{Z}_2[x]}{(x^2)}$ *and* $|I| \leq 6$*;*
(vii) $\frac{R}{I} \cong \mathbb{Z}_9$*, and* $|I| \leq 3$*;*
(viii) $\frac{R}{I} \cong \frac{\mathbb{Z}_3[x]}{(x^2)}$ *and* $|I| \leq 3$*;*
(ix) $\frac{R}{I} \cong \mathbb{Z}_8$*, and* $|I| = 2$*;*
(x) $\frac{R}{I} \cong \frac{\mathbb{Z}_2[x]}{(x^3)}$ *and* $|I| = 2$*;*
(xi) $\frac{R}{I} \cong \frac{\mathbb{Z}_4[x]}{(x^2-2,x^3)}$ *and* $|I| = 2$*.*

Proof. Assume that $\omega(\Gamma(\frac{R}{I})) \leq 2$ and $\bar{\gamma}(\Gamma_I(R)) \leq 1$. Let $\frac{R}{I} \cong R_1 \times R_2 \times \cdots \times R_k$, where R_i is a finite local ring for every i. By Lemma 2.8, $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\frac{R}{I}$ is local or $\frac{R}{I}$ is a product of two local rings. If $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\omega(\Gamma(\frac{R}{I})) = 3$, a contradiction.

Suppose that $\frac{R}{I} \cong R_1 \times R_2$. As seen in the proof of Theorem 2.10, $R_1 \cong \mathbb{Z}_2$. By Lemma 2.7, R_2 is either a field or satisfies $|V(\Gamma(R_2))| = 1$, otherwise $\bar{\gamma}(\Gamma_I(R)) > 1$. It follows that $\frac{R}{I}$ is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$ and $\mathbb{Z}_2 \times \mathbb{F}_q$, $q \geq 3$.

If $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\Gamma(\frac{R}{I}) \cong K_2$. Since $\frac{R}{I}$ has no nilpotent elements, $\Gamma_I(R) \cong K_{t,t}$, $|I| = t$. By Lemma 1.1, $|I| \leq 3$.

If $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{F}_q$, $q \geq 3$, then $\Gamma(\frac{R}{I}) \cong K_{1,q}$. Since $\frac{R}{I}$ has no nilpotent elements, $\Gamma_I(R) \cong K_{t,tq}$, $|I| = t$, it follows that $|I| = 2$, by Lemma 1.1.

If $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, then $K_{1,3}$ is a subgraph of $\frac{R}{I}$. By Lemma 1.1, $|I| = 2$ and by Theorem 2.1, $\Gamma_I(R)$ is planar.

Finally assume that $\frac{R}{I}$ is local with maximal ideal $\frac{M}{I}$. By Theorem 2.10, $\Gamma(\frac{R}{I})$ is planar, so we need to consider only the rings in Theorem 2.2.

Case 1: $(\frac{M}{I})^2 = (0)$. Then $\Gamma(\frac{R}{I}) \cong K_t$ if $|\frac{M}{I}| = t+1$, so that $\Gamma(\frac{R}{I})$ is K_1 or K_2 as $\omega(\Gamma(\frac{R}{I})) \leq 2$, it follows that $\frac{R}{I}$ is isomorphic to the following rings: $\mathbb{Z}_4$, $\frac{\mathbb{Z}_2[x]}{(x^2)}$, $\mathbb{Z}_9$ and $\frac{\mathbb{Z}_3[x]}{(x^2)}$. If $\frac{R}{I}$ is isomorphic to $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{(x^2)}$, then $\Gamma_I(R) \cong K_t$ if $|I| = t$. By Lemma 1.1, $|I| \leq 6$. If $\frac{R}{I}$ is isomorphic to $\mathbb{Z}_9$ or $\frac{\mathbb{Z}_3[x]}{(x^2)}$, then every element $a \in Z(\frac{R}{I})$ is nilpotent, by Remark 2.5, $\Gamma_I(R) \cong K_{2t}$ if $|I| = t$, it follows that $|I| \leq 3$, by Lemma 1.1.

Case 2: $(\frac{M}{I})^2 \neq (0)$. By Lemma 2.6, $|\frac{R}{M}| = 2$. That is, $|V(\Gamma(\frac{R}{I}))| = 2^n - 1$ for some positive integer n and so by Theorem 2.2, $|V(\Gamma(\frac{R}{I}))| = 1, 3, 7$. From the above, $|V(\Gamma(\frac{R}{I}))| = 3, 7$. However, if $|V(\Gamma(\frac{R}{I}))| = 7$, then by Theorem 2.2, $\frac{R}{I}$ is isomorphic to one of the following rings: $\mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,y^2)}$, $\frac{\mathbb{Z}_2[x,y]}{(x^3,xy,y^2-x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2)}$, $\frac{\mathbb{Z}_4[x]}{(x^3,x^2-2x)}$, $\frac{\mathbb{Z}_4[x,y]}{(x^2,y^2,xy-2)}$, $\frac{\mathbb{Z}_4[x,y]}{(x^3,x^2-2,xy,y^2-2)}$ and $\frac{\mathbb{Z}_8[x]}{(x^2-4,2x)}$. In each of these cases, ideal based zero-divisor graphs $\Gamma(\frac{R}{I})$ contains K_3, a contradiction. Thus $|V(\Gamma(\frac{R}{I}))| = 3$ and so $\Gamma(\frac{R}{I}) \cong K_{1,2}$. Therefore, $\frac{R}{I}$ is isomorphic to one of the following rings: $\mathbb{Z}_8$, $\frac{\mathbb{Z}_2[x]}{(x^3)}$, $\frac{\mathbb{Z}_4[x]}{(x^3,x^2-2)}$. Then $K_{t,2t}$ is a subgraph of $\Gamma_I(R)$ if $|I| = t$. By Lemma 1.1, $|I| = 2$ and by Theorem 2.1, $\Gamma_I(R)$ is planar. $\square$

Theorem 2.12. *Let R be a finite ring and I be a non-zero ideal of R. Suppose that $\omega(\Gamma(\frac{R}{I})) = 3$. Then $\bar{\gamma}(\Gamma_I(R)) \leq 1$ if and only if $|I| = 2$ and $\frac{R}{I}$ is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_{16}$, $\frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$, $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\frac{\mathbb{F}_4[x]}{(x^2)}$, and $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$.*

Proof. Suppose that $\bar{\gamma}(\Gamma_I(R)) \leq 1$. Since R is finite, let $\frac{R}{I} \cong R_1 \times R_2 \times \cdots \times R_k$, where R_i is a finite local ring for every i. From Lemma 2.8, $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\frac{R}{I}$ is local or $\frac{R}{I}$ is a product of two local rings.

Assume that $\frac{R}{I} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $K_{1,3}$ is a subgraph of $\Gamma(\frac{R}{I})$. If $|I| \geq 3$, Lemma 1.1 implies $\bar{\gamma}(\Gamma_I(R)) > 1$. Therefore, let $|I| = 2$ and also $\omega(\Gamma(\frac{R}{I})) = 3$. By Theorem 2.1, corresponding ideal based zero-divisor graph $\Gamma_I(R)$ is not planar. Remark 2.5 gives that graphs $\Gamma_I(R)$ and $\Gamma_{(0)\times(0)\times(0)\times\mathbb{Z}_2}(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2)$ are isomorphic. Note that the vertex set of $\Gamma_I(R)$ consists of the following vertices: $u_1 = (1,0,0,0)$, $u_2 = (0,1,0,0)$, $v_1 = (1,0,0,1)$, $v_2 = (0,1,0,1)$, $v_3 = (0,0,1,1)$, $v_4 = (0,1,1,0)$, $v_5 = (0,1,1,1)$,

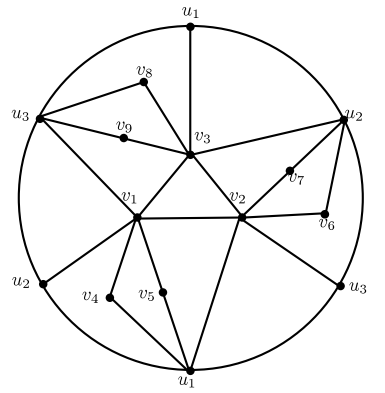

Fig. 1. An embedding of $\Gamma_I(R)$ in $\overline{S}_1$.

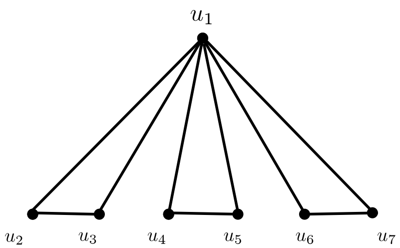

Fig. 2. Graph isomorphic to $\Gamma(\frac{R}{I})$.

$v_6 = (1,0,1,0)$, $v_7 = (1,0,1,1)$, $v_8 = (1,1,0,0)$ and $v_9 = (1,1,0,1)$. Note that Fig. 1 shows explicitly an embedding of $\Gamma_I(R)$ to $\overline{S}_1$.

If $\frac{R}{I}$ is not local and that $\frac{R}{I} \cong R_1 \times R_2$. From the proof of Theorem 2.10, $R_1 \cong \mathbb{Z}_2$. Moreover, if R_2 is either a field or satisfies $|V(\Gamma(R_2))| = 1$ which is a contradiction to the assumption $\omega(\Gamma(\frac{R}{I})) = 3$. If $|V(\Gamma(R_2))| \geq 2$, then by Lemma 2.7, $K_{2,2}$ is a subgraph of $\Gamma(\frac{R}{I})$, it follows that $\bar{\gamma}(\Gamma_I(R)) > 1$, a contradiction.

Finally assume that $\frac{R}{I}$ is local with maximal ideal $\frac{M}{I}$. As earlier Theorem 2.10, $\Gamma(\frac{R}{I})$ is planar, so only consider the rings in Theorem 2.2.

Case 1: $(\frac{M}{I})^2 = (0)$. Then $\Gamma(\frac{R}{I}) \cong K_t$ if $|\frac{M}{I}| = t+1$, so that $\Gamma(\frac{R}{I})$ is K_3 as $\omega(\Gamma(\frac{R}{I})) = 3$. By Theorem 2.2, $\frac{R}{I}$ is isomorphic to $\frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$ or $\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$ or $\frac{\mathbb{F}_4[x]}{(x^2)}$ or $\frac{\mathbb{Z}_4[x]}{(x^2+x+1)}$. Moreover, every element $a \in Z(\frac{R}{I})$ is nilpotent. Now Remark 2.5 implies that $\Gamma_I(R) \cong K_{3t}$ if $|I| = t$ and it follows that $\bar{\gamma}(\Gamma_I(R)) = 1$ if and only if $|I| = 2$.

Case 2: $(\frac{M}{I})^2 \neq (0)$. By Lemma 2.6, $|\frac{R}{M}| = 2$. That is, $|V(\Gamma(\frac{R}{I}))| = 2^n - 1$ for some positive integer n and so by Theorem 2.2, $|V(\Gamma(\frac{R}{I}))| = 1, 3, 7$. From the above, $|V(\Gamma(\frac{R}{I}))| = 7$. Then by Theorem 2.2, $\frac{R}{I}$ is isomorphic to one of the following local rings: $\frac{\mathbb{Z}_2[x,y]}{(x^2,y^2)}$, $\frac{\mathbb{Z}_4[x,y]}{(x^2,y^2,xy-2)}$, $\frac{\mathbb{Z}_4[x]}{(x^2)}$ and $\mathbb{Z}_{16}$.

If $\frac{R}{I}$ is isomorphic to one of the following: $\frac{\mathbb{Z}_2[x,y]}{(x^2,y^2)}$, $\frac{\mathbb{Z}_4[x,y]}{(x^2,y^2,xy-2)}$ and $\frac{\mathbb{Z}_4[x]}{(x^2)}$, then there exist three distinct elements u_1, u_2, $u_3 \in V(\Gamma(\frac{R}{I}))$ such that $u_1^2 = u_2^2 = u_3^2 = 0$ and the graph $\Gamma(\frac{R}{I})$ is isomorphic to Fig. 2.

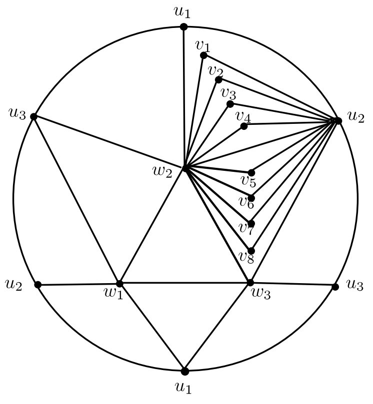

Fig. 3. An embedding of $\Gamma_I(R)$ in $\overline{S}_1$.

If $|I| = 2$, then the corresponding ideal based graph can be viewed as $\Gamma_{(0)\times\mathbb{Z}_2}(\frac{R}{I} \times \mathbb{Z}_2)$. Note that $H_1 \cong K_5$ and $H_2 \cong K_{3,3}$ are subgraphs of $\Gamma_{(0)\times\mathbb{Z}_2}(\frac{R}{I} \times \mathbb{Z}_2)$ with the vertex sets $\{(u_1,0),(u_2,0),(u_2,1),(u_3,0),(u_3,1)\}$ and $\{(u_1,0),(u_1,1),(u_4,0),(u_4,1),(u_5,0),(u_5,1)\}$ respectively. Clearly $V(H_1) \cap V(H_2) = \{(u_1,0)\}$ and so Lemma 1.2 gives $\bar{\gamma}(\Gamma_I(R)) > 1$. If $\frac{R}{I}$ is isomorphic to $\mathbb{Z}_{16}$, then $\Gamma_I(R)$ is not planar, as $\omega(\Gamma(\frac{R}{I})) = 3$ and by Theorem 2.1. Since $\Gamma_I(R)$ and $\Gamma_{(0)\times\mathbb{Z}_2}(\mathbb{Z}_{16}\times\mathbb{Z}_2)$ are the same, then the vertex set of $\Gamma_{(0)\times\mathbb{Z}_2}(\mathbb{Z}_{16}\times\mathbb{Z}_2)$ consists of the following vertices: $u_1 = (4,0)$, $u_2 = (8,0)$, $u_3 = (12,0)$, $v_1 = (4,1)$, $v_2 = (8,1)$, $v_3 = (12,1)$, $v_4 = (2,0)$, $v_5 = (6,0)$, $v_6 = (10,0)$, $v_7 = (14,0)$, $v_8 = (2,1)$, $v_9 = (6,1)$, $v_{10} = (10,1)$ and $v_{11} = (14,1)$. Fig. 3 shows explicitly an embedding of $\Gamma_I(R)$ to $\bar{S}_1$. $\square$

3. Crosscap one in the general case

Our main investigation of this section is to give a sufficient condition for a ring R and a non-zero ideal I of R with the property that $\bar{\gamma}(\Gamma_I(R)) \leq 1$.

Proposition 3.1. *Let P_1 and P_2 be prime ideals of R and $I = P_1 \cap P_2$. If $\bar{\gamma}(\Gamma_I(R)) \leq 1$, then $|P_1 - I| \leq 3$ or $|P_2 - I| \leq 3$.*

Proof. Since $\Gamma_I(R)$ is not empty, I is not equal to P_1 and P_2. Suppose that $|P_1 - I| \geq 4$ and $|P_2 - I| \geq 4$. Let $P_1 - I = \{u_1, u_2, \ldots, u_m\}$ and $P_1 - I = \{v_1, v_2, \ldots, v_n\}$ such that $m, n \geq 4$. Then $u_i v_j \in I$ for all i, j and so $K_{4,4}$ is a subgraph of $\Gamma_I(R)$. Hence by Lemma 1.1, $\bar{\gamma}(\Gamma_I(R)) > 1$. $\square$

Proposition 3.2. *Let I be a non-zero ideal of R. If $|Ass(\frac{R}{I})| \leq 3$, then $\bar{\gamma}(\Gamma_I(R)) \leq 1$.*

Proof. On the contrary assume that $|Ass(\frac{R}{I})| \geq 4$. By Lemma 2.1 in [1], K_4 is a subgraph of $\Gamma(\frac{R}{I})$. It follows that $K_{4,4}$ is a subgraph of $\Gamma_I(R)$ as $|I| \geq 2$. Therefore $\bar{\gamma}(\Gamma_I(R)) > 1$. $\square$

We conclude this paper with the following Theorem.

Theorem 3.3. *Let I be a non-zero ideal of R. If $\bar{\gamma}(\Gamma_I(R)) \leq 1$, then $\omega(\Gamma(\frac{R}{I})) \leq 3$ and either $|I| \leq 3$ or $|V(\Gamma(\frac{R}{I}))| = 1$ with $|I| \leq 6$.*

Proof. Suppose that $\bar{\gamma}(\Gamma_I(R)) \leq 1$. First we show that $\omega(\Gamma(\frac{R}{I})) \leq 3$. If $\omega(\Gamma(\frac{R}{I})) \geq 4$, then K_4 is a subgraph of $\Gamma(\frac{R}{I})$. Since $|I| \geq 2$ and Remark 2.5, $K_{2,2,2,2}$ is a subgraph of $\Gamma_I(R)$ which implies $K_{4,4}$ is a subgraph of $\Gamma_I(R)$ and so $\bar{\gamma}(\Gamma_I(R)) > 1$, a contradiction. Now if $\Gamma(\frac{R}{I})$ consists of exactly one vertex, then $\Gamma_I(R) \cong K_t$ if $|I| = t+1$, so that Lemma 1.1 yields that $|I| \leq 6$. If $\Gamma(\frac{R}{I})$ consists of at least two adjacent vertices $a + I, b + I$ with $|I| \geq 4$, then it is easy to verify that $V_1 = \{a, a + i_1, a + i_2, a + i_3\}$ and $V_1 = \{b, b + i_1, b + i_2, b + i_3\}$ define two subsets of the vertex set $|V(\Gamma(\frac{R}{I}))|$ yielding a subgraph isomorphic to $K_{4,4}$ for all distinct non-zero elements i_1, i_2 and i_3 in I. However this would contradict the crosscap of $\Gamma_I(R)$. Thus, if $\Gamma(\frac{R}{I})$ has more than one vertex, we must have $|I| \leq 3$. $\square$

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