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Copula conditional tail expectation for multivariate financial risks

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Abstract. Our goal in this paper is to propose an alternative risk measure which takes into account the fluctuations of losses and possible correlations between random variables. This new notion of risk measures, that we call Copula Conditional Tail Expectation describes the expected amount of risk that can be experienced given that a potential bivariate risk exceeds a bivariate threshold value, and provides an important measure for right-tail risk. An application to real financial data is given.

Keywords: Conditional tail expectation; Positive quadrant dependence; Copulas; Dependence measure; Risk management; Market models

Mathematics Subject Classification: 62P05; 62H20; 91B26; 91B30

1. Introduction

In actuarial science, several risk measures have been proposed, namely: the Value-at-Risk (VaR), the expected shortfall or the conditional tail expectation (CTE), the distorted risk measures (DRM) and recently the copula distorted risk measure (CDRM) as a risk measure which takes into account the fluctuations dependence between random variables (rv), see [3]. The CTE in risk analysis represents the conditional expected loss given that the loss exceeds

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its VaR and provides an important measure for right-tail risk. In this paper we will only consider a rv with finite mean. For a real number α in $(0, 1)$, the CTE of a risk X is given by

$$\mathbb{CTE}(\alpha) := \mathbb{E}[X|\, X > VaR_X(\alpha)], \tag{1.1}$$

where $VaR_X(\alpha) := \inf\{x : F(x) \geq \alpha\}$ is the quantile of order α pertaining to distribution function (df) F. In practice the expectation of X is computed when the conditional event α is fixed (to be equal to 95% or 99% for example).

Suppose now that we deal with a couple of random losses (X_1, X_2). It is clear that the CTE of X_1 is unrelated to X_2. If we had to control the overflow of the two risks X_1 and X_2 at the same time, CTE does not answer the problem, then we need another formulation of CTE which takes into account the excess of the two risks X_1 and X_2. Then we deal with the amount

$$\mathbb{E}\left[X_1|\, X_1 > VaR_{X_1}(\alpha), X_2 > VaR_{X_2}(t)\right]. \tag{1.2}$$

If the couple of random losses (X_1, X_2) are independent rv's then the amount (1.2) defined only the CTE of a univariate risk, X_1 for a fixed conditional event α. Therefore the case of independence is not important.

In recent years dependence is beginning to play an important role in the world of risk. The increasing complexity of insurance and financial activity products has led to increased actuarial and financial interest in the modeling of dependent risks. While independence can be defined in only one way, dependence can be formulated in infinite ways. Therefore the assumption of independence makes the treatment easier. Nevertheless, in applications dependence is the rule and independence is the exception. For more details see [12].

The copulas is a function that completely describes the dependence structure. It contains all the information to link the marginal distributions to their joint distribution. To obtain a valid multivariate df, we combine several marginal df's, or a different distributional family, with any copula function. Using Sklar's theorem [37], we can construct a bivariate distribution with arbitrary marginal distributions. Thus, for the purposes of statistical modeling, it is desirable to have a large collection of copulas at one's disposal. A great many examples of copulas can be found in the literature, most are members of families with one or more real parameter. For a formal treatment of copulas and their properties, see the monographs by Hutchinson and Lai [26], Dall'Aglio et al. [10], Joe [27], the conference proceedings edited by Beneš and Štěpán [2], Cuadras et al. [9], Dhaene et al. [15] and the textbook of Nelsen [31].

Recently in finance, insurance and risk management have emphasized the importance of positive or negative quadrant dependence notions (PQD or NQD) introduced by Lehmann [28], in different areas of applied probability and statistics, as an example, see [13,14]. Two rv's are said to be PQD when the probability that they are simultaneously large (or small) is at least as great as it would be where they are independent. In terms of copula, if their copula is greater than their product, i.e., $C(u_1, u_2) > u_1 u_2$ or, simply $C > C^{\perp}$, where $C^{\perp}$ denotes the product copula. For the sake of brevity, we will restrict ourselves to concepts of positive dependence.

The main idea of this paper is to use the information of dependence between PQD or NQD risks to quantifying insurance losses and measuring financial risk assessments, we propose a risk measure defined by:

$$\mathbb{CCTE}_{X_1}(t) := \mathbb{E}\left[X_1|\, X_1 > VaR_{X_1}(\alpha), X_2 > VaR_{X_2}(t)\right].$$

We will call this new risk measure by the *Copula Conditional Tail Expectation* (CCTE), like a risk measure which measures the conditional expectation given two dependent losses exceeds $VaR_{X_1}(\alpha)$ and $VaR_{X_2}(t)$ for a fixed $\alpha \geq 0.9$ and $t \in (0, 1)$ usually with $t > 0.9$. Again, CCTE satisfies all the desirable properties of a coherent risk measure [1]. The notion of the copula in risk measure field has recently been considered by several authors, see for instance [3,11,16,17] and recently in [7,29].

This risk measure can give a good quantification of losses when we have combined dependents risk, this dependence can influence the losses of interested risks. Therefore, quantifying the risk of our position is useful to decide if it is acceptable or not. For this reason, we use the all information about this interest risk. The dependence of our risk on other risks is one of important information that we must take into consideration.

The rest of the paper is organized as follows. In Section 2, we give an explicit formulation of the new notion copula conditional tail expectation risk measure in bivariate case. In Section 3 we present some illustrative examples to explain how to use the new CCTE measure. Application to real financial data is given in Section 4. Concluding notes are given in Section 5. Proofs are relegated to Appendix.

2. Copula conditional tail expectation

A risk measure quantifies the risk exposure in a way that is meaningful for the problem at hand. The most commonly used risk measure in finance and insurance are VaR and CTE (also known as the Tail-VaR or expected shortfall). The risk measure is simply the loss size for which there is a small (e.g. 1%) probability of exceeding. For some time, it has been recognized that this measure suffers from serious deficiencies if the losses are not normally distributed.

According to Artzner et al. [1] and Wirch and Hardy [38], the conditional tail expectation of a rv X at its $VaR_X(\alpha)$ is defined by:

$$\mathbb{CTE}_X(\alpha) = \frac{1}{1 - F_X(VaR_X(\alpha))} \int_{VaR_X(\alpha)}^{\infty} x dF_X(x),$$

where F_X is the df of X.

Since X is continuous, then $F_X(VaR_X(\alpha)) = \alpha$, it follows that for all $0 < \alpha < 1$

$$\mathbb{CTE}_X(\alpha) = \frac{1}{1 - \alpha} \int_{\alpha}^{1} VaR_X(u)\, du. \tag{2.3}$$

The CTE can be larger than the VaR measure for the same value of level α described above since it can be thought of as the sum of the quantile $VaR_X(\alpha)$ and the expected excess loss. Tail-VaR is a coherent risk measure in the sense of Artzner et al. [1]. For application of this kind of coherent risk measure we refer to Artzner et al. [1] and Wirch and Hardy [38].

Thus the CTE is nothing, see [34], but the mathematical transcription of the concept of "average loss in the worst $100(1 - \alpha)\%$ cases", defining by $\nu = VaR_X(\alpha)$ a critical loss threshold corresponding to some confidence level α, $\mathbb{CTE}_X(\alpha)$ provides a cushion against the mean value of losses exceeding the critical threshold ν.

Now, assume that X_1 and X_2 are dependent with joint df H and continuous margins F_{X_i}, $i = 1, 2$, respectively. Through this paper, we call X_1 the *target risk* and X_2 the *associated risk*. In this case, the problem becomes different and its resolution requires more than the usual background.

Our contribution is to introduce the copula notion to provide more flexibility to the CTE of rv's in terms of loss and dependence structure. For comprehensive details on copulas one may consult the textbook of Nelsen [31].

According to Sklar's Theorem [37], there exists a unique copula $C : [0,1]^d \to [0,1]$ such that

$$H(x_1, x_2) = C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right). \tag{2.4}$$

The formula of CTE focuses only on the average of loss. For this we should think of another formula to be more inclusive, this formula must take into consideration the dependence structure and the behavior of margin tails. These two aspects have an important influence when quantifying risks. On the other hand, if the correlation factor is neglected, the calculation of the CTE follows from formula (2.3), which only focuses on the target risk.

Now, by considering the correlation between the target and the associated risks, we define a new notion of CTE called *Copula Conditional Tail Expectation* (CCTE) given in (1.2), this notion led to give a risk measurement focused on the target risk and the link between target and associated risk.

Let us denote the survival functions by $\overline{F}_{X_i}(x_i) = \mathbb{P}(X_i > x_i)$, $i = 1, 2$, and the joint survival function by $\overline{H}(x_1, x_2) = \mathbb{P}(X_1 > x_1, X_2 > x_2)$. The copula function $\overline{C}$ which couples $\overline{H}$ to $\overline{F}_{X_i}$, $i = 1, 2$ via $\overline{H}(x_1, x_2) = \overline{C}(\overline{F}_{X_1}(x_1), \overline{F}_{X_2}(x_2))$ is called the survival copula of (X_1, X_2). Furthermore, $\overline{C}$ is a copula, and

$$\overline{C}(u, v) := u + v - 1 + C(1 - u, 1 - v), \tag{2.5}$$

where C is the (ordinary) copula of X_1 and X_2. For more details on the survival copula function see, Section 2.6 in [31].

If we suppose that C is absolutely continuous with density c, we can rewrite for all s and t in $(0, 1)$

$$\overline{C}(1 - s, 1 - t) = \int_s^1 J_t(u)\, du$$

where

$$J_t(u) := \int_t^1 c(u, v)\, dv. \tag{2.6}$$

So for the fixed level $s = \alpha$, we have

$$\overline{C}(1 - \alpha, 1 - t) = 1 - \alpha - t + C(\alpha, t). \tag{2.7}$$

The CCTE of the target risk X_1 computed under a fixed conditional risk probability $\overline{C}(1 - \alpha, 1 - t)$ with respect to the associated risk X_2 is given in the following proposition.

Proposition 2.1. *Let (X_1, X_2) a bivariate rv with joint df represented by the copula C. Assume that X_2 has a finite mean and df F_{X_1}. Then for a fixed α and for all t in $(0, 1)$, the copula conditional tail expected of X_1 is given by*

$$\mathbb{CCTE}_{X_1}(t) = \frac{\int_\alpha^1 J_t(u)\, F_{X_1}^{-1}(u)\, du}{\int_\alpha^1 J_t(u)\, du}, \tag{2.8}$$

where $J_t(\cdot)$ is given in (2.6) and $F_{X_1}^{-1}$ is the quantile function of F_{X_1}.

This notion does not depend on the df of the associated risks, but it depends only on the copula function and the df of target risk.

Next, in Section 3, we will prove that the risk when we consider the correlation between PQD risks is greater than in the case of a single one. That means, for all α, t in (0, 1) then

$$\mathbb{CCTE}_{X_1}(t) \geq \mathbb{CTE}_{X_1}(\alpha). \tag{2.9}$$

Notice that in the NQD rv's we have the reverse inequality of (2.9) and the CCTE coincides with CTE measures in the non-dependence case, i.e. the copula $C = C^{\perp}$.

3. Illustration examples

3.1. CCTE via Farlie–Gumbel–Morgenstern copulas

One of the most important parametric families of copulas is the Farlie–Gumbel–Morgenstern (FGM) family defined as

$$C_\theta^{FGM}(u, v) = uv + \theta uv(1-u)(1-v), \quad u, v \in [0, 1], \tag{3.10}$$

where $\theta \in [-1, 1]$. The family was discussed by Morgenstern [30], Gumbel [23] and Farlie [19].

The copula given in (3.10) is PQD for $\theta \in (0, 1]$ and NQD for $\theta \in [-1, 0)$. In practical applications this copula has been shown to be somewhat limited, for copula dependence parameter $\theta \in [-1, 1]$, Spearman's correlation $\rho \in [-1/3, 1/3]$ and Kendall's $\tau \in [-2/9, 2/9]$, for more details on copulas see, for example, [31].

Members of the FGM family are symmetric, i.e., $C_\theta^{FGM}(u, v) = C_\theta^{FGM}(v, u)$ for all (u, v) in $[0, 1]^2$ and have the lower and upper tail dependence coefficients equal to 0.

A pair (X, Y) of rv's is said to be exchangeable if the vectors (X, Y) and (Y, X) are identically distributed. Note that, in applications, exchangeability may not always be a realistic assumption. For identically distributed continuous rv's, exchangeability is equivalent to the symmetry of the FGM copula.

For practical purposes, we consider copula families with only positive dependence. Furthermore, risk models are often designed to model positive dependence, since in some sense it is the "dangerous" dependence: assets (or risks) move in the same direction during periods of extreme events, see [18].

Consider the bivariate loss PQD rv's (X_i, Y), $i = 1, 2, 3$, having continuous marginal df's $F_{X_i}(x)$ and $G_Y(y)$ and joint df $H_{X_i,Y}(x, y)$ represented by the FGM copula of parameters θ_i, respectively for $i = 1, 2, 3$

$$H_{X_i,Y}(x, y) = C_{\theta_i}^{FGM}(F_{X_i}(x), G_Y(y)).$$

The marginal survival functions $\overline{F}_{X_i}(x)$, $i = 1, 2, 3$ and $\overline{G}_Y(y)$ are given by

$$\overline{F}_{X_i}(x) = \begin{cases} (1+x)^{-\gamma}, & x \geq 0, \\ 1, & x < 0, \end{cases} \quad \text{and} \quad \overline{G}_Y(y) = \begin{cases} (1+y)^{-\gamma}, & y \geq 0, \\ 1, & y < 0 \end{cases} \tag{3.11}$$

where $\gamma > 0$ is called the Pareto index, the case $\gamma \in (1, 2)$ means that X_i have heavy-tailed distributions, so that X_i and Y have identical Pareto dfs.

For each couple (X_i, Y), $i = 1, 2, 3$, we propose $\theta_1 = 0.01$, $\theta_2 = 0.5$ and $\theta_3 = 1$, respectively. The choice of parameters $\theta_i, i = 1, 2, 3$ corresponds respectively to the weak, medium and the high dependence.

In this example, among the target risks X_i we will choose the less risky with the associated risk Y. The $\mathbb{CTE}$s and the VaRs of X_i for a fixed level $s = \alpha$ are the same and are given respectively by

$$\mathbb{CTE}_{X_i}(\alpha) = \frac{\gamma(1-\alpha)^{-1/\gamma}}{\gamma - 1} \tag{3.12}$$

and

$$VaR_{X_i}(\alpha) = (1-\alpha)^{-1/\gamma}, \tag{3.13}$$

for $i = 1, 2, 3$.

For a fixed $s = \alpha$, we have that

$$\overline{C}(1-\alpha, 1-t) = 1 - \alpha - t + \alpha t + \theta_i \alpha t(1-\alpha)(1-t). \tag{3.14}$$

Now, we calculate

$$\begin{aligned}\int_\alpha^1 J_t(u) F_{X_i}^{-1}(u)\, du &= \int_\alpha^1 (1-u)^{-1/\gamma}(\theta_i - 2u\theta_i - 2v\theta_i + 4uv\theta_i + 1)\, du dv \\ &= \int_t^1 (\theta_i - 2\theta_i v + 1)\, dv \int_\alpha^1 (1-u)^{-1/\gamma} du \\ &+ 2\theta_i \int_t^1 (2v-1)\, dv \int_\alpha^1 u(1-u)^{-1/\gamma} du,\end{aligned}$$

then

$$\begin{aligned}\int_\alpha^1 J_t(u) F_{X_i}^{-1}(u)\, du &= \gamma \frac{(1-t)(2\gamma + t\theta_i - 2t\theta_i\alpha + 2t\theta_i\alpha\gamma - 1)}{2\gamma^2 - 3\gamma + 1} \\ &\times (1-\alpha)^{1-1/\gamma}.\end{aligned} \tag{3.15}$$

Finally, by substitution of (3.14) and (3.15) in (2.8) we get

$$\mathbb{CCTE}_{X_i}(t) = \gamma \frac{2\gamma + t\theta_i - 2t\alpha\theta_i + 2t\alpha\gamma\theta_i - 1}{(t\alpha\theta_i + 1)\left(2\gamma^2 - 3\gamma + 1\right)}(1-\alpha)^{-1/\gamma}. \tag{3.16}$$

We have in Table 3.1 and Fig. 3.1 the comparison of the riskiness of X_1, X_2 and X_3. Recall that, the $\mathbb{CTE}$'s risk measure of X_i at level α is the same in all cases. Note that $\mathbb{CCTE}$ coincides with $\mathbb{CTE}$ in the independence cases ($\theta_1 = 0$). The $\mathbb{CCTE}$ of the loss X_3 is riskier than X_2 and X_1 but not very significant, in the 6th column of Table 3.1, the relative difference between 64.7946 and 64.633 is only about 0.025%. This is because FGM copula does not take into account the dependence in the upper and the lower tail ($\lambda_L = \lambda_U = 0$). In this case, we cannot clearly confirm which is the most dangerous risk.

3.2. CCTE via Archimedean copulas

A bivariate copula is said to be Archimedean (see, [22]) if it can be expressed by

$$C(u, v) = \psi^{[-1]}(\psi(u) + \psi(v)),$$

where ψ, called the generator of C, is a continuous strictly decreasing convex function from $[0, 1]$ to $[0, \infty]$ such that $\psi(1) = 0$ with $\psi^{[-1]}$ denotes the *pseudo-inverse* of ψ, that is

$$\psi^{[-1]}(t) = \begin{cases}\psi^{-1}(t), & \text{for} \quad t \in [0, \psi(0)], \\ 0, & \text{for} \quad t \geq \psi(0).\end{cases}$$

Table 3.1
Risk measures of dependent Pareto (1.5) rv's with FGM copula.

α	0.9000	0.9225	0.9450	0.9675	0.9900
$VaR_{X_i}(\alpha)$	4.6415	5.5013	6.9144	9.8192	21.5443
$\mathbb{CTE}_{X_i}(\alpha)$	13.9247	16.5039	20.7433	29.4577	64.6330
t	$\mathbb{CCTE}_{X_1}(t)$, $\theta = 0.01$				
0.9000	13.9309	16.5096	20.7484	29.4619	64.6359
0.9225	13.9311	16.5097	20.7485	29.4620	64.6359
0.9450	13.9312	16.5099	20.7487	29.4621	64.6360
0.9675	13.9314	16.5100	20.7488	29.4623	64.6361
0.9900	13.9316	16.5101	20.7489	29.4624	64.6362
t	$\mathbb{CCTE}_{X_2}(t)$, $\theta = 0.5$				
0.9000	14.1477	16.7072	20.9234	29.60778	64.7336
0.9225	14.1517	16.7108	20.9266	29.61038	64.7353
0.9450	14.1555	16.7143	20.9297	29.61293	64.7370
0.9675	14.1594	16.7178	20.9327	29.61545	64.7387
0.9900	14.1631	16.7212	20.9357	29.61793	64.7404
t	$\mathbb{CCTE}_{X_3}(t)$, $\theta = 1$				
0.9000	14.2709	16.8183	21.0208	29.6880	64.7868
0.9225	14.2756	16.8226	21.0245	29.6910	64.7888
0.9450	14.2803	16.8267	21.0281	29.6940	64.7908
0.9675	14.2848	16.8308	21.0316	29.6969	64.7927
0.9900	14.2892	16.8348	21.0351	29.6997	64.7946

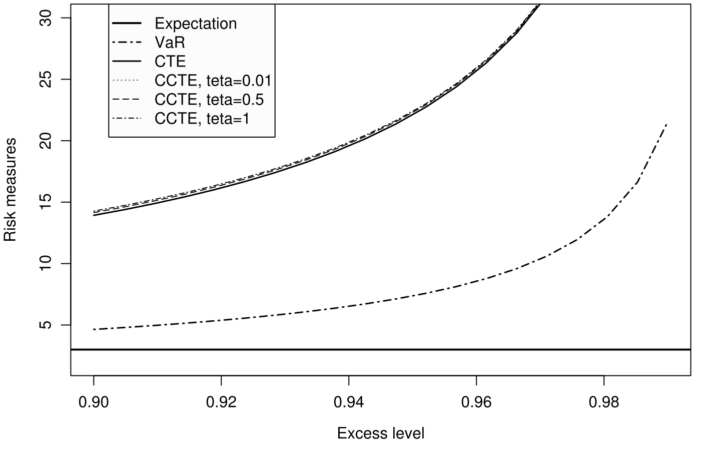

Fig. 3.1. $\mathbb{CCTE}$, $\mathbb{CTE}$ and VaR risks measures of PQD Pareto (1.5) rv's with FGM copula and $0.9 \leq \alpha = t \leq 0.99$.

When $\psi(0) = \infty$, the generator ψ and C are said to be *strict* and therefore $\psi^{[-1]} = \psi^{-1}$. All notions of positive dependence that appeared in the literature, including the weakest one of PQD as defined by Lehmann [28], require the generator to be strict.

Archimedean copulas are widely used in applications due to their simple form, a variety of dependence structures and other "nice" properties. For example, in the Actuarial field: the idea arose indirectly in [4] and was developed in [5,32]. A survey of Actuarial applications is in [20].

For an Archimedean copula, Kendall's tau can be evaluated directly from the generator of the copula, as shown in [22]

$$\tau = 4 \int_0^1 \frac{\psi(u)}{\psi'(u)} du + 1 \tag{3.17}$$

where $\psi'(u)$ exists since the generator is convex. This is another "nice" feature of Archimedean copulas. As for tail dependency, as shown in [27] the coefficient of upper tail dependency is

$$\lambda_U = 2 - 2 \lim_{s \to 0^+} \frac{\psi(u)}{\psi'(2u)}$$

and the coefficient of lower tail dependency is

$$\lambda_L = 2 \lim_{s \to +\infty} \frac{\psi(u)}{\psi'(2u)}.$$

A collection of twenty-two one-parameter families of Archimedean copulas can be found in Table 4.1 of Nelsen [31].

Notice that in the case of Archimedean copula the copula conditional tail expectation has not an explicit formula, so we give by the following Corollary the expression of CCTE in terms of the generator.

Corollary 3.1. *Let C be an Archimedean copula absolutely continuous with generator ψ, then for a fixed α and t in* $(0, 1)$

$$J_t(u) = 1 - \frac{\psi'(u)}{\psi'(C(u,t))}. \tag{3.18}$$

Thus the CCTE of target risk X_2 in terms of Archimedean copula generator with respect to threshold $0 < t < 1$, is given by

$$\mathbb{CCTE}_{X_1}(t) = \frac{1}{\overline{C}(1-\alpha, 1-t)} \left((1-\alpha)\,\mathbb{CTE}_{X_1}(\alpha) - \int_\alpha^1 \frac{\psi'(u) F_{X_1}^{-1}(u)}{\psi'(C(u,t))} du \right).$$

Note that in practice we can easily fit copula-based models with the maximum likelihood method or to estimate the dependence parameter by the relationship between Kendall's tau of the data and the generator of the Archimedean copula given in (3.17) under the specified copula model.

In the following section, we give some examples to explain how to calculate and compare the CCTE with other risk measures such as VaR and CTE.

3.2.1. CCTE via Clayton copula

In the following example, we consider the bivariate Clayton copula, which is a member of the class of Archimedean copula, with the dependence parameter θ in $[-1, \infty) \setminus \{0\}$.

The Clayton family was first proposed by Clayton [4] and studied by Oakes [32,33], Cox and Oakes [8] and Cook and Johnson [5,6]. The Clayton copula has been used to study correlated risks, it has the form

$$C_\theta^C(u, v) := \left[\max\left(u^{-\theta} + v^{-\theta} - 1, 0\right)\right]^{-1/\theta}. \tag{3.19}$$

For $\theta > 0$ the copulas are strict and the copula expression simplifies to

$$C_\theta^C(u, v) = \left(u^{-\theta} + v^{-\theta} - 1\right)^{-1/\theta}. \tag{3.20}$$

Table 3.2
Upper tail, Kendall's tau and Clayton copula parameters used in calculation of risk measures.

$\overline{\lambda}_L$	θ_i	τ
0.250	0.5	0.200
0.707	2	0.500
0.943	12	0.857

Asymmetric tail dependence is prevalent if the probability of joint extreme (left) negative realizations differs from that of joint extreme (right) positive realizations. It can be seen that the Clayton copula assigns a higher probability to joint extreme negative events than to joint extreme positive events. The Clayton copula is said to display lower tail dependence $\lambda_L = 2^{-1/\theta}$, while it displays zero upper tail dependence $\lambda_U = 0$, for $\theta \geq 0$. The converse can be said about the Gumbel copula (displaying upper but zero lower tail dependence). The margins become independent as θ approaches zero, while for $\theta \to 1$, the Clayton copula arrives at the comonotonicity copula. For $\theta = -1$ we obtain the Fréchet–Hoeffding lower bound and the copula attains the Fréchet upper bound as θ approaches infinity.

We take the same example as in Section 3.1, we may now represent the joint df's H_i, $i = 1, 2, 3$, respectively, by the Clayton copulas $C_{\theta_i}^C$ given in (3.20).

The relationship between Kendall's tau τ and the Clayton copula is given by

$$\tau = \theta / (\theta + 2), \tag{3.21}$$

we select a different dependent parameter corresponding to several levels of positive dependency summarized in Table 3.2 for a weakness, a moderate and a strong positive association, to calculate and compare the CCTEs of X_i, $i = 1, 2, 3$.

The CTEs and VaRs of X_i are the same and are given respectively by (3.12) and (3.13), for $i = 1, 2, 3$. The CCTE of the rv's X_i with respect to the threshold t is given by

$$\begin{aligned}\mathbb{CCTE}_{X_i}(t) = \frac{1}{\overline{C}_{\theta_i}^C(1-\alpha, 1-t)} &\left(\frac{\gamma (1-\alpha)^{-1/\gamma+1}}{(\gamma - 1)} \right. \\ &\left. - \int_\alpha^1 \frac{\left(t^{-\theta_i} + u^{-\theta_i} - 1\right)^{-1-1/\theta_i}}{(1-u)^{1/\gamma} u^{\theta_i+1}} du \right). \end{aligned} \tag{3.22}$$

Table 3.3 and Fig. 3.2 show that the loss X_3 is clearly considerably riskier than X_2 and X_1, in the 6th column of Table 3.3, the relative difference between 66.3802 and 64.6330 is about 2.63%.

Clayton copula is best suited for applications in which two outcomes are likely to experience low values together, since the dependence is strong in the lower tail and weak in the upper tail.

3.2.2. CCTE via Gumbel copula

The Gumbel family has been introduced by Gumbel [24]. Since it has been discussed in [25], it is also known as the Gumbel–Hougaard family. The Gumbel copula is an asymmetric Archimedean copula given by

$$C_\theta^G(u, v) = \exp\left\{ -\left[(-\ln u)^\theta + (-\ln v)^\theta \right]^{1/\theta} \right\},$$

Table 3.3
Risk measures of dependent Pareto (1.5) rv's with Clayton copula.

α	0.9000	0.9225	0.9450	0.9675	0.9900
$VaR_{X_i}(\alpha)$	4.6415	5.5013	6.9144	9.8192	21.5443
$\mathbb{CTE}_{X_i}(\alpha)$	13.9247	16.5039	20.7433	29.4577	64.6330
t	$\mathbb{CCTE}_{X_1}(t)$, $\theta = 0.5$				
0.9000	14.0887	16.6529	20.8749	29.5669	64.7060
0.9225	14.0928	16.6566	20.8782	29.5697	64.7078
0.9450	14.0969	16.6604	20.8815	29.5724	64.7097
0.9675	14.1010	16.6641	20.8848	29.5751	64.7115
0.9900	14.1051	16.6678	20.8880	29.5779	64.7133
t	$\mathbb{CCTE}_{X_2}(t)$, $\theta = 2$				
0.9000	14.5006	17.0238	21.1992	29.8337	64.8826
0.9225	14.5361	17.0562	21.2279	29.8577	64.8987
0.9450	14.5726	17.0895	21.2575	29.8824	64.9153
0.9675	14.6101	17.1239	21.2880	29.9079	64.9324
0.9900	14.6486	17.1592	21.3195	29.9342	64.9501
t	$\mathbb{CCTE}_{X_3}(t)$, $\theta = 12$				
0.9000	15.6051	17.9134	21.8883	30.3313	65.1690
0.9225	16.1180	18.3667	22.2741	30.6377	65.3635
0.9450	16.7436	18.9301	22.7627	31.0332	65.6192
0.9675	17.4948	19.6187	23.3719	31.5369	65.9518
0.9900	18.3837	20.4476	24.1199	32.1694	66.3802

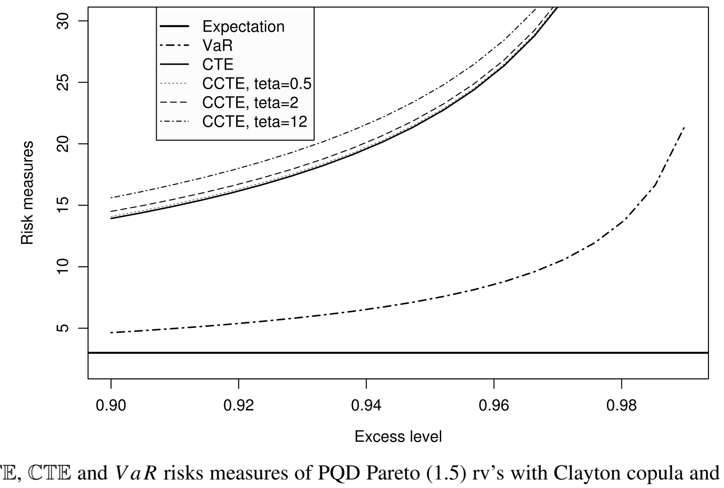

Fig. 3.2. $\mathbb{CCTE}$, $\mathbb{CTE}$ and VaR risks measures of PQD Pareto (1.5) rv's with Clayton copula and $0.9 \leq \alpha = t \leq 0.99$.

its generator is

$$\psi_\theta(t) = (-\ln t)^\theta.$$

The dependence parameter is restricted to the interval $[1, \infty)$. It follows that the Gumbel family can represent independence and "positive" dependence only since the lower and upper bounds for its parameter correspond to the product copula and the upper Fréchet bound. The Gumbel copula families are often used for modeling heavy dependencies in right tail.

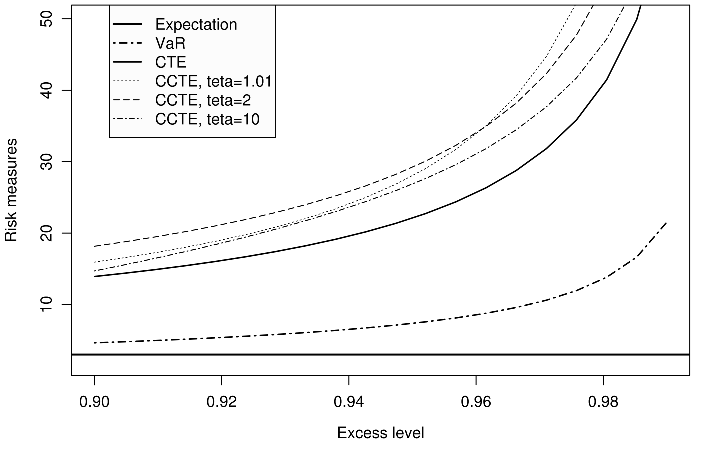

Fig. 3.3. $\mathbb{CCTE}$, $\mathbb{CTE}$ and VaR risks measures of PQD Pareto (1.5) rv's with Gumbel copula and $0.9 \leq \alpha = t \leq 0.99$.

Table 3.4
Upper tail, Kendall's tau and Gumbel copula parameters used in calculate of risk measures.

λ_U	θ_i	τ
0.013	1.01	0.009
0.585	2	0.500
0.928	10	0.900

It exhibits strong right (upper) tail dependence $\lambda_U = 2 - 2^{1/\theta}$ and relatively weak left (lower) tail dependence $\lambda_L = 0$. If outcomes are known to be strongly correlated with high values, but less correlated at low values, then the Gumbel copula will be an appropriate choice.

We give the CCTE of rv's X_i, $i = 1, 2, 3$ in terms of Gumbel copula by

$$\mathbb{CCTE}_{X_i}(t) = \frac{1}{\overline{C}_{\theta_i}^{G}(1-\alpha, 1-t)} \left(\frac{\gamma(1-\alpha)^{1-1/\gamma}}{\gamma - 1} - \int_{\alpha}^{1} u^{-1}(1-u)^{-1/\alpha}(-\ln u)^{\theta_i - 1} C_{\theta_i}^{G}(u,t) \left(-\ln\left(C_{\theta_i}^{G}(u,t)\right)\right)^{1-\theta_i} du \right), \tag{3.23}$$

where $\overline{C}_{\theta_i}^{G}(\alpha, t) = \alpha + t - 1 + C_{\theta_i}^{G}(1-\alpha, 1-t)$.

By the relationship between Kendall's tau τ and the Gumbel copula parameter θ given by:

$$\tau = (\theta - 1)/\theta,$$

we select the values of θ_i corresponding respectively to a weak, a moderate and a strong positive association which is summarized in Table 3.4.

Table 3.5 and Fig. 3.3 show that the loss X_3 is considerably riskier than X_2 and X_1, in the 6th column of Table 3.5, the relative difference between 112.1868 and 69.6017 is about 61.184%.

Table 3.5
Risk measures of PQD Pareto (1.5) rv's with Gumbel copula.

α	0.9000	0.9225	0.9450	0.9675	0.9900
$VaR_{X_i}(\alpha)$	4.6415	5.5013	6.9144	9.8192	21.5443
$\mathbb{CTE}_{X_i}(\alpha)$	13.9247	16.5039	20.7433	29.4577	64.6330
t	$\mathbb{CCTE}_{X_1}(t)$, $\theta = 1.01$				
0.9000	15.9370	18.8793	23.6990	33.5569	72.9927
0.9225	16.4850	19.5288	24.5076	34.6672	75.1339
0.9450	17.4102	20.6250	25.8737	36.5349	78.6453
0.9675	19.3659	22.9487	28.7606	40.4546	85.7265
0.9900	25.0078	33.6905	40.5881	56.2757	112.1868
t	$\mathbb{CCTE}_{X_2}(t)$, $\theta = 2$				
0.9000	18.1581	20.2092	23.8421	31.8490	66.0876
0.9225	19.7693	21.6536	25.0597	32.7667	66.6063
0.9450	22.6911	24.3385	27.3837	34.5437	67.5834
0.9675	28.9506	30.6075	33.0707	39.1284	70.0747
0.9900	52.9293	53.7426	55.2768	59.2078	86.3853
t	$\mathbb{CCTE}_{X_3}(t)$, $\theta = 10$				
0.9000	13.7652	15.6122	19.3784	29.4577	64.6330
0.9225	16.6944	16.6265	19.4465	29.4585	64.6330
0.9450	23.3388	21.9025	20.8214	29.4800	64.6330
0.9675	39.4830	36.9244	32.8079	31.6923	64.6331
0.9900	128.3195	120.0009	106.5448	95.7376	69.6017

Returning to our example given in Section 3.1, by modeling the dependence structure of two rv's with a survival Gumbel copula, there is a high probability that the two variables are increasing at the same time.

Remark 3.1. The survival Gumbel copula can measure the lower tail dependence instead of the upper tail dependence as compared to Gumbel copula. This is appropriate for analyzing tail dependence structure since it explores all possibilities of copula functions in measuring dependencies. In this case $\lambda_U = \overline{\lambda}_L$, where $\overline{\lambda}_L$ is the upper tail dependence of the survival Gumbel copula. The survival copula also has the same property and dependence range as their original copula functions.

The CCTE of rv's X_i, $i = 1, 2, 3$ in terms of survival Gumbel copula is given by

$$\mathbb{CCTE}_{X_i}(t) = \frac{1}{C^G_{\theta_i}(\alpha, t)} \Bigg(\frac{\gamma(1-\alpha)^{1-1/\gamma}}{\gamma - 1} - \int_\alpha^1 u^{-1}(1-u)^{-1/\gamma}(-\ln u)^{\theta_i - 1}\overline{C}^G_{\theta_i}(u,t) \left(-\ln\left(\overline{C}^G_{\theta_i}(u,t)\right)\right)^{1-\theta_i} du \Bigg).$$

Note that we have modeled the joint df with the survival Gumbel copula instead of the Gumbel copula and we compare with the Gumbel copula (the previous example). So the

Table 3.6
Risk measures of dependent Pareto (1.5) rv's with FGM copula.

α	0.9000	0.9225	0.9450	0.9675	0.9900
$VaR_{X_i}(\alpha)$	4.6415	5.5013	6.9144	9.8192	21.5443
$\mathbb{CTE}_{X_i}(\alpha)$	13.9247	16.5039	20.7433	29.4577	64.6330
t	$\mathbb{CCTE}_{X_1}(t)$, $\theta = 1.01$				
0.9000	0.1786	0.1603	0.1398	0.1149	0.0761
0.9225	0.1354	0.1215	0.1060	0.0871	0.0577
0.9450	0.0941	0.0844	0.0737	0.0605	0.0401
0.9675	0.0545	0.0489	0.0427	0.0351	0.0233
0.9900	0.0165	0.0148	0.0129	0.0106	0.0070
t	$\mathbb{CCTE}_{X_2}(t)$, $\theta = 2$				
0.9000	0.8301	0.7791	0.7177	0.6331	0.4695
0.9225	0.7173	0.6749	0.6241	0.5543	0.4175
0.9450	0.5891	0.5561	0.5167	0.4627	0.3558
0.9675	0.4330	0.4109	0.3845	0.3483	0.2758
0.9900	0.2099	0.2013	0.1911	0.1773	0.1486
t	$\mathbb{CCTE}_{X_3}(t)$, $\theta = 10$				
0.9000	1.3070	1.2244	1.0989	0.9062	0.5632
0.9225	1.2099	1.1465	1.0501	0.8791	0.5492
0.9450	1.0710	1.0318	0.9683	0.8410	0.5352
0.9675	0.8683	0.8451	0.8162	0.7535	0.5173
0.9900	0.5186	0.5059	0.4936	0.4805	0.4269

comparison will be the contrast (recall Remark 3.1), that means, the small value gives more riskiness.

Table 3.6 shows that all $\mathbb{CCTE}_{X_i}(t) < \mathbb{CTE}_{X_i}(\alpha)$ for $i = 1, 2, 3$ and $0.9 \leq t \leq 0.99$, in this case we cannot take a decision about the riskiness of the target risk. Nevertheless, we can get an idea of the comparison in this case. So in the survival Gumbel copula model, we have only the lower tail dependence. Now it is natural to consider that the risk thresholds be in $0 < t \leq 0.1$ places of $0.9 \leq t \leq 0.99$, in this case, we obtain the same reasoning given in the case of the Gumbel copula see Table 3.7.

4. Application

The relationships between the copula parameter and Kendall's tau permitted us to compute the θ value assuming a Gumbel, Clayton copula. Once endowed with the parameter value, we are able to compute any joint probability between the stock indices. For instance, we analyzed 500 observations from four European stock indices return series calculated by $\log(X_{t+1}/X_t)$ for the period 1991 to November 1992 (see, Fig. 4.4), available in "QRM and data sets packages" of the R software, it contains the daily closing prices of major European stock indices: Germany DAX (Ibis), Switzerland SMI, France CAC and UK FTSE. The data are sampled in business time, i.e., weekends and holidays are omitted. Table 4.8 summaries the Kendall's tau between the four Market Index returns.

The Lévy-stable distribution offers a reasonable improvement to the alternative distributions, each stable distribution $S_\gamma(\sigma; \beta; \mu)$ has the stability index γ that can be treated as the main parameter, when we make an investment decision, skewness parameter β, in the

Table 3.7
Risk measures of PQD Pareto (1.5) rv's with Gumbel copula.

α	0.9000	0.9225	0.9450	0.9675	0.9900
$VaR_{X_i}(\alpha)$	4.6415	5.5013	6.9144	9.8192	21.5443
$\mathbb{CTE}_{X_i}(\alpha)$	13.9247	16.5039	20.7433	29.4577	64.6330
t	$\mathbb{CCTE}_{X_1}(t)$, $\theta = 0.01$				
0.9000	15.9370	18.8793	23.6990	33.5569	72.9927
0.9225	16.4850	19.5288	24.5076	34.6672	75.1339
0.9450	17.4102	20.6250	25.8737	36.5349	78.6453
0.9675	19.3659	22.9487	28.7606	40.4546	85.7265
0.9900	25.0078	33.6905	40.5881	56.2757	112.1868
t	$\mathbb{CCTE}_{X_2}(t)$, $\theta = 2$				
0.9000	18.1581	20.2092	23.8421	31.8490	66.0876
0.9225	19.7693	21.6536	25.0597	32.7667	66.6063
0.9450	22.6911	24.3385	27.3837	34.5437	67.5834
0.9675	28.9506	30.6075	33.0707	39.1284	70.0747
0.9900	52.9293	53.7426	55.2768	59.2078	86.3853
t	$\mathbb{CCTE}_{X_3}(t)$, $\theta = 10$				
0.9000	13.7652	15.6122	19.3784	29.4577	64.6330
0.9225	16.6944	16.6265	19.4465	29.4585	64.6330
0.9450	23.3388	21.9025	20.8214	29.4800	64.6330
0.9675	39.4830	36.9244	32.8079	31.6923	64.6331
0.9900	128.3195	120.0009	106.5448	95.7376	69.6017

Table 4.8
Kendall's tau matrix estimates from four European stock indices returns.

Variable	DAX	SMI	CAC	FTSE
DAX	1.0000	0.4087	0.3695	0.2913
SMI	0.4087	1.0000	0.3547	0.4075
CAC	0.3695	0.3547	1.0000	0.3670
FTSE	0.2913	0.4075	0.3670	1.0000

Table 4.9
Maximum likelihood fit of four-parameters stable distribution to four European stock indices returns data.

	DAX	SMI	CAC	FTSE
γ	1.6420	1.8480	1.6930	1.8740
β	0.1470	0.1100	−0.0380	0.9500
σ	0.0046	0.0046	0.0062	0.0054
μ	−0.0002	0.0006	0.0004	−0.0005

range $[-1, 1]$, scale parameter σ and shift parameter μ. In models that use financial data, it is generally assumed that $\gamma \in (1, 2]$. By using the "fBasics" package in R software, based on the maximum likelihood estimators to fit the parameters of a df of the four Market Index returns, the results are summarized in Table 4.9.

The Lévy-stable distribution has Pareto-type tails, it is like a power function, i.e., F is regularly varying (at infinity) with index $(-\gamma)$, meaning that $\overline{F}(x) = x^{-\gamma} L(x)$ as x

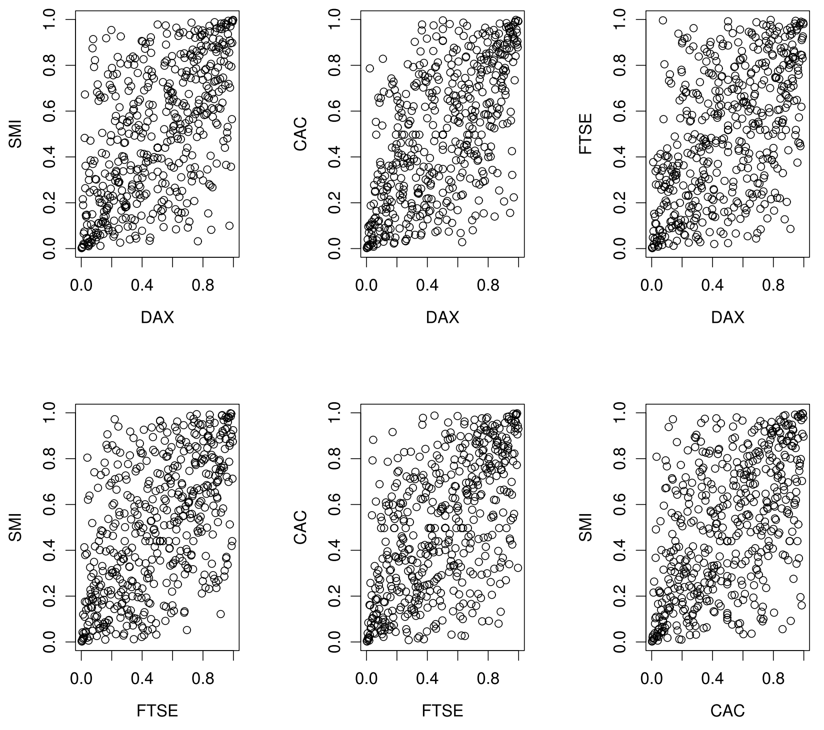

Fig. 4.4. Scatterplots of 500 pseudo-observations drawn from four European stock indices returns.

becomes large, where $L > 0$ is a slowly varying function, which can be interpreted as slower than any power function (see, [35] and [36] for a technical treatment of regular variation).

In Table 4.10 we show the results of bivariate goodness of fit tests (see, [21]) for four different copula families, two elliptic: the Gaussian and the Student t with 1 degree of freedom, and two Archimedean: Clayton and Gumbel copulas. For each of the previous cases, the copulas are reflection symmetric only in two dimensions. All the copula simulations are obtained by the use of the copula R package.

For the majority of the pairs compared with the goodness of fit test are rejected in Gumbel and Clayton copulas cases and accepted by Gaussian copula and t-copula, if one compares with the greatest p-value which close to 1 we choose the t-copula.

Next, we consider the four Market Index returns fitted by t-copula given by

$$C_{\rho,\upsilon}(u,v) = t_{\rho,\upsilon}\left(t_\upsilon^{-1}(u), t_\upsilon^{-1}(v)\right)$$

where υ is the degree-of-freedom parameter, t_υ^{-1} is the inverse of the univariate standard Student-t df, and $t_{\rho,\upsilon}$ is the bivariate standard Student-t distribution parametrized by the correlation parameterρ and υ. The density of the bivariate t-copula is given by

$$c_{\rho,\upsilon}(u,v) = \frac{\upsilon}{2\sqrt{1-\rho^2}} \frac{\Gamma(\upsilon/2)^2}{\Gamma((\upsilon+1)/2)^2} \frac{\left(1+\frac{x^2+y^2-2\rho xy}{\upsilon(1-\rho^2)}\right)^{-(\upsilon+2)/2}}{\left(\left(1+\frac{x^2}{\upsilon}\right)\left(1+\frac{y^2}{\upsilon}\right)\right)^{-(\upsilon+1)/2}},$$

where $x = t_\upsilon^{-1}(u)$, $y = t_\upsilon^{-1}(v)$ and Γ is the Gamma function.

Table 4.10
p-value of bootstrap-based goodness-of-fit test of Gumbel, Clayton, Gaussian and t copula of dimension 2, with 'method' = "Sn", 'estim.method' = "itau".

Variable	SMI	FTSE	CAC	Copula
DAX	0.0019	0.0015	0.0005	Gumbel
	0.0004	0.0005	0.0004	Clayton
	0.1543	**0.2572**	**0.2662**	Gaussian
	0.4381	**0.2942**	**0.3302**	t
SMI	–	0.0004	0.0004	Gumbel
	–	0.0004	0.0004	Clayton
	–	**0.4071**	**0.2283**	Gaussian
	–	**0.3390**	**0.5220**	t
FTSE	–	–	0.0394	Gumbel
	–	–	0.0004	Clayton
	–	–	**0.3941**	Gaussian
	–	–	**0.5230**	t

Table 4.11
Fitted t-copula parameter ρ corresponding to Kendall's tau and $\upsilon = 1$.

Variable	DAX	SMI	CAC	FTSE
DAX	∞	0.5945	0.6344	0.5498
SMI	0.5945	∞	0.5610	0.5781
CAC	0.6344	0.5610	∞	0.5974
FTSE	0.5498	0.5781	0.5974	∞

Table 4.12
CCTE's Risk measures for $\alpha = 0.9$ and $t = 0.9$ with t-copula.

Variable	DAX	SMI	CAC	FTSE
DAX	–	0.617	0.677	0.666
SMI	0.617	–	0.842	0.624
CAC	0.677	0.842	–	0.590
FTSE	0.666	0.624	0.590	–

By assuming that t-copula represents our four dependences structure, we obtain the fitted dependence parameters of the six bivariate joint dfs, presented in Table 4.11.

By using Eqs. (2.8) with t-copula, we calculate for a fixed level $\alpha = t = 0.9$ the CCTE's risk measures for all cases, the results are summarized in Table 4.12.

In Table 4.12, the smallest value gives the lowest risk. So, the less risky couple (X, Y) is: (CAC, FTST), where X is the target risk and Y is the associated risk.

5. Conclusion notes

For a good investment it is better to divide the capital of investment in more than one market, but the most important question is that if these markets are linked and if one of them collapses, does the rest of the interrelated market collapse as well?

Tables 3.1, 3.3 and 3.5 show that the CCTEs become larger if dependency increases. However, CTE and VaR are neither increasing nor decreasing as the correlation increases.

Therefore, to reduce the risk, in preference for this market to be independent, or preferably for the investors to choose the independent markets or the less dependent one to invest their money.

In this paper, we give a new risk measure called copula conditional tail expectation which preserves the property of coherence. This measure is apt to understand the relationships among multivariate assets and to help us greatly about how best to position our investments and enhance our financial risk protection.

Acknowledgment

The authors want to thank the anonymous reviewer for the helpful comments and suggestions used to improve this paper.

Appendix

Proof of Proposition 2.1. By calculating we have

$$\begin{aligned}
&\mathbb{P}\left(X_1 \le x | X_1 \ge VaR_{X_1}(s), X_2 \ge VaR_{X_2}(t)\right) \\
&= \frac{\mathbb{P}\left(X_1 \le x, X_1 > VaR_{X_1}(s), X_2 > VaR_{X_2}(t)\right)}{\mathbb{P}\left(X_1 > VaR_{X_1}(s), X_2 > VaR_{X_2}(t)\right)} \\
&= \frac{\mathbb{P}\left(VaR_{X_1}(s) < X_1 \le x, X_2 \ge VaR_{X_2}(t)\right)}{\mathbb{P}\left(X_1 > VaR_{X_1}(s), X_2 > VaR_{X_2}(t)\right)} \\
&= \frac{\mathbb{P}\left(VaR_{X_1}(s) < X_1 \le x, X_2 \ge VaR_{X_2}(t)\right)}{1 - \mathbb{P}\{X_1 \le F_{X_1}^{-1}(s)\} - \mathbb{P}\{X_2 \le F_{X_2}^{-1}(t)\} + \mathbb{P}\{X_1 \le F_{X_1}^{-1}(s), X_2 \le F_{X_2}^{-1}(t)\}} \\
&= \frac{\mathbb{P}\left(VaR_X(s) < X \le x, Y \ge VaR_Y(t)\right)}{1 - \mathbb{P}\{F_{X_1}(X) \le s\} - \mathbb{P}\{F_{X_2}(X_2) \le t\} + \mathbb{P}\{F_{X_1}(X_1) \le s, F_{X_2}(X_2) \le t\}}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
1 - \mathbb{P}\{F_{X_1}(X) \le s\} - \mathbb{P}\{F_{X_2}(X_2) \le t\} + \mathbb{P}\{F_{X_1}(X_1) \le s, F_{X_2}(X_2) \le t\} \\
= 1 - s - t + C(s,t) \\
= \overline{C}(1-s, 1-t),
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{P}\left(X_1 \le x | X_1 \ge VaR_{X_1}(s), X_2 \ge VaR_{X_2}(t)\right) \\
&= \frac{1}{\overline{C}(1-s, 1-t)} \int_{VaR_{X_2}(t)}^{\infty} \int_{VaR_{X_1}(s)}^{x} \frac{\partial^2 C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right)}{\partial x_1 \partial x_2} dx_1 dx_2.
\end{aligned}$$

Then for a fixed level $s = \alpha$, the CCTE is given by

$$\begin{aligned}
\mathbb{CCTE}_{X_1}(t) = &\frac{1}{\overline{C}(1-\alpha, 1-t)} \int_{VaR_{X_1}(\alpha)}^{\infty} \int_{VaR_{X_2}(t)}^{\infty} x_1 \frac{\partial^2 C\left(F_{X_1}(x_1), F_{X_2}(x_2)\right)}{\partial x_1 \partial x_2} \\
&\times dx_1 dx_2.
\end{aligned}$$

We suppose that the densities of F_{X_i}, $i = 1, 2$ are f_{X_i}, respectively, then

$$\mathbb{CCTE}_{X_1}(t) = \frac{1}{\overline{C}(1-\alpha, 1-t)} \int_{VaR_{X_1}(\alpha)}^{\infty} \int_{VaR_{X_2}(t)}^{\infty} x_1 c\left(F_{X_1}(x_1), F_{X_2}(x_2)\right)$$

$$\times f_{X_1}(x_1) f_{X_2}(x_2) \, dx_1 dx_2.$$

Transforming by $F_{X_1}(x_1) = u$ and $F_{X_2}(x_2) = v$, we get

$$\begin{aligned}\mathbb{CCTE}_{X_1}(t) &= \frac{1}{\overline{C}(1-\alpha, 1-t)} \int_t^1 \int_\alpha^1 F_{X_1}^{-1}(u) \, c(u, v) \, du dv. \\ &= \frac{1}{\overline{C}(1-\alpha, 1-t)} \int_\alpha^1 F_{X_1}^{-1}(u) \left(\int_t^1 c(u, v) \, dv \right) du.\end{aligned}$$

By (2.6) it follow that

$$\mathbb{CCTE}_{X_1}(t) = \frac{\int_\alpha^1 J_t(u) F_{X_1}^{-1}(u) \, du}{\int_\alpha^1 J_t(u) \, du}.$$

This close the proof of Proposition 2.1. □

Proof of Corollary 3.1. Let us denote by

$$C_u(u, v) := \frac{\partial C(u, v)}{\partial u}$$

then by (2.6), we have

$$\begin{aligned}J_t(u) &= \int_t^1 c(u, v) \, dv = C_u(u, v)]_t^1 \\ &= C_u(u, 1) - C_u(u, t).\end{aligned}$$

So, C is Archimedean copula, then

$$C_u(u, v) = \frac{\psi'(u)}{\psi'(C(u, v))}.$$

Finally, we get (3.18) by the property of copula that is $C(u, 1) = u$. □

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