Cowling–Price type theorem related to Bessel–Struve transform

SOUMEYA HAMEM [a], LOTFI KAMOUN [a,*], SELMA NEGZAOUI [b]

[a] Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, 5019 Monastir, Tunisia
[b] Preparatory Institute of Engineering Studies of Monastir, University of Monastir, 5000 Monastir, Tunisia

Abstract. The purpose of this paper is to establish an analogue of Cowling–Price theorem for the Bessel–Struve transform. Also, we provide Hardy's type theorem associated with this transform.

1. INTRODUCTION

The uncertainty principle states that a nonzero function and its Fourier transform cannot both be sharply localized. In the language of quantum mechanics, this principle says that an observer cannot simultaneously and precisely determine the values of position and momentum of a quantum particle. A mathematical formulation of these physical ideas was first developed by Heisenberg [8] in 1927. Later, precisely in 1933, Hardy [7] has obtained a theorem concerning the decay of a measurable function f on $\mathbb{R}$ and its Fourier transform $\mathcal{F}f$ at infinity. This theorem can be stated as follows

Theorem 1.1. *Let $a > 0, b > 0, C > 0$ and let f be a measurable function on $\mathbb{R}$ such that*

* Corresponding author. Tel.: +216 73500274; fax: +216 73500278.
E-mail addresses: soumeya.hamem@yahoo.fr (S. Hamem), kamoun.lotfi@planet.tn (L. Kamoun), selma.negzaoui@issatgb.rnu.tn (S. Negzaoui).
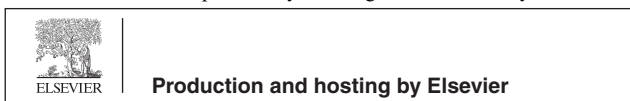
$$|f(x)| \leqslant Ce^{-ax^2} \quad \text{and} \quad |\mathcal{F}(f)(y)| \leqslant Ce^{-by^2}, \quad \text{a.e} \quad x, y \in \mathbb{R}. \tag{1}$$

We have,

(1) *If $ab > \frac{1}{4}$ then $f = 0$ almost everywhere.*
(2) *If $ab < \frac{1}{4}$ then infinitely nonzero functions satisfy conditions (1).*
(3) *If $ab = \frac{1}{4}$ then $f(x) = const.e^{-ax^2}$.*

An L^p version of Hardy's theorem has been obtained in 1983 by Cowling and Price [3]. Precisely, they have obtained the following theorem

Theorem 1.2. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\|e_a f\|_p < +\infty \quad \text{and} \quad \|e_b \mathcal{F}(f)\|_q < +\infty, \tag{2}$$

where $a > 0, b > 0, e_a(x) = e^{ax^2}$, $1 \leqslant p, q \leqslant +\infty$ and $min(p, q) < +\infty$. We have,

(1) If $ab \geqslant \frac{1}{4}$ then $f = 0$ almost everywhere.
(2) If $ab < \frac{1}{4}$ then there exist infinitely many linearly independent functions satisfying conditions (2).

In this paper, we consider the Bessel–Struve transform, for $\alpha > -\frac{1}{2}$,

$$\mathcal{F}_{B,S}^{\alpha}(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{\alpha}(-i\lambda x)d\mu_{\alpha}(x),$$

where Φ_{α} is the Bessel–Struve kernel given by

$$\Phi_{\alpha}(x) = j_{\alpha}(ix) - ih_{\alpha}(ix).$$

j_{α} and h_{α} are respectively the normalized Bessel and Struve functions of index α. These kernels are given as follows

$$j_{\alpha}(z) = 2^{\alpha}\Gamma(\alpha + 1)z^{-\alpha}J_{\alpha}(z) = \Gamma(\alpha + 1)\sum_{n=0}^{+\infty}\frac{(-1)^n(z/2)^{2n}}{n!\Gamma(n + \alpha + 1)}$$

and

$$h_{\alpha}(z) = 2^{\alpha}\Gamma(\alpha + 1)z^{-\alpha}\mathbf{H}_{\alpha}(z) = \Gamma(\alpha + 1)\sum_{n=0}^{+\infty}\frac{(-1)^n(z/2)^{2n+1}}{\Gamma\left(n + \frac{3}{2}\right)\Gamma\left(n + \alpha + \frac{3}{2}\right)}.$$

By proceeding as the same way of the paper of Gallardo and Trimche [6], we establish for Bessel–Struve transform the analogue of Cowling–Price's and Hardy's theorems. The paper [6] deals with Dunkl operators. The method of this paper has been used in the case of the Dunkl–Bessel differential difference operator by Mejjaoli and Trimèche [10] and in the case of Chébli–Trimèche operators by Trimèche [15] to obtain similar results. Also, in the case of Chébli–Trimèche operators, Bouattour and Trimèche in [2] have obtained the analogue of Cowling–Price's and Hardy's theorems via Beurling–Hörmander's theorem. Many authors have established the analogous of Cowling–Price's and Hardy's theorems in other various settings of harmonic analysis (see for instance [1,4,5,11–13]).

The outline of the content of the paper is as follows.

In the second section, we deal with harmonic analysis associated with the Bessel–Struve operator. We consider the Bessel–Struve transform $\mathcal{F}_{BS}$ which is related to Weyl integral transform W_α by the relation

$$\forall f \in L^1_\alpha(\mathbb{R}), \quad \mathcal{F}^\alpha_{B,S}(f) = \mathcal{F} \circ W_\alpha(f).$$

This relationship allows us to deduce the main results knowing their analogue for classical Fourier transform $\mathcal{F}$. These transforms have been introduced firstly by Trimèche [14] in connection with the Dunkl theory.

The third section, is devoted to Cowling–Price's theorem for the Bessel–Struve transform. We establish that for all p, $q \in [1, +\infty]$ and at least one of them is finite, if f is a measurable function on $\mathbb{R}$ such that

$$e^{ax^2} f \in L^p_\alpha(\mathbb{R}) \quad \text{and} \quad e^{by^2} \mathcal{F}^\alpha_{B,S} f \in L^q_\alpha(\mathbb{R}),$$

where $a > 0$, $b > 0$, then if $ab \geqslant 1/4$, $f = 0$ almost everywhere and there are infinitely many nonzero functions f satisfying these conditions if $ab < 1/4$.

In the fourth section, we study an analogue of Theorem 1.1 associated with the Bessel–Struve transform. In particular, if $ab = \frac{1}{4}$ and α a half integer, the functions satisfying the hypotheses of Hardy's theorem for Bessel–Struve transform are precisely determined by $f(x) = const.e^{-ax^2}$.

Throughout the paper, C designates a real number which can differ from line to other.

2. BESSEL–STRUVE TRANSFORM

We consider the Bessel–Struve operator $\ell_\alpha, \alpha > -\frac{1}{2}$, defined on $C^\infty(\mathbb{R})$ by

$$\ell_\alpha u(x) = \frac{d^2 u}{dx^2}(x) + \frac{2\alpha + 1}{x}\left[\frac{du}{dx}(x) - \frac{du}{dx}(0)\right]. \tag{3}$$

For $\lambda \in \mathbb{C}$, the differential equation:

$$\begin{cases} \ell_\alpha u(x) = \lambda^2 u(x) \\ u(0) = 1, \quad u'(0) = \frac{\lambda \Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+3/2)}. \end{cases}$$

possesses a unique solution denoted $\Phi_\alpha(\lambda)$. This eigenfunction, called the Bessel–Struve kernel, is given by :

$$\Phi_\alpha(\lambda x) = j_\alpha(i\lambda x) - ih_\alpha(i\lambda x), \quad x \in \mathbb{R}. \tag{4}$$

The kernel Φ_α possesses the following integral representation:

$$\forall x \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}, \quad \Phi_\alpha(\lambda x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} e^{\lambda x t} dt. \tag{5}$$

Let $p \in [1, +\infty]$, we denote by $L^p_\alpha(\mathbb{R})$, the space of real-valued functions f, measurable on $\mathbb{R}$ such that

$$\|f\|_{p,\alpha} = \left(\int_{\mathbb{R}} |f(x)|^p d\mu_\alpha(x) \right)^{\frac{1}{p}} < +\infty, \quad \text{if} \quad p < +\infty,$$

where

$$d\mu_\alpha(x) = A(x)dx \quad \text{and} \quad A(x) = |x|^{2\alpha+1},$$

$$\|f\|_{\infty,\alpha} = \text{ess sup}_{x \in \mathbb{R}} |f(x)| < +\infty.$$

The Bessel–Struve kernel Φ_α is related to the exponential function by

$$\forall x \in \mathbb{R}, \quad \forall \lambda \in \mathbb{C}, \quad \Phi_\alpha(\lambda x) = \chi_\alpha(e^{\lambda \cdot})(x),$$

where χ_α is the Bessel–Struve intertwining operator (see [9]).

The function $\Phi_\alpha(i\lambda\cdot)$ plays the role of the exponential function $e^{i\lambda\cdot}$ in the classical Fourier analysis. In fact, one can introduce the Bessel–Struve transform, in terms of Bessel–Struve kernel as follows

Definition 2.1. The Bessel–Struve transform is defined on $L_\alpha^1(\mathbb{R})$ by

$$\forall \lambda \in \mathbb{R}, \quad \mathcal{F}_{B,S}^\alpha(f)(\lambda) = \int_{\mathbb{R}} f(x) \quad \Phi_\alpha(-i\lambda x)d\mu_\alpha(x). \tag{6}$$

Remark 2.1. We notice that if f is an even function then the Bessel–Struve transform $\mathcal{F}_{B,S}^\alpha$ coincides with Hankel transform denoted $\mathcal{H}_\alpha$ given by

$$\mathcal{H}_\alpha(f)(\lambda) = \int_0^{+\infty} f(t) j_\alpha(\lambda t) t^{2\alpha+1} dt.$$

Definition 2.2. For $f \in L_\alpha^1(\mathbb{R})$ with bounded support, the integral transform W_α, given by

$$W_\alpha f(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} \int_{|x|}^{+\infty} (y^2 - x^2)^{\alpha-\frac{1}{2}} y \, f(sgn(x)y)dy, \quad x \in \mathbb{R} \setminus \{0\}, \tag{7}$$

is called Weyl integral transform associated with Bessel–Struve operator.

Remark 2.2. By a change of variable, $W_\alpha f$ can be written

$$W_\alpha f(x) = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)} |x|^{2\alpha+1} \int_1^{+\infty} (t^2 - 1)^{\alpha-\frac{1}{2}} t f(tx)dt, \quad x \in \mathbb{R} \setminus \{0\}. \tag{8}$$

If we denote $a_\alpha = \frac{2\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma\left(\alpha+\frac{1}{2}\right)}$ and $dv_x(y) = a_\alpha \chi_{]|x|,+\infty[}(y)(y^2 - x^2)^{\alpha-\frac{1}{2}} y dy$ we can write

$$W_\alpha f(x) = \int_{\mathbb{R}} f(sgn(x)y)dv_x(y). \tag{9}$$

Proposition 2.1. W_α is a bounded operator from $L_\alpha^1(\mathbb{R})$ to $L^1(\mathbb{R})$, where $L^1(\mathbb{R})$ is the space of Lebesgue-integrable functions.

Proof. Let $f \in L_\alpha^1(\mathbb{R})$. Using Fubini–Tonelli's theorem and a change of variable, we get:

$$\int_{\mathbb{R}} \left(\int_1^{+\infty} (t^2 - 1)^{\alpha - \frac{1}{2}} t |f(tx)| dt \right) |x|^{2\alpha + 1} dx$$
$$= \left(\int_{\mathbb{R}} |f(x)| |x|^{2\alpha + 1} \, dx \right) \left(\int_1^{+\infty} (t^2 - 1)^{\alpha - \frac{1}{2}} t^{-2\alpha - 1} dt \right) < +\infty.$$

Invoking relation (8), we deduce that $\int_{\mathbb{R}} |W_\alpha f(y)| dy < +\infty$ and

$$\|W_\alpha(f)\|_1 \leqslant C\|f\|_{1,\alpha}. \qquad \square \tag{10}$$

Proposition 2.2. *We have*

$$\forall f \in L_\alpha^1(\mathbb{R}), \quad \mathcal{F}_{B,S}^\alpha(f) = \mathcal{F} \circ W_\alpha(f), \tag{11}$$

where $\mathcal{F}$ is the classical Fourier transform defined on $L^1(\mathbb{R})$ by

$$\mathcal{F}(g)(\lambda) = \int_{\mathbb{R}} g(x) e^{-i\lambda x} dx.$$

Proof. Let $f \in L_\alpha^1(\mathbb{R})$ then $W_\alpha(f) \in L^1(\mathbb{R})$ and $\mathcal{F} \circ W_\alpha(f)$ are well defined.

Using Chasles relation and a change of variable, we get

$$\mathcal{F} \circ W_\alpha(f)(\lambda) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma\left(\alpha + \frac{1}{2}\right)} \int_0^{+\infty} \int_y^{+\infty} (x^2 - y^2)^{\alpha - \frac{1}{2}} x f(x) dx e^{-i\lambda y} dy + \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma\left(\alpha + \frac{1}{2}\right)}$$
$$\times \int_0^{+\infty} \int_y^{+\infty} (x^2 - y^2)^{\alpha - \frac{1}{2}} x f(-x) dx e^{i\lambda y} dy.$$

From Fubini's theorem, we obtain

$$\mathcal{F} \circ W_\alpha(f)(\lambda) = \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma\left(\alpha + \frac{1}{2}\right)} \int_0^{+\infty} \int_0^x (x^2 - y^2)^{\alpha - \frac{1}{2}} e^{-i\lambda y} dy x f(x) dx + \frac{2\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma\left(\alpha + \frac{1}{2}\right)}$$
$$\times \int_0^{+\infty} \int_0^x (x^2 - y^2)^{\alpha - \frac{1}{2}} e^{i\lambda y} dy x f(-x) dx.$$

By a change of variable and using relation (5), we find that

$$\mathcal{F} \circ W_\alpha(f)(\lambda) = \int_0^{+\infty} f(x) \Phi_\alpha(-i\lambda x) |x|^{2\alpha + 1} dx + \int_0^{+\infty} f(-x) \Phi_\alpha(i\lambda x) |x|^{2\alpha + 1} dx.$$

A change of variable and Chasles relation imply

$$\mathcal{F} \circ W_\alpha(f)(\lambda) = \int_{-\infty}^{+\infty} f(x) \Phi_\alpha(-i\lambda x) |x|^{2\alpha + 1} dx = \mathcal{F}_{BS}(f)(\lambda). \qquad \square$$

In the next section, we need the following lemmas of Phragmen–Lindlöf type that we get using the same technique as in [3].

Lemma 2.1. *Let* $p \in [1, +\infty)$ *and let* h *be an entire function on* $\mathbb{C}$. *Assume that*

(1) $\forall z \in \mathbb{C}, |h(z)| \leqslant Me^{a(\Re(z))^2}$
(2) $\|h_{|\mathbb{R}}\|_{\alpha,p} < +\infty$, *for some positive constants* a *and* M. *Then* $h = 0$.

Lemma 2.2. *Let* h *be an entire function on* $\mathbb{C}$ *such that*

(1) $\forall z \in \mathbb{C}, |h(z)| \leqslant Me^{a(\Re(z))^2}$
(2) $\|h_{|\mathbb{R}}\|_{\alpha,\infty} < +\infty$, *for some positive constants* a *and* M. *Then* h *is a constant on* $\mathbb{C}$.

3. COWLING–PRICE THEOREM ASSOCIATED WITH THE BESSEL–STRUVE TRANSFORM

We denote e_a the function given by $e_a(x) = e^{ax^2}$, $x \in \mathbb{R}$.

Proposition 3.1. *Let* $a > 0$. *The Weyl integral transform verifies*

$$W_\alpha(e_{-a}) = Ce_{-a}, \tag{12}$$

where $C = \frac{\Gamma(\alpha+1)}{2\sqrt{\pi}a^{\alpha+\frac{1}{2}}}$.

Proof. From the relation (11) and remark 2.1, we obtain

$$W_\alpha(e_{-a}) = \mathcal{F}^{-1} \circ \mathcal{H}_\alpha(e_{-a}). \tag{13}$$

But we have

$$\mathcal{H}_\alpha(e_{-a})(y) = \frac{\Gamma(\alpha+1)}{2a^{\alpha+1}} e^{\frac{-y^2}{4a}}. \tag{14}$$

By applying the classical inverse Fourier transform to the relation (14), we obtain (12). $\square$

Proposition 3.2. *Let* $p \in [1, +\infty]$, $a > 0$ *and let* f *be a measurable function on* $\mathbb{R}$ *such that* $\|e_a f\|_{p,\alpha} < +\infty$. *Then*

$$\|e_a W_\alpha(f)\|_p < +\infty.$$

Proof. First case : $p \in (1, +\infty)$. Let $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

From Hölder inequality, we get

$$\int_{\mathbb{R}} |f(x)||x|^{2\alpha+1}dx \leqslant \left(\int_{\mathbb{R}} e^{apx^2}|f(x)|^p|x|^{2\alpha+1}dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} e^{-aqx^2}|x|^{2\alpha+1}dx\right)^{\frac{1}{q}}$$

$$\leqslant \|e_a f\|_{p,\alpha} \left(\int_{\mathbb{R}} e^{-aqx^2}|x|^{2\alpha+1}dx\right)^{\frac{1}{q}} < +\infty.$$

So, it follows that $f \in L^1_\alpha(\mathbb{R})$. Then from Proposition 2.1, the function $W_\alpha(f)$ is defined almost everywhere on $\mathbb{R}$. We have

$$\|e_a W_\alpha(f)\|^p_p = \int_{\mathbb{R}} \left| e^{ax^2} \left(a_\alpha \int_{|x|}^{+\infty} (y^2 - x^2)^{\alpha-\frac{1}{2}} y f(sgn(x)y) dy \right) \right|^p dx$$

$$\leqslant \int_{\mathbb{R}} e^{apx^2} \left(\int_{\mathbb{R}} e^{ay^2} |f(sgn(x)y)| e^{-ay^2} \, d\upsilon_x(y) \right)^p dx.$$

Let $G(x) = \left(\int_{\mathbb{R}} e^{ay^2} |f(sgn(x)y)| e^{-ay^2} d\upsilon_x(y) \right)^p$.

Using Hölder inequality, we get

$$G(x) \leqslant \left(\int_{\mathbb{R}} e^{apy^2} |f(sgn(x)y)|^p d\upsilon_x(y) \right) \left(\int_{\mathbb{R}} e^{-aqy^2} d\upsilon_x(y) \right)^{\frac{p}{q}}.$$

From the relation (9), we have

$$G(x) \leqslant W_\alpha(e_{ap} |f|^p)(x) (W_\alpha(e_{-aq})(x))^{\frac{p}{q}}.$$

Since $W_\alpha(e_{-aq})(x) = C e^{-aqx^2}$. Then by using (10), one gets

$$\|e_a W_\alpha(f)\|^p_p \leqslant C \int_{\mathbb{R}} W_\alpha(e_{ap} |f|^p)(x) dx \leqslant C \|e_a f\|^p_{p,\alpha} < +\infty.$$

Second case: $p = +\infty$. We have

$$\int_{\mathbb{R}} |f(x)| |x|^{2\alpha+1} dx \leqslant \|e_a f\|_{\infty,\alpha} \int_{\mathbb{R}} e^{-ax^2} |x|^{2\alpha+1} dx < +\infty.$$

Consequently $W_\alpha(f)$ is defined almost everywhere on $\mathbb{R}$ and we have

$$|W_\alpha(f)(x)| \leqslant \int_{\mathbb{R}} e^{ay^2} |f(sgn(x)y)| e^{-ay^2} d\upsilon_x(y) \leqslant W_\alpha(e_{-a})(x) \|e_a f\|_{\infty,\alpha}.$$

From Proposition 3.1, we obtain for almost everywhere $x \in \mathbb{R}$ that

$$e^{ax^2} |W_\alpha(f)(x)| \leqslant C \|e_a f\|_{\infty,\alpha}$$

which gives the result.

Third case: $p = 1$. We have

$$\int_{\mathbb{R}} |f(x)| |x|^{2\alpha+1} dx \leqslant \|e_a f\|_{1,\alpha} < +\infty$$

which implies that $W_\alpha(f)$ is defined almost everywhere on $\mathbb{R}$ and we have

$$\|e_a W_\alpha f\|_1 \leqslant a_\alpha \int_{\mathbb{R}} e^{ax^2} \left(\int_{|x|}^{+\infty} (y^2 - x^2)^{\alpha-\frac{1}{2}} e^{ay^2} |f(sgn(x)y)| \right) y e^{-ay^2} dy dx$$

$$\leqslant a_\alpha \int_{\mathbb{R}} \left(\int_{|x|}^{+\infty} (y^2 - x^2)^{\alpha-\frac{1}{2}} e^{ay^2} |f(sgn(x)y)| \right) y dy dx = \|W_\alpha f\|_1.$$

Then, using relation (10), we get

$$\|e_a W_\alpha f\|_1 \leqslant C \int_{\mathbb{R}} e^{ax^2} |f|(x) d\mu_\alpha(x). \qquad \square$$

Lemma 3.1. *Let $p \in [1, +\infty]$ and let f be a measurable function on $\mathbb{R}$ such that $\|e_a f\|_{p,\alpha} < +\infty$ for some $a > 0$. Then the function defined on $\mathbb{C}$ by*

$$\mathcal{F}_{B,S}^\alpha(f)(z) = \int_{\mathbb{R}} f(x)\Phi_\alpha(-izx) d\mu_\alpha(x) \qquad (15)$$

is well defined and entire on $\mathbb{C}$. Moreover, for all $\xi, \eta \in \mathbb{R}$, we have

$$|\mathcal{F}_{B,S}^\alpha(f)(\xi + i\eta)| \leqslant C e^{\frac{\eta^2}{4a}}. \qquad (16)$$

Proof. From analyticity theorem under the integral sign, we deduce that the function defined on $\mathbb{C}$ by (15) is well defined and entire on $\mathbb{C}$.

On the other hand, since $f \in L_\alpha^1(\mathbb{R})$ then

$W_\alpha(f) \in L^1(\mathbb{R})$ and the function $z \mapsto \int_{\mathbb{R}} W_\alpha(f)(x)e^{-ixz}dx$ is well defined and entire on $\mathbb{C}$.

Consequently, from (11) we deduce that for all $\xi, \eta \in \mathbb{R}$, we have

$$\mathcal{F}_{B,S}^\alpha(f)(\xi + i\eta) = \mathcal{F} \circ W_\alpha(f)(\xi + i\eta), = \int_{\mathbb{R}} W_\alpha(f)(x)e^{-ix(\xi+i\eta)}dx.$$

Thus

$$|\mathcal{F}_{B,S}^\alpha(f)(\xi + i\eta)| \leqslant e^{\frac{\eta^2}{4a}} \int_{\mathbb{R}} e^{ax^2} W_\alpha(f)(x)e^{-a(x-\frac{\eta}{2a})^2}dx.$$

Using Proposition 3.2, we obtain the relation (16). $\square$

Theorem 3.1. *Let f be a measurable function on $\mathbb{R}$ such that*

$$\|e_a f\|_{p,\alpha} < +\infty \quad \text{and} \quad \|e_b \mathcal{F}_{B,S}^\alpha(f)\|_{q,\alpha} < +\infty \qquad (17)$$

for some constants $a > 0$, $b > 0$, $1 \leqslant p, q \leqslant +\infty$ and at least one of p and q is finite. We have

(1) *If $ab \geqslant \frac{1}{4}$ then $f = 0$ almost everywhere.*
(2) *If $ab < \frac{1}{4}$ then for all $\delta \in]a, \frac{1}{4b}[$, the functions having the form $f(x) = P(x)e^{-\delta x^2}$, where P is an even polynomial on $\mathbb{R}$, satisfy relation (17).*

Proof. Assume that p is finite. Firstly, suppose that $ab \geqslant \frac{1}{4}$.

Consider the function h defined on $\mathbb{C}$ by

$$h(z) = e^{\frac{z^2}{4a}} \mathcal{F}_{B,S}^\alpha(f)(z). \qquad (18)$$

This function is entire on $\mathbb{C}$ and using (16), we obtain

$$\forall z \in \mathbb{C}, \quad |h(z)| \leqslant Ce^{\frac{(\Re(z))^2}{4a}}. \tag{19}$$

In the following, we distinguish two cases for the number q.

First case: $q < +\infty$. We have

$$\|h_{|\mathbb{R}}\|_{q,\alpha}^q = \int_{\mathbb{R}} \left| e^{\frac{y^2}{4a}} \mathcal{F}_{B,S}^\alpha(f)(y) \right|^q |y|^{2\alpha+1} dy = \int_{\mathbb{R}} \left| e^{by^2} \mathcal{F}_{B,S}^\alpha(f)(y) \right|^q e^{(\frac{1}{4a}-b)y^2 q} |y|^{2\alpha+1} dy.$$

Using the fact that $ab \geqslant \frac{1}{4}$ and the hypothesis (17), we obtain

$$\|h_{|\mathbb{R}}\|_{q,\alpha} \leqslant \|e_b \mathcal{F}_{B,S}^\alpha(f)\|_{q,\alpha} < +\infty. \tag{20}$$

From the relations (19) and (20) and lemma 2.1, it follows that $h(z) = 0$ for all $z \in \mathbb{C}$. Thus $\mathcal{F}_{B,S}^\alpha(f)(y) = 0$ for all $y \in \mathbb{R}$. The injectivity of $\mathcal{F}_{B,S}^\alpha$ implies that $f = 0$ almost everywhere on $\mathbb{R}$.

Second case: $q = +\infty$. We have

$$\|h_{|\mathbb{R}}\|_{\infty,\alpha} = \operatorname{ess\,sup}_{y \in \mathbb{R}} \left| e^{\frac{y^2}{4a}} \mathcal{F}_{B,S}^\alpha(f)(y) \right| = \operatorname{ess\,sup}_{y \in \mathbb{R}} \left\{ \left| e^{by^2} \mathcal{F}_{B,S}^\alpha(f)(y) \right| e^{(\frac{1}{4a}-b)y^2 q} \right\}.$$

For $ab > \frac{1}{4}$, we get from (17)

$$\|h_{|\mathbb{R}}\|_{\infty,\alpha} \leqslant \left\| e_b \mathcal{F}_{B,S}^\alpha(f)(y) \right\|_{\alpha,\infty} < +\infty. \tag{21}$$

Using relations (19),(21) and lemma 2.2, there exists a positive constant C such that for all $y \in \mathbb{R}, h(y) = C$. On the other hand, from (18), we have

$$\forall y \in \mathbb{R}, \quad \mathcal{F}_{B,S}^\alpha(f)(y) = Ce^{\frac{y^2}{4a}}. \tag{22}$$

But the assumption on $\mathcal{F}_{B,S}^\alpha(f)$ is expressed as

$$|\mathcal{F}_{B,S}^\alpha(f)(y)| \leqslant M\ e^{-by^2} \quad \text{a.e} \quad y \in \mathbb{R}, \tag{23}$$

for some constant $M > 0$. The continuity of $\mathcal{F}_{B,S}^\alpha(f)$ on $\mathbb{R}$ shows that the inequality (23) holds everywhere. By (22) and (23) this is impossible since $ab > \frac{1}{4}$ unless if $C = 0$. Thus $\mathcal{F}_{B,S}^\alpha(f)(y) = 0$ everywhere and then $f = 0$ almost everywhere on $\mathbb{R}$.

For $ab = \frac{1}{4}$, using Lemma 2.2, relations (11) and (17), we deduce that the function $W_\alpha(f)$ verifies

$$\|e_a W_\alpha(f)\|_p < +\infty \quad \text{and} \quad \|e_b \mathcal{F}(W_\alpha(f))\|_\infty < +\infty.$$

Therefore, from Theorem 1.2, we get that $W_\alpha f = 0$ almost everywhere and then $f = 0$ almost everywhere.

We suppose now that $ab < \frac{1}{4}$. Let $f(x) = P(x)\,e^{-\delta x^2}$, where P is an even polynomial and $\delta \in]a, \frac{1}{4b}[$.

We have $x \mapsto e^{ax^2} f(x) = P(x) e^{(a-\delta)x^2}$ belongs to $L_\alpha^p(\mathbb{R})$.

Since f is an even function, $\mathcal{F}_{B,S}^\alpha(f)(x) = \mathcal{H}_\alpha(f)(x) = Q(x) e^{-\frac{1}{4\delta}x^2}$. $\quad\square$

4. AN ANALOGUE OF HARDY'S THEOREM

Lemma 4.1. *Let a > 0 and let f be a continuous function on $\mathbb{R}$ such that*

$$\forall x \in \mathbb{R}, \quad |f(x)| \leqslant Ce^{-ax^2}. \tag{24}$$

Then $W_\alpha(f)$ is of class C^1 on $\mathbb{R} \setminus \{0\}$ and verifies

$$\forall x \in \mathbb{R} \setminus \{0\}, \quad [W_{\frac{1}{2}}f]'(x) = -xf(x)$$

and

$$\forall \alpha > \frac{1}{2}, \quad \forall x \in \mathbb{R} \setminus \{0\}, \quad [W_\alpha f]'(x) = -2\alpha x W_{\alpha-1} f(x).$$

Proof. Let f be a continuous function on $\mathbb{R}$ verifying (24), we have

$$W_{\frac{1}{2}}f(x) = \begin{cases} \int_x^{+\infty} yf(y)dy & \text{if } x > 0 \\ \int_{-x}^{+\infty} yf(-y)dy & \text{if } x < 0 \end{cases}$$

We deduce that

$$\forall x \in \mathbb{R} \setminus \{0\}, \quad [W_{\frac{1}{2}}f]'(x) = -xf(x).$$

Now, we take $\alpha > \frac{1}{2}$. Using theorem of derivation and relation (7) we get

$$[W_\alpha f]'(x) = \begin{cases} \frac{-4\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\frac{1}{2})} x \int_y^{+\infty} (y^2 - x^2)^{\alpha-\frac{3}{2}} yf(y)dy & \text{if } x > 0 \\ \frac{-4\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha-\frac{1}{2})} x \int_{-x}^{+\infty} (y^2 - x^2)^{\alpha-\frac{3}{2}} yf(-y)dy & \text{if } x < 0 \end{cases}$$

which gives the wanted result. $\square$

Proposition 4.1. *For $\alpha = k + \frac{1}{2}$, $k \in \mathbb{N}$, let f be a continuous function on $\mathbb{R}$ verifying (24). Then W_α is of class C^{k+1} on $\mathbb{R} \setminus \{0\}$ and we have*

$$V_\alpha \circ W_\alpha(f) = f,$$

where

$$V_\alpha f(x) = (-1)^{k+1} \frac{2^{2k+1}k!}{(2k+1)!} \left(\frac{d}{dx^2}\right)^{k+1} (f(x)), \quad x \in \mathbb{R} \setminus \{0\}$$

and $\frac{d}{dx^2} = \frac{1}{2x}\frac{d}{dx}$.

Proof. From Lemma 4.1, $W_{\frac{1}{2}}f$ is of class C^1 on $\mathbb{R} \setminus \{0\}$ and we have

$V_{\frac{1}{2}} \circ W_{\frac{1}{2}}(f) = f$. Assume $W_{k-\frac{1}{2}}$ is C^{k-1} on $\mathbb{R} \setminus \{0\}$ and $V_{k-\frac{1}{2}} \circ W_{k-\frac{1}{2}}(f) = f$. From Lemma 4.1, $\frac{d}{dx^2} W_{k+\frac{1}{2}} = -\frac{2k+1}{2} W_{k-\frac{1}{2}}f$.

$V_{k+\frac{1}{2}} \circ W_{k+\frac{1}{2}}f(x) = (-1)^k \frac{2^{2k}k!}{(2k)!} \left(\frac{d}{dx^2}\right)^k W_{k-\frac{1}{2}}f = V_{k-\frac{1}{2}} \circ W_{k-\frac{1}{2}}(f) = f$ which completes the proof. $\square$

Theorem 4.1. *Let a > 0, b > 0. We consider f a measurable function on $\mathbb{R}$ such that*

$$|f(x)| \leqslant Ce^{-ax^2} \quad \text{and} \quad |\mathcal{F}^{\alpha}_{B,S}(f)(y)| \leqslant Ce^{-by^2}, \quad \text{a.e} \quad x, y \in \mathbb{R}. \tag{25}$$

We have,

(1) *If $ab > \frac{1}{4}$ then $f = 0$ almost everywhere.*
(2) *If $ab < \frac{1}{4}$ then infinitely nonzero functions satisfy the conditions (25).*
(3) *If $ab = \frac{1}{4}$ and $\alpha = k + \frac{1}{2}$, $k \in \mathbb{N}$, then the continuous functions f satisfying condition (25) are exactly the functions having the form $f(x) = const.e^{-ax^2}$.*

Proof.

(1) If $ab > \frac{1}{4}$ then the second case of the proof of Theorem 3.1 gives the result.
(2) If $ab < \frac{1}{4}$ then the function defined in 2) of Theorem 3.1 clearly satisfies also the condition (25).
(3) Assume that $ab = \frac{1}{4}$ and consider f a continuous function on $\mathbb{R}$ verifying the relation (25). Using the expression of W_{α} given by relation (7), we get

$$|W_{\alpha}f(x)| \leqslant CW_{\alpha}(e^{-ax^2}).$$

From Proposition 3.1, we get $|W_{\alpha}f(x)| \leqslant Ce^{-ax^2}$.

Furthermore, relation (11) implies that

$$|\mathcal{F}(W_{\alpha}f)(y)| \leqslant Ce^{-by^2}.$$

Using Theorem 1.1, we obtain that $W_{\alpha}(f)(x) = const.e^{-ax^2}$.
From Proposition 4.1, for $\alpha = k + \frac{1}{2}$, we get $f = V_{k+\frac{1}{2}}(const.\ e_{-a})$.

Since $\frac{d}{dx^2}(e^{-ax^2}) = -ae^{-ax^2}$, we obtain $f(x) = const.\ e^{-ax^2}$. $\square$

Remark 4.1. Without restriction of the parameter α, we are not able to characterize all functions satisfying condition (25). Nonetheless the function $f(x) = const.\ e^{-ax^2}$ satisfies this condition.

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