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Classification of derivation algebras in low dimensions

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Abstract. A left-symmetric algebra A is said to be a derivation algebra if all its left and right multiplications are derivations of the Lie algebra associated to A. In this paper, we shall study derivation algebras over $\mathbb{R}$ and then classify almost all those of dimensions ≤ 4.

Keywords: Extensions of left-symmetric algebras; Left-invariant affine connections; Novikov algebras; Derivation algebras

Mathematics Subject Classification: 17A30; 17B30; 53C05

1. Introduction

The notion of a left-symmetric algebra appeared for the first time in the work of Koszul [10] and Vinberg [14] concerning bounded homogeneous domains and convex homogeneous cones, respectively. A *left-symmetric algebra* A over a field $\mathbb{F}$ is a finite-dimensional vector space over $\mathbb{F}$ together with a product, said to be left-symmetric and denoted by $x \cdot y$, which satisfies the following condition

$$(x \cdot y) \cdot z - (y \cdot x) \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z), \quad \text{for all } x, y, z \in A. \tag{1}$$

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In this case, the commutator

$$[x, y] = x \cdot y - y \cdot x, \tag{2}$$

defines a structure of Lie algebra on A, called *the associated* (or *the sub-adjacent*) *Lie algebra*. On the other hand, if $\mathcal{G}$ is a Lie algebra with a left-symmetric product $\cdot$ satisfying (2), then we say that this left-symmetric structure is compatible with the Lie structure of $\mathcal{G}$. Throughout this paper, we will restrict to the case $\mathbb{F} = \mathbb{R}$.

If A is a left-symmetric algebra whose associated Lie algebra is $\mathcal{G}$, and if L_x and R_x denote the left and right multiplications, respectively, then (1) can be stated in terms of L_x as follows

$$\left[L_x, L_y\right] = L_{[x,y]}, \quad \text{for all } x, y \in A,$$

or, in other words, the linear map $L : \mathcal{G} \to End\,(A)$ is a representation of Lie algebras.

On the other hand, if G is a simply connected Lie group with a left-invariant affine connection ∇, then define a product $\cdot$ on the Lie algebra $\mathcal{G}$ of G by

$$x \cdot y = \nabla_x y,$$

for all $x, y \in \mathcal{G}$. It is clear that the flatness and the torsion-freeness of ∇ correspond to the conditions (1) and (2), respectively. Conversely, If G is a simply connected Lie group with Lie algebra $\mathcal{G}$ and $x \cdot y$ denotes a left-symmetric product on $\mathcal{G}$ compatible with the Lie structure, then the left-invariant connection given by $\nabla_x y = x \cdot y$ defines a left-invariant affine connection ∇ on G. In other words, for a given simply connected Lie group G with Lie algebra $\mathcal{G}$, there is a one-to-one correspondence between left-invariant affine connections on G and compatible left-symmetric products on $\mathcal{G}$.

Before passing to the next section, we briefly recall some basic definitions. Let A be a left-symmetric algebra A. We say that A is simple if it is non-zero and has no proper two-sided ideal. A is semisimple if it is a direct sum of simple left-symmetric algebras. We say that A is *complete* (or *transitive*) if R_x is a nilpotent operator for all $x \in A$. It turns out that, for a given simply connected Lie group G with Lie algebra $\mathcal{G}$, the complete left-invariant affine structures on G are in one-to-one correspondence with the complete left-symmetric structures on $\mathcal{G}$ compatible with the Lie structure. It is also known that an n-dimensional simply connected Lie group admits a complete left-invariant affine structure if and only if it acts simply transitively on $\mathbb{R}^n$ by affine transformations [9]. A simply connected Lie group which is acting simply transitively on $\mathbb{R}^n$ by affine transformations must be solvable [1]. It is well known that not every solvable (even nilpotent) Lie group can admit an affine structure [2].

In this paper, we are concerned with a special class of left-invariant affine connections on real Lie groups considered first in [11] and called adapted to the automorphism structure of the Lie group. Namely, if ∇ is a left-invariant affine connection a given a Lie group G with Lie algebra $\mathcal{G}$, we say that ∇ is adapted to the automorphism structure of G if and only if the linear mapping $\theta : \mathcal{G} \to End\,(\mathcal{G})$ defined by $\theta\,(x) = \nabla_x$ takes its values in the algebra $Der\,(\mathcal{G})$ of all the derivations of $\mathcal{G}$. In terms of left-symmetric structures, we are concerned with the class of finite-dimensional real left-symmetric algebras A satisfying the identity

$$(x \cdot y) \cdot z = (z \cdot y) \cdot x, \quad \text{for all } x, y, z \in A. \tag{3}$$

A left-symmetric algebra A with the associated Lie algebra $\mathcal{G}$ is called a *derivation algebra* if it obeys the identity (3). This is also equivalent to say that left and right multiplications L_x and R_x are derivations of $\mathcal{G}$.

The paper is organized as follows. In Section 2, we will briefly recall some necessary definitions and basic results on left-symmetric algebras and their extensions. In Section 3, we will determine all the commutative associative real algebras of dimension ≤ 4. In Section 4, we will provide some interesting results concerning derivation algebras, and we will discuss the relationships of derivation algebras with other special left-symmetric algebras such as Novikov algebras and left-symmetric algebras satisfying the condition $[x, y] \cdot z = 0$. We will also give the classification of derivation algebras of dimension ≤ 3. In Section 5, using the extensions of left-symmetric algebras, we will classify almost all derivation algebras over $\mathbb{R}$ of dimension 4.

2. Extensions of left-symmetric algebras

To our knowledge, the notion of extensions of left-symmetric algebras has been considered for the first time in [9]. Suppose that a vector space extension $\widetilde{A}$ of a left-symmetric algebra A by another left-symmetric algebra E is given. We want to define a left-symmetric structure on $\widetilde{A}$ in terms of the left-symmetric structures given on A and E. In other words, we want to define a left-symmetric product on $\widetilde{A}$ for which E becomes a two-sided ideal in $\widetilde{A}$ such that $\widetilde{A}/E \cong A$; or equivalently,

$$0 \to E \to \widetilde{A} \to A \to 0$$

becomes a short exact sequence of left-symmetric algebras.

Theorem 1 (*[9]*). *There exists a left-symmetric structure on $\widetilde{A}$ extending a left-symmetric algebra A by a left-symmetric algebra E if and only if there exist two linear maps $\lambda, \rho : A \to End(E)$ and a bilinear map $g : A \times A \to E$ such that, for all $x, y, z \in A$ and $a, b \in E$, the following conditions are satisfied.*

1. $\lambda_x (a \cdot b) = \lambda_x(a) \cdot b + a \cdot \lambda_x(b) - \rho_x(a) \cdot b,$
2. $\rho_x ([a, b]) = a \cdot \rho_x(b) - b \cdot \rho_x(a),$
3. $[\lambda_x, \lambda_y] - \lambda_{[x,y]} = L_{g(x,y)-g(y,x)},$
4. $[\lambda_x, \rho_y]+ \ \rho_y \circ \rho_x - \rho_{x\cdot y} = R_{g(x,y)}$
5. $g (x, y \cdot z) - g (y, x \cdot z) + \lambda_x (g (y, z)) - \lambda_y (g (x, z)) - g ([x, y] , z) \ -\rho_z\big(g (x, y) - g (y, x)\big) = 0.$

If the conditions of Theorem 1 are fulfilled, then the extended left-symmetric product on $\widetilde{A} \cong A \times E$ is given by

$$(x, a) \cdot (y, b) = \big(x \cdot y, a \cdot b + \lambda_x (b) + \rho_y (a) + g (x, y)\big) . \tag{4}$$

Let K be a left-symmetric algebra, and suppose that a K-bimodule V is known. We denote by $L^p (K, V)$ the space of all p-linear maps from K to V, and we define two coboundary operators $\delta_1 : L^1 (K, V) \to L^2 (K, V)$ and $\delta_2 : L^2 (K, V) \to L^3 (K, V)$ as follows:

For a linear map $h \in L^1 (K, V)$ we set

$$\delta_1 h (x, y) = \rho_y (h (x)) + \lambda_x (h (y)) - h (x \cdot y) , \tag{5}$$

and for a bilinear map $g \in L^2(K, V)$ we set

$$\delta_2 g(x, y, z) = g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y, z)) - \lambda_y(g(x, z)) \\ - g([x, y], z) - \rho_z(g(x, y) - g(y, x)) \quad (6)$$

where λ and ρ are linear maps $\lambda, \rho : K \to End(V)$.

It is straightforward to check that $\delta_2 \circ \delta_1 = 0$. Therefore, if we set $Z^2_{\lambda,\rho}(K, V) = \ker \delta_2$ and $B^2_{\lambda,\rho}(K, V) = \operatorname{Im} \delta_1$, we can define a notion of second cohomology for the actions λ and ρ by simply setting $H^2_{\lambda,\rho}(K, V) = Z^2_{\lambda,\rho}(K, V) / B^2_{\lambda,\rho}(K, V)$. As in the case of Lie algebras, we can prove the following (see [9]).

Proposition 2. *For given linear maps $\lambda, \rho : K \to End(V)$, the equivalent classes of extensions*

$$0 \to V \to A \to K \to 0$$

of K by V are in one-to-one correspondence with the elements of the second cohomology group $H^2_{\lambda,\rho}(K, V)$.

A left-symmetric algebras extension

$$0 \to E \overset{i}{\to} \widetilde{A} \overset{\pi}{\to} A \to 0$$

is called central if and only if $i(E) \subseteq C(\widetilde{A})$ where

$$C(\widetilde{A}) = \left\{x \in \widetilde{A} : x \cdot y = y \cdot x = 0\right\}$$

is the center of $\widetilde{A}$. In particular, the extension is central whenever E is a trivial A-bimodule (i.e., $\lambda = \rho = 0$).

We say that the extension is exact if and only if $i(E) = C(\widetilde{A})$. It is easy to verify (see [9]) that the extension is exact if and only if $I_{[g]} = 0$, where

$$I_{[g]} = \{x \in A : x \cdot y = y \cdot x = 0 \text{ and } g(x, y) = g(y, x) = 0 \text{ for all } y \in A\}.$$

We observe that $I_{[g]}$ is depends only on the cohomology class of g, that is $I_{[g]}$ is well defined.

In case E is a trivial A-bimodule, we denote the central extension corresponding to the class $[g] \in H^2(A, E)$ by $(\widetilde{A}, [g])$.

Let $(\widetilde{A}, [g])$ and $\left(\widetilde{A}', \left[g'\right]\right)$ be two central extensions of A by E, and $\mu \in Aut(E) = GL(E)$ and $\eta \in Aut(A)$, where $Aut(E)$ and $Aut(A)$ are the groups of left-symmetric automorphisms of E and K, respectively. It is clear that if, $h \in L^1(A, E)$, then the linear mapping $\psi : \widetilde{A} \to \widetilde{A}'$ defined by $\psi(x, a) = (\eta(x), \mu(a) + h(x))$ is an isomorphism provided $g'(\eta(x), \eta(y)) = \mu(g(x, y)) + \delta_1 h(x, y)$ for all $(x, y) \in A \times A$, i.e., $\eta^* \left[g'\right] = \mu_*[g]$.

This allows us to define an action of the group $G = Aut(E) \times Aut(A)$ on $H^2(A, E)$ by setting

$$(\mu, \eta) \cdot [g] = \mu_* \eta^* [g]$$

or equivalently, $(\mu, \eta) \cdot g(x, y) = \mu(g(\eta(x), \eta(y)))$ for all $x, y \in A$.

Denoting the set of all exact central extensions of A by E by

$$H^2_{ex}(A, E) = \left\{[g] \in H^2(A, E) : I_{[g]} = 0\right\}$$

and the orbit of $[g]$ by $G_{[g]}$, it turns out that the following result is valid (see [9]).

Proposition 3. *Let* $[g]$ *and* $\left[g'\right]$ *be two classes in* $H_{ex}^2(A, E)$. *Then, the central extensions* $\left(\widetilde{A}, [g]\right)$ *and* $\left(\widetilde{A}', \left[g'\right]\right)$ *are isomorphic if and only if* $G_{[g]} = G_{\left[g'\right]}$. *In other words, the classification of the exact central extensions of* A *by* E *is, up to left-symmetric isomorphism, the orbit space of* $H_{ex}^2(A, E)$ *under the natural action of* $G = Aut(E) \times Aut(A)$.

We close this section with the following useful result (see [5]).

Proposition 4. *Let* $0 \to I \to A \to J \to 0$ *be an exact sequence of left-symmetric algebras. Then,* A *is complete if and only if* I *and* J *are both complete.*

3. Commutative associative algebras

It is easy to prove that a left-symmetric algebra A is commutative (i.e., $x \cdot y = y \cdot x$, for all $x, y \in A$) if and only if its associated Lie algebra is abelian. In this case, A is necessarily associative.

We will determine all the commutative associative real algebras of dimension ≤ 4. First, we note that, up to an isomorphism, there are exactly two commutative real algebras of dimension 1: The trivial algebra in which $x \cdot y = 0$ for all $x, y \in A$, and the field $\mathbb{R}$ itself. We will denote them by $\mathbb{R}_0$ and $\mathbb{R}_1$, respectively. Note here that we will often write $\mathbb{R}_1 = \mathbb{R}e_0$, with $e_0 \cdot e_0 = e_0$.

In dimensions 2 and 3, commutative associative real algebras are described by the following lemmas that we state without proofs (see [13], Proposition 4.1 and Theorem 5.2).

Lemma 5. *Every two-dimensional commutative associative algebra* A *over* $\mathbb{R}$ *is isomorphic to one of the following algebras*

$A_{2,1} : e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2,$
$A_{2,2} : e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2 \cdot e_1 = e_2,$
$A_{2,3} : e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, e_2 \cdot e_2 = -e_1,$
$A_{2,4} : e_1 \cdot e_1 = e_1,$
$A_{2,5} : e_1 \cdot e_1 = e_2,$
$A_{2,6} : e_i \cdot e_j = 0,\ i, j = 1, 2.$

If A *is complete, then* A *is isomorphic to* $A_{2,i}$, $i = 5, 6$.

Lemma 6. *Every three-dimensional commutative algebra* A *over* $\mathbb{R}$ *is isomorphic to one of the following algebras*

$A_{3,1} : e_1 \cdot e_i = e_i \cdot e_1 = e_i,\ e_2 \cdot e_2 = e_2, e_3 \cdot e_3 = e_3, i = 1, 2, 3,$
$A_{3,2} : e_1 \cdot e_i = e_i \cdot e_1 = e_i,\ e_2 \cdot e_2 = e_2, e_3 \cdot e_3 = e_2 - e_1,\ i = 1, 2, 3,$
$A_{3,3} : e_1 \cdot e_i = e_i \cdot e_1 = e_i,\ e_2 \cdot e_2 = e_2,\ i = 1, 2, 3,$
$A_{3,4} : e_1 \cdot e_i = e_i \cdot e_1 = e_i,\ e_3 \cdot e_3 = e_2,\ i = 1, 2, 3,$
$A_{3,5} : e_1 \cdot e_i = e_i \cdot e_1 = e_i,\ i = 1, 2, 3,$
$A_{3,6} : e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2,$
$A_{3,7} : e_1 \cdot e_1 = e_1,\ e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, e_2 \cdot e_2 = -e_1,$
$A_{3,8} : e_1 \cdot e_1 = e_1,$
$A_{3,9} : e_1 \cdot e_1 = e_1,\ e_1 \cdot e_2 = e_2 \cdot e_1 = e_2,$
$A_{3,10} : e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_3,$

$A_{3,11} : e_1 \cdot e_1 = e_2, e_3 \cdot e_3 = e_2,$
$A_{3,12} : e_1 \cdot e_1 = e_2, e_3 \cdot e_3 = -e_2,$
$A_{3,13} : e_1 \cdot e_1 = e_2,$
$A_{3,14} : e_1 \cdot e_1 = e_2,\ e_1 \cdot e_2 = e_2 \cdot e_1 = e_3,$
$A_{3,15} : e_i \cdot e_j = 0,\ i, j = 1, 2, 3.$

If A is complete, then A is isomorphic to $A_{3,i}, i = 11, \ldots\ldots, 15$.

In dimension 4, the classification of complete commutative associative real algebras was given in [9]. To classify the incomplete commutative associative real algebras of dimension 4, we proceed according to the following lemma that we quote from [3].

Lemma 7. *Let A be a commutative associative algebra over $\mathbb{R}$. Suppose that A is not a direct sum of proper non-trivial ideals. Then A is either*

- *complete, or*
- *equal to $\widetilde{B}$ for a complete B, where $\widetilde{B} = B \oplus \langle 1 \rangle$ with $b \cdot 1 = 1 \cdot b = b$ for all $b \in B$, or*
- *equal to $A_{2,3} \oplus \mathcal{R}(A)$, where $A_{2,3}$ is the algebra given in Lemma 5 and $\mathcal{R}(A)$ is the radical of A.*

With this lemma in hand, and based on the classification of commutative associative real algebras of dimension 3, we obtain the following result.

Proposition 8. *Every four-dimensional commutative associative algebra A over $\mathbb{R}$ is isomorphic to one of the following algebras*

$A_{4,1} = \widetilde{A_{3,11}} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_2 \cdot e_2 = e_3 \cdot e_3 = e_4, \quad 1 \le i \le 4,$
$A_{4,2} = \widetilde{A_{3,12}} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_2 \cdot e_2 = e_4,\ e_3 \cdot e_3 = -e_4, \quad 1 \le i \le 4,$
$A_{4,3} = \widetilde{A_{3,13}} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_2 \cdot e_2 = e_3,\ 1 \le i \le 4,$
$A_{4,4} = \widetilde{A_{3,14}} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_2 \cdot e_2 = e_3,\ e_2 \cdot e_3 = e_3 \cdot e_2 = e_4, \quad 1 \le i \le 4,$
$A_{4,5} = \widetilde{A_{3,15}} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, \quad 1 \le i \le 4,$
$A_{4,6} = A_{3,1} \oplus \mathbb{R}_0 : e_i \cdot e_i = e_i, \quad 1 \le i \le 3,$
$A_{4,7} = A_{3,2} \oplus \mathbb{R}_0 : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_2 \cdot e_2 = -e_1,\ e_3 \cdot e_3 = e_3, \quad 1 \le i \le 2,$
$A_{4,8} = A_{3,3} \oplus \mathbb{R}_0 : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_3 \cdot e_3 = e_3, \quad 1 \le i \le 2,$
$A_{4,9} = A_{3,4} \oplus \mathbb{R}_0 : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_3 \cdot e_3 = e_2, \quad 1 \le i \le 3,$
$A_{4,10} = A_{3,5} \oplus \mathbb{R}_0 : e_1 \cdot e_i = e_i \cdot e_1 = e_i, \quad 1 \le i \le 3,$
$A_{4,11} = A_{3,6} \oplus \mathbb{R}_0 : e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2,$
$A_{4,12} = A_{3,7} \oplus \mathbb{R}_0 : e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, e_2 \cdot e_2 = -e_1,$
$A_{4,13} = A_{3,8} \oplus \mathbb{R}_0 : e_1 \cdot e_1 = e_1,$
$A_{4,14} = A_{3,9} \oplus \mathbb{R}_0 : e_1 \cdot e_i = e_i \cdot e_1 = e_i, \quad 1 \le i \le 2,$
$A_{4,15} = A_{3,10} \oplus \mathbb{R}_0 : e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_3,$
$A_{4,16} = A_{2,1} \oplus A_{2,5} : e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2, e_3 \cdot e_3 = e_4,$
$A_{4,17} = A_{2,2} \oplus A_{2,5} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_3 \cdot e_3 = e_4, \quad 1 \le i \le 2,$
$A_{4,18} = A_{2,3} \oplus A_{2,5} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_2 \cdot e_2 = -e_1,\ e_3 \cdot e_3 = e_4, \quad 1 \le i \le 2,$
$A_{4,19} = A_{3,1} \oplus \mathbb{R}_1 : e_i \cdot e_i = e_i, \quad 1 \le i \le 4,$
$A_{4,20} = A_{3,4} \oplus \mathbb{R}_1 : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_3 \cdot e_3 = e_2, e_4 \cdot e_4 = e_4, \quad 1 \le i \le 3,$
$A_{4,21} = A_{3,5} \oplus \mathbb{R}_1 : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_4 \cdot e_4 = e_4, \quad 1 \le i \le 3,$
$A_{4,22} = A_{3,11} \oplus \mathbb{R}_1 : e_1 \cdot e_1 = e_2, e_3 \cdot e_3 = e_2, e_4 \cdot e_4 = e_4,$

$A_{4,23} = A_{3,12} \oplus \mathbb{R}_1 : e_1 \cdot e_1 = e_2, e_3 \cdot e_3 = -e_2, e_4 \cdot e_4 = e_4,$
$A_{4,24} = A_{3,14} \oplus \mathbb{R}_1 : e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = e_2 \cdot e_1 = e_3, e_4 \cdot e_4 = e_4,$
$A_{4,25} = A_{2,2} \oplus A_{2,1} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_3 \cdot e_3 = e_3, e_4 \cdot e_4 = e_4, \quad 1 \leq i \leq 2,$
$A_{4,26} = A_{2,3} \oplus A_{2,1} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_2 \cdot e_2 = -e_1, e_3 \cdot e_3 = e_3, e_4 \cdot e_4 = e_4, 1 \leq i \leq 2,$
$A_{4,27} = A_{2,2} \oplus A_{2,2} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_3 \cdot e_3 = e_3, e_3 \cdot e_4 = e_4 \cdot e_3 = e_4, 1 \leq i \leq 2,$
$A_{4,28} = A_{2,2} \oplus A_{2,3} : e_1 \cdot e_i = e_i \cdot e_1 = e_i, e_3 \cdot e_3 = e_3, e_3 \cdot e_4 = e_4 \cdot e_3 = e_4, e_4 \cdot e_4 = -e_3, 1 \leq i \leq 2,$
$A_{4,29} = A_{2,3} \oplus A_{2,3} : e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, e_2 \cdot e_2 = -e_1, e_3 \cdot e_3 = e_3, e_3 \cdot e_4 = e_4 \cdot e_3 = e_4, e_4 \cdot e_4 = -e_3,$
$A_{4,30} : e_1 \cdot e_1 = e_2, e_3 \cdot e_3 = e_2, e_4 \cdot e_4 = e_2,$
$A_{4,31} : e_1 \cdot e_1 = e_2 \; e_3 \cdot e_3 = e_2, e_4 \cdot e_4 = -e_2,$
$A_{4,32} : e_1 \cdot e_1 = e_2, e_3 \cdot e_3 = e_4, e_3 \cdot e_4 = e_4 \cdot e_3 = e_2,$
$A_{4,33} : e_2 \cdot e_4 = e_4 \cdot e_2 = e_1, e_3 \cdot e_3 = e_1, e_3 \cdot e_4 = e_4 \cdot e_3 = e_2, e_4 \cdot e_4 = e_3,$
$A_{4,34} : e_1 \cdot e_1 = e_2, e_3 \cdot e_3 = e_2,$
$A_{4,35} : e_1 \cdot e_1 = e_2, e_3 \cdot e_3 = -e_2,$
$A_{4,36} : e_1 \cdot e_1 = e_2, e_1 \cdot e_3 = e_3 \cdot e_1 = e_4, e_3 \cdot e_3 = -e_2,$
$A_{4,37} : e_1 \cdot e_1 = e_2, e_3 \cdot e_3 = e_4,$
$A_{4,38} : e_1 \cdot e_1 = e_2, e_1 \cdot e_3 = e_3 \cdot e_1 = e_4,$
$A_{4,39} : e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = e_2 \cdot e_1 = e_3,$
$A_{4,40} : e_1 \cdot e_1 = e_2,$
$A_{4,41} : e_i \cdot e_j = 0, \quad i, j = 1, 2, 3, 4.$

If A is complete,then A is isomorphic to $A_{4,i}, i = 30, \ldots, 41$.

4. Derivation algebras

As we mentioned in the introduction, a left-symmetric algebra A with associated Lie algebra $\mathcal{G}$ is called a *derivation algebra* if it obeys identity (3), or equivalently, if left and right multiplications L_x and R_x are derivations of $\mathcal{G}$. This is also equivalent to say that $R_x \circ R_y = L_{yx}$ for all $x, y \in A$. In the particular case when L_x and R_x are inner derivations, we say that A is an *inner derivation algebra* (see [11]).

It turns out that the Lie group of a derivation (resp. inner derivation) algebra admits a left-invariant affine connection adapted to its automorphism (resp. interior automorphism) structure.

If A is an inner derivation algebra, then it is easy to verify that the left-symmetric product is given by

$$x \cdot y = ad_{f(x)} y = [f(x), y], \text{ for all } x, y \in A,$$

where f is an endomorphism of the vector space A satisfying the conditions:

1. $[x, y] = [f(x), y] + [x, f(x)],$
2. $f([x, y]) - [f(x), f(y)] \in \mathcal{Z}(\mathcal{G}).$

Example 9. Every commutative associative algebra is a derivation algebra, which is inner if and only if it is trivial (that is, with zero product).

Example 10. Let $\mathcal{G}$ be a two-step nilpotent Lie algebra. Then, the product

$$x \cdot y = \frac{1}{2}[x, y], \quad \text{for all } x, y \in \mathcal{G},$$

defines an inner derivation structure on $\mathcal{G}$ compatible with the Lie structure of $\mathcal{G}$.

Remark 11. It turns out that a Lie algebra $\mathcal{G}$ which is a semi-direct product of two abelian Lie algebras admits an inner derivation structure that is compatible with the Lie structure of $\mathcal{G}$ (see [11]). This implies that every solvable Lie algebra of dimension ≤ 3 admits a derivation structure. This is not true in dimension 4. The counterexample given in [11] is a three-step solvable Lie algebra of dimension 4 which does not admit any derivation structure.

In [11], it was shown that every two-step solvable Lie algebra with trivial center admits a derivation structure. For example, let $\mathcal{G}$ be the Lie algebra with basis $\{e_0, e_1, e_2, \ldots .e_n\}$ for which the Lie brackets are defined by $[e_0, e_i] = \lambda_i e_i$, $i \geq 1$, where λ_i are pairwise distinct with $\lambda_i \in \mathbb{R} \setminus \{0\}$. This is a two-step solvable with trivial center, and it was shown in [11] that the product given by

$$e_0 \cdot e_0 = \sum_{k=1}^{n} \alpha_k e_k, \;\; \alpha_k \in \mathbb{R}, \quad \text{and } e_0 \cdot e_i = \lambda_i e_i \;\text{ for all } i \geq 1,$$

defines a derivation structure on $\mathcal{G}$ compatible with the Lie structure.

We consider here the set $\mathfrak{G}$ of Lie algebras $\mathcal{G}$ with the following property: For any elements x, y in $\mathcal{G}$ the bracket $[x, y]$ is a linear combination of x and y. It is shown in [12] that this condition is equivalent to the existence of a codimension one abelian ideal E of $\mathcal{G}$ and an element $e_0 \in \mathcal{G} \setminus E$ such that $[e_0, x] = x$ for all $x \in E$. It is clear that $\mathcal{G}$ is a two-step solvable Lie algebra with trivial center, and it is also clear that there exists a basis $\{e_0, e_1, e_2, \ldots .e_n\}$, of $\mathcal{G}$ for which the Lie brackets are defined by $[e_0, e_i] = e_i$, for all $i \geq 1$.

Proposition 12. *Let $\mathcal{G}$ belongs to $\mathfrak{G}$, and let $\{e_0, e_1, e_2, \ldots .e_n\}$ be a basis of $\mathcal{G}$ such that $[e_0, e_i] = e_i$, for all $i \geq 1$. Then, any derivation structure on $\mathcal{G}$ compatible with the Lie structure is isomorphic to one of the following derivation structures:*

1. $e_0 \cdot e_i = e_i$, *for all* $i \geq 1$,
2. $e_0 \cdot e_i = e_i + \sum_{k=1, k \neq i}^{n} \alpha_k e_k, \;\; e_i \cdot e_0 = \sum_{k=1, k \neq i}^{n} \alpha_k e_k$.

In [11], it has been announced that the Lie algebra associated to a derivation algebra is necessarily solvable. Afterwards, we give a proof for this result. For, we need first to state the following result (see [11]). Recall here that given a left-symmetric algebra A, we consider the set

$$T(A) = \{x \in A : x \cdot y = 0, \quad \text{for all } y \in A\}.$$

Since $T(A)$ is both a right ideal of A and a Lie ideal of the associated Lie algebra $\mathcal{G}_A$, it follows that $T(A)$ is a two-sided ideal of A.

Theorem 13. *A derivation algebra A over a field $\mathbb{F}$ has a unique decomposition as a direct sum of two-sided ideals A_0 and A_*, where A_0 is a complete algebra containing the derived ideal $\mathcal{D}(A) = [A, A]$ and A_* is commutative with identity and contained in the center of A. Moreover, we have that $T(A) \subseteq A_0$, with $T(A) = 0$ if and only if $A_0 = 0$.*

We also need the following definitions and lemmas. Recall that an ideal of a Lie algebra $\mathcal{G}$ is characteristic if it is invariant under every derivation of $\mathcal{G}$. Let A be a derivation algebra with associated Lie algebra $\mathcal{G}$. Let I be an ideal of A and let $\mathcal{I}$ be the Lie subalgebra of $\mathcal{G}$ associated to I. If $\mathcal{I}$ is a characteristic ideal of $\mathcal{G}$ associated to A, then all the left and right multiplications L_a and R_a of A are derivations that leave the ideal $\mathcal{I}$ invariant. Hence, $\mathcal{I}$ is a two-sided ideal of A.

Lemma 14. *If A is a left-symmetric algebra and I is a left (or right) ideal of A such that the derived ideal $\mathcal{D}(A) = [A, A]$ is contained in I, then I is a two-sided ideal and A/I is commutative.*

Proof. Assume, without loss of generality, that I is a left ideal of A. Let $a \in A$ and $x \in I$. Then $a \cdot x$ and $[a, x] \in I$. Since $x \cdot a = a \cdot x - [a, x]$, we deduce that $x \cdot a \in I$, that is I is a two-sided ideal. Now, remark that for all $\overline{x} = (x + I)$ and $\overline{y} = (y + I) \in A/I$, we have $[\overline{x}, \overline{y}] = ([x, y] + I) = \overline{0}$. That is, A/I is commutative. ■

Given a left-symmetric algebra A over a field $\mathbb{F}$, one defines *the radical* $R(A)$ of A to be the largest left ideal contained in the subset

$$I(A) = \{a \in A : tr(R_a) = 0\}.$$

It turns out that $R(A)$ is nothing but the largest complete left ideal of A. This has been proved in [8] for the case $\mathbb{F} = \mathbb{C}$, and in [4] for $\mathbb{F} = \mathbb{R}$.

In general, the radical of an arbitrary left-symmetric algebra is not a two-sided ideal (cf. [8]). For derivation algebras, we have the following result.

Lemma 15. *If A is a derivation algebra, then its radical $\mathcal{R}(A)$ is a two-sided ideal. In particular, A is an extension of a commutative associative algebra by a complete derivation algebra.*

Proof. Let A be a derivation algebra. By Theorem 13, A has a unique decomposition as a direct sum of two ideals

$$A = A_0 \oplus A_*,$$

where A_0 is a complete algebra containing the derived ideal $\mathcal{D}(A) = [A, A]$, and A_* is an algebra with identity and contained in the center $Z(A)$ of A. Since $\mathcal{D}(A)$ is contained in $A_0 = \cap_{a\in A} \ker\left(R_a^n\right)$, then R_a is nilpotent for all $a \in \mathcal{D}(A)$, that is, $\mathrm{tr} R_a = 0$ for all $a \in \mathcal{D}(A)$, and hence $\mathcal{D}(A) \subset I(A)$. On the other hand, since $\mathcal{D}(A)$ is a characteristic ideal of a derivation algebra A, then it is a two-sided ideal. Hence $\mathcal{D}(A) \subset \mathcal{R}(A)$, as $\mathcal{R}(A)$ is the largest left ideal of A contained in $I(A)$. Now, using Lemma 14, we conclude that $\mathcal{R}(A)$ is a two-sided ideal. ■

We are now in position to prove the result announced above.

Proposition 16. *Let A be a derivation algebra. Then, the Lie algebra $\mathcal{G}$ associated to A is solvable.*

Proof. Let A be a derivation algebra, and let $\mathcal{G}$ be its associated Lie algebra. Since the radical $\mathcal{R}(A)$ of A is a complete left-symmetric algebra, it follows by [1] that the Lie algebra

$\mathcal{H}$ associated to $\mathcal{R}(A)$ is solvable. On the other hand, the Lie algebra associated to $A/\mathcal{R}(A)$ is abelian by Lemmas 14 and 15. It follows that $\mathcal{G}$ is solvable. ■

Remark 17. It is easy to show that the Lie algebra associated to an inner derivation algebra is two-step solvable. Moreover, all examples of derivation algebras we know turn out to have their associated Lie algebras two-step solvable. In fact, at the end of this paper, we will deduce from the classification of derivation algebras of dimension 4 that the Lie algebra associated to a derivation algebra of dimension 4 is necessarily two-step solvable. Furthermore, up to dimension 6, we could not find any derivation structure on a solvable Lie algebra with order of solvability greater than two, and consequently the question as to whether the Lie algebra associated to a derivation algebra is two-step solvable is still an open question.

About Theorem 13, we can deduce from this result the following remarkable consequence.

Corollary 18. *Let A be a derivation algebra that is non-commutative or complete. Then, $T(A)$ is nontrivial. In particular A is not simple.*

Proof. By Theorem 13, $T(A) = 0$ if and only $A_0 = 0$ if and only if $A = A_*$, and hence A is commutative and incomplete. ■

In fact, simple derivations algebras over $\mathbb{R}$ are well known (see [6]) .

Proposition 19. *A simple derivation algebra over $\mathbb{R}$ is isomorphic to either $A_{2,3}: e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, e_2 \cdot e_2 = -e_1$ or the field $\mathbb{R}$.*

In [11] (see also [6]) it was shown that for a derivation algebra A, the center $Z(A)$ of A is a two-sided ideal of A which coincides with the center of the associated Lie algebra of A. In this setting, we have the following immediate consequence of Theorem 13.

Corollary 20. *Let A be a derivation algebra whose associated Lie algebra $\mathcal{G}$ is solvable with trivial center. Then, A is complete.*

Two other immediate but important consequences of Theorem 13 are the following Propositions. Recall here that a Lie algebra $\mathcal{G}$ is said to be nonsingular k-step nilpotent whenever the kth element $\mathcal{C}^{k-1}\mathcal{G}$ of lower central series of $\mathcal{G}$ coincides with $\mathcal{Z}(\mathcal{G})$.

Proposition 21. *Let A be a derivation algebra with associated Lie algebra $\mathcal{G}$. If the center $\mathcal{Z}(\mathcal{G})$ of $\mathcal{G}$ is contained in its derived ideal $\mathcal{D}\mathcal{G}$. Then, A is complete.*

Proof. By Theorem 13, we have $A_* \subseteq Z(A) \subseteq A_0$, which implies that $A_* = \{0\}$ and hence $A = A_0$, that is A is complete. ■

Proposition 22. *Let A be a derivation algebra whose associated Lie algebra $\mathcal{G}$ is nonsingular k-step nilpotent. Then, A is complete.*

Proof. In virtue of Theorem 13 and the fact that $\mathcal{G}$ is nonsingular k-step nilpotent, we have

$$A_* \subseteq \mathcal{Z}(\mathcal{G}) = \mathcal{C}^{k-1}\mathcal{G} = \mathcal{Z}(\mathcal{G}) \cap [A, A] \subseteq [A, A] \subseteq A_0,$$

from which we deduce that $A_* = \{0\}$ and hence $A = A_0$, that is A is complete. ■

Corollary 23. *Let A be a derivation algebra whose associated Lie algebra is the Heisenberg algebra $\mathcal{H}_{2n+1}$. Then, A is complete.*

4.1. Relationships of derivation algebras with other special left-symmetric algebras

A left-symmetric algebra A is a *Novikov algebra* if it satisfies the identity

$$(x \cdot y) \cdot z = (x \cdot z) \cdot y, \text{ for all } x, y, z \in A. \tag{7}$$

In terms of left and right multiplications, (7) is equivalent to the formula

$$\left[R_x, R_y\right] = 0, \quad \text{for all } x, y \in A.$$

In [6], the first author has studied a remarkable class of left-symmetric algebras. Namely, the class of left-symmetric algebras A satisfying the identity

$$(x \cdot y) \cdot z = (y \cdot x) \cdot z, \text{ for all } x, y, z \in A, \tag{8}$$

or equivalently

$$\left[L_x, L_y\right] = 0, \quad \text{for all } x, y \in A.$$

It is clear that every commutative associative algebra is Novikov and satisfying (8). Although identities (3), (7), and (8) are pairwise independent, the following remarkable fact was proved in [6].

Proposition 24. *If a left-symmetric algebra A satisfies any two of the conditions (3), (7), (8), then A also satisfies the third condition.*

4.2. Derivation algebras of dimension ≤ 3

In this subsection, we will classify derivation algebras over $\mathbb{R}$ of dimension ≤ 3. The classification of commutative associative derivation algebras over $\mathbb{R}$ of dimension ≤ 3 was given in Section 3 (see Lemmas 5 and 6).

We know that the Lie algebra $aff(\mathbb{R})$ of the affine group of the real line is the unique non-abelian solvable Lie algebra of dimension 2. It has a basis $\{e_1, e_2\}$ satisfying $[e_1, e_2] = e_2$. The non-abelian Lie algebras over $\mathbb{R}$ of dimension 3 are also well known [12]. They are:

$\mathcal{H}_3 : [e_1, e_2] = e_3,$
$\mathcal{E}(2) : [e_1, e_2] = e_3, [e_1, e_3] = -e_2,$
$\mathcal{E}(1, 1) : [e_1, e_2] = e_2, [e_1, e_3] = -e_3,$
$\mathcal{G}_{3,1} = \mathbb{R} \times aff(\mathbb{R}) : [e_1, e_2] = e_2,$
$\mathcal{G}_{3,2} : [e_1, e_2] = e_2, [e_1, e_3] = e_3,$
$\mathcal{G}_{3,3} : [e_1, e_2] = e_2 + e_3, [e_1, e_3] = e_3,$
$\mathcal{G}_{3,4}^{\mu} : [e_1, e_2] = e_2, [e_1, e_3] = \mu e_3, 0 < |\mu| < 1,$
$\mathcal{G}_{3,5}^{\zeta} : [e_1, e_2] = e_2 + \zeta e_3, [e_1, e_3] = -\zeta e_2 + e_3, \zeta > 0.$

Note here that the center of each of the Lie algebras $aff(\mathbb{R})$, $\mathcal{E}(2)$, $\mathcal{E}(1, 1)$, $\mathcal{G}_{3,2}$, $\mathcal{G}_{3,3}$, $\mathcal{G}_{3,4}^{\mu}$ and $\mathcal{G}_{3,5}^{\zeta}$ is trivial. It follows by Corollary 20 that any derivation structure on any of these Lie algebras is complete. Also, Corollary 23 implies that any derivation structure on $\mathcal{H}_3$ is complete. Thus, from the classification list of complete left-symmetric structures on solvable Lie algebras over $\mathbb{R}$ of dimension ≤ 3 given in [7], we obtain the following classification results.

Proposition 25. *Up to a left-symmetric algebra isomorphism, there exists one and only one derivation structure on* $aff(\mathbb{R})$. *It is given by the relation* $e_1 \cdot e_2 = e_2$, *and the corresponding derivation algebra will be denoted by* N_2.

Remark 26. We can easily verify that N_2 is inner and satisfies (8) (hence, Novikov).

Proposition 27. *Let* A *be a complete non-commutative derivation algebra of dimension* 3 *over* $\mathbb{R}$. *Then,* A *is isomorphic to one of the following left-symmetric algebras:*

$$\begin{aligned}
&H_{3,1} : e_1 \cdot e_1 = pe_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_2 = e_3,\ \ p \in \mathbb{R},\\
&H_{3,2} : e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = me_3, e_2 \cdot e_1 = (m-1)\, e_3,\ \ m \in \mathbb{R},\\
&H_{3,3} : e_1 \cdot e_2 = \tfrac{1}{2}e_3, e_2 \cdot e_1 = -\tfrac{1}{2}e_3,\\
&I_{3,0} : e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = -e_3,\\
&J_{3,0} : e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = -e_2,\\
&N_{3,0} : e_1 \cdot e_2 = e_2,\\
&N_{3,1} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_2,\\
&B_{3,0} : e_1 \cdot e_2 = e_2,\ \ e_1 \cdot e_3 = e_3,\\
&B_{3,1} : e_1 \cdot e_2 = e_2 + e_3, e_2 \cdot e_1 = e_3, e_1 \cdot e_3 = e_3,\\
&C_{3,1} : e_1 \cdot e_2 = e_2 + e_3, e_1 \cdot e_3 = e_3,\\
&C_{3,t} : e_1 \cdot e_2 = e_2 + te_3, e_1 \cdot e_3 = e_3, e_2 \cdot e_1 = (t-1)\, e_3,\ \ t \neq 1,\\
&D_{3,1}(\mu) : e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = \mu e_3,\ \ 0 < |\mu| < 1,\\
&E_{3,1}(\zeta) : e_1 \cdot e_2 = e_2 + \zeta e_3, e_1 \cdot e_3 = -\zeta e_2 + e_3,\ \ \zeta > 0.
\end{aligned}$$

Remark 28. In Proposition 27, we note that:

1. The left-symmetric algebras $H_{3,1}$, $H_{3,2}$, $H_{3,3}$, $I_{3,0}$ $J_{3,0}$, $N_{3,0}$, $N_{3,1}$, $B_{3,0}$, $C_{3,1}$, $D_{3,1}(\mu)$ and $E_{3,1}(\zeta)$ satisfy identity (8), and hence all of them are Novikov.
2. The algebras $H_{3,1}$, $H_{3,3}$, $I_{3,0}$ $J_{3,0}$, $N_{3,0}$, $B_{3,0}$, $C_{3,1}$, $D_{3,1}(\mu)$ and $E_{3,1}(\zeta)$ are inner derivation.

Now, to complete the classification of derivation structures on solvable Lie algebras over $\mathbb{R}$ of dimension ≤ 3, it remains to find all the incomplete derivation structures on the Lie algebra $\mathcal{G}_{3,1}$. In fact, there exists one and only one incomplete derivation structure on $\mathcal{G}_{3,1}$, given in the following result which is an application of Theorem 13.

Theorem 29. *Let* A *be an incomplete non-commutative derivation algebra over* $\mathbb{R}$ *of dimension* 3. *Then,* A *is the direct sum* $A \cong \mathbb{R}_1 \oplus N_2$, *where* $\mathbb{R}_1$ *is the field and* $N_2 = \langle e_1, e_2 : e_1 \cdot e_2 = e_2 \rangle$

Proof. Let A be a derivation algebra. Then, by Theorem 13, A has a unique decomposition as a direct sum of two ideals A_0 and $A_*$where A_0 is a complete algebra containing the derived ideal $[A, A]$ and A_* is an algebra with identity and contained in the center $Z(A)$ of A.

If $\dim A_* = 1$, then A_* is the field $\mathbb{R}_1$ and A_0 is a non-commutative complete derivation algebra of dimension 2, that is, $A_0 \cong N_2$.

If $\dim A_* = 2$, then A_0 is the trivial algebra $\mathbb{R}_0$. Since, A_* is commutative, then so is A, a contradiction. ■

Remark 30. In view of Theorem 29, there is a basis $\{e_1, e_2, e_3\}$ of the Lie algebra $\mathcal{G}_{3,1}$ in which the incomplete derivation structure on $\mathcal{G}_{3,1}$ is defined by the following multiplication table

$$e_1 \cdot e_2 = e_2, \;\; e_3 \cdot e_3 = e_3.$$

This structure is not inner as it is incomplete. However, it satisfies (8) and thus Novikov.

We close this section by the following corollaries which give the classification of derivation algebras of dimension ≤ 3.

Corollary 31. *Any two-dimensional derivation algebra over $\mathbb{R}$ is isomorphic to one of the algebras given in Lemma 5 or Proposition 25.*

Corollary 32. *Any three-dimensional derivation algebra over $\mathbb{R}$ is isomorphic to one of the algebras given in Lemma 6, Proposition 27, or Theorem 29.*

5. Derivation algebras of dimension 4

The aim of this section is to classify derivation algebras over $\mathbb{R}$ of dimension 4. For, let A be a derivation algebra over $\mathbb{R}$ of dimension 4 whose associated Lie algebra is $\mathcal{G}$. If $\mathcal{G}$ is abelian, then the derivation structures on $\mathcal{G}$ are just given by commutative associative algebras of dimension 4 given in Proposition 8. So, it remains to classify all derivation algebras on non-abelian Lie algebras of dimension 4. Toward this, we start with the following result which gives the classification of all incomplete non-commutative derivation algebras of dimension 4.

Theorem 33. *Let A be an incomplete non-commutative derivation algebra over $\mathbb{R}$ of dimension 4. Then, A is either a direct sum of the form $A \cong \mathbb{R}_1 \oplus A_3$, where $\mathbb{R}_1$ is the field and A_3 is a complete non-commutative derivation algebra of dimension 3, or A is a direct sum of the form $A \cong A_2 \oplus N_2$, where A_2 is an incomplete commutative algebra of dimension 2 and $N_2 = \langle e_1, e_2 : e_1 \cdot e_2 = e_2 \rangle$.*

Proof. Let A be an incomplete non-commutative derivation algebra of dimension 4. Then $A = A_0 \oplus A_*$, where A_0 and A_* are the two-sided ideals in Theorem 13. Assume first that $\dim A_* = 1$. Then, in this case, $A \cong \mathbb{R}_1 \oplus A_3$, where $\mathbb{R}_1$ is the field and A_3 is a complete non-commutative derivation algebra of dimension 3.

If $\dim A_* = 2$, then A_* is isomorphic to $A_{2,i}$, $1 \leq i \leq 4$, where $A_{2,i}$ is one of the commutative algebras of dimension 2 given in Lemma 5. In this case, since A_0 is a complete non-commutative derivation algebra of dimension 2, we must have $A_0 \cong N_2$. Thus, $A \cong A_{2,i} \oplus N_2$ for some $i \in \{1, \dots, 4\}$.

If $\dim A_* = 3$, then A_0 is the trivial algebra $\mathbb{R}_0$. Since A_* is commutative, it follows that A is commutative, which is a contradiction. ■

In light of Theorem 33, we have the following result.

Proposition 34. *Let A be an incomplete non-commutative derivation algebra over $\mathbb{R}$ of dimension 4. Then, A is isomorphic to one of the following left-symmetric algebras:*

$\mathbb{R}_1 \oplus H_{3,1} : e_1 \cdot e_1 = pe_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_2 = e_3, e_4 \cdot e_4 = e_4, p \in \mathbb{R},$
$\mathbb{R}_1 \oplus H_{3,2} : e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = me_3, e_2 \cdot e_1 = (m-1)\, e_3, e_4 \cdot e_4 = e_4,\ m \in \mathbb{R},$
$\mathbb{R}_1 \oplus H_{3,3} : e_1 \cdot e_2 = \frac{1}{2}e_3, e_2 \cdot e_1 = -\frac{1}{2}e_3, e_4 \cdot e_4 = e_4,$
$\mathbb{R}_1 \oplus I_{3,0} : e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = -e_3, e_4 \cdot e_4 = e_4,$
$\mathbb{R}_1 \oplus J_{3,0} : e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = -e_2, e_4 \cdot e_4 = e_4,$
$\mathbb{R}_1 \oplus N_{3,0} = A_{2,4} \oplus N_2 : e_1 \cdot e_2 = e_2, e_4 \cdot e_4 = e_4,$
$\mathbb{R}_1 \oplus N_{3,1} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_2, e_4 \cdot e_4 = e_4,$
$\mathbb{R}_1 \oplus B_{3,0} : e_1 \cdot e_2 = e_2,\ e_1 \cdot e_3 = e_3, e_4 \cdot e_4 = e_4,$
$\mathbb{R}_1 \oplus B_{3,1} : e_1 \cdot e_2 = e_2 + e_3, e_2 \cdot e_1 = e_3, e_1 \cdot e_3 = e_3, e_4 \cdot e_4 = e_4,$
$\mathbb{R}_1 \oplus C_{3,1} : e_1 \cdot e_2 = e_2 + e_3, e_1 \cdot e_3 = e_3, e_4 \cdot e_4 = e_4,$
$\mathbb{R}_1 \oplus C_{3,t} : e_1 \cdot e_2 = e_2 + te_3, e_1 \cdot e_3 = e_3, e_2 \cdot e_1 = (t-1)\, e_3, e_4 \cdot e_4 = e_4,\ t \neq 1,$
$\mathbb{R}_1 \oplus D_{3,1}(\mu) : e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = \mu e_3, e_4 \cdot e_4 = e_4,\ 0 < |\mu| < 1,$
$\mathbb{R}_1 \oplus E_{3,1}(\zeta) : e_1 \cdot e_2 = e_2 + \zeta e_3, e_1 \cdot e_3 = -\zeta e_2 + e_3, e_4 \cdot e_4 = e_4,\ \zeta > 0,$
$A_{2,1} \oplus N_2 : e_1 \cdot e_1 = e_1, e_2 \cdot e_2 = e_2, e_3 \cdot e_4 = e_4,$
$A_{2,2} \oplus N_2 : e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, e_3 \cdot e_4 = e_4,$
$A_{2,3} \oplus N_2 : e_1 \cdot e_1 = e_1, e_1 \cdot e_2 = e_2 \cdot e_1 = e_2, e_2 \cdot e_2 = -e_1, e_3 \cdot e_4 = e_4.$

Remark 35. We note that, except for the algebras $\mathbb{R}_1 \oplus B_{3,1}$ and $\mathbb{R}_1 \oplus C_{3,t}$, each of the derivation algebras appearing in Proposition 34 satisfies identity (8) (hence, Novikov).

To complete our work, we need to study the case where A is a complete non-commutative derivation algebra over $\mathbb{R}$ of dimension 4. Next, we will discuss the problem of extension of a derivation algebra by another derivation algebra. Suppose that a vector space extension $\widetilde{A}$ of a left-symmetric algebra A by another left-symmetric algebra E is given. We want to define a derivation structure on $\widetilde{A}$ in terms of the left-symmetric structures given on A and E. In other words, we want to define a derivation structure on $\widetilde{A}$ for which E becomes a two-sided ideal in $\widetilde{A}$ such that $\widetilde{A}/E \cong A$. Equivalently,

$$0 \to E \to \widetilde{A} \to A \to 0$$

becomes a short exact sequence of left-symmetric algebras. The product

$$(x, a) \cdot (y, b) = \big(x \cdot y, a \cdot b + \lambda_x(b) + \rho_y(a) + g(x, y)\big), \tag{9}$$

defines a derivation structure on $\widetilde{A} = A \times E$ if and only if the conditions 1–5 in Theorem 1 are satisfied and

$$((x, a) \cdot (y, b)) \cdot (z, c) = ((z, c) \cdot (y, b)) \cdot (x, a),$$

for all $x, y, z \in A$ and $a, b, c \in E$.

By a direct computation we get the following result.

Proposition 36. *The product (9) defines a derivation structure on $\widetilde{A}$ if and only if A and E are derivation algebras, the conditions 1–5 in Theorem 1 are satisfied, and in addition the following conditions hold for all $x, y, z \in A$ and $a, b, c \in E$:*

1. $L_{\rho_x(a)} = R_a \circ \rho_x,$
2. $\rho_x \circ L_a = R_a \circ \lambda_x,$
3. $\rho_x \circ \lambda_y = \rho_y \circ \lambda_x,$
4. $[\lambda_x, \lambda_y] + \rho_y \circ \rho_x - \lambda_{x \cdot y} = L_{g(x,y)},$
5. $g(x \cdot y, z) - g(z \cdot y, x) - \rho_x(g(z, y)) - \rho_z(g(x, y)) = 0.$

As a special case, we assume that the left-symmetric structure on A is commutative. In this case, we have the following result.

Corollary 37. *Suppose that the left-symmetric structure on A is commutative. Then,* (9) *defines a derivation structure on $\widetilde{A}$ if and only if the following conditions hold:*

1. $\lambda_x(a \cdot b) = \lambda_x(a) \cdot b + a \cdot \lambda_x(b) - \rho_x(a) \cdot b$,
2. $\rho_x([a, b]) = a \cdot \rho_x(b) - b \cdot \rho_x(a)$,
3. $[\lambda_x, \rho_y] + \rho_y \circ \rho_x - \rho_{x \cdot y} = R_{g(x;y)}$,
4. $L_{\rho_x(a)} = R_a \circ \rho_x$,
5. $\rho_x \circ L_a = R_a \circ \lambda_x$,
6. $\rho_x \circ \lambda_y = \rho_y \circ \lambda_x$,
7. $\rho_x \circ \rho_y - \lambda_{y \cdot x} = L_{g(x,y)}$,
8. $g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y, z)) - \lambda_y(g(x, z)) - \rho_z(g(x, y) - g(y; x)) = 0$,
9. $g(x \cdot y, z) - g(z \cdot y; x) - \rho_x(g(z, y)) - \rho_z(g(x, y)) = 0$.

5.1. *The classification*

The classification of all complete non-commutative derivation algebras of dimension 4 is not difficult to accomplish. It can be achieved using the extensions of left-symmetric algebras, but the list will be very large. For this reason, we will restrict here to the complete non-commutative derivation algebras of dimension 4 that are obtained as follows. Let A be a complete non-commutative derivation algebra of dimension 4. Lemma 15 and Proposition 4 show that A can be obtained as an extension of a complete commutative associative algebra J by a complete derivation algebra I. We will assume here that J is not trivial (that is, $\dim J \geq 1$). Therefore, we get a short exact sequence of left-symmetric algebras:

$$0 \to I \xrightarrow{i} A \xrightarrow{\pi} J \to 0. \tag{10}$$

Since A is supposed to be non-commutative, we have $\dim J = 1$, 2, or 3. Therefore, according to the dimension of J or, equivalently, the dimension of I, we have three cases to consider.

Case 1. $\dim I = 1$.

In this case, I is isomorphic to the trivial algebra $\mathbb{R}_0$, and the Lie algebra associated to J is isomorphic to $\mathbb{R}^3$. Then, J is one of the algebras $A_{3,i}$, $11 \leq i \leq 15$, where the $A_{3,i}$ are the real three-dimensional complete commutative algebras given in Lemma 6.

The extended left-symmetric product on $A = A_{3,i} \bigoplus \mathbb{R}_0$ given by (9) turns out to take the simplified form

$$(x, a) \cdot (y, b) = \big(x \cdot y, b\lambda_x + a\rho_y + g(x, y)\big), \tag{11}$$

for all $x, y \in A_{3,i}$ and $a, b \in \mathbb{R}$. Here $\lambda_x, \rho_y \in End(I) \cong \mathbb{R}$, so we can identify λ_x and ρ_y with real numbers that we denote by λ_x and ρ_y as well.

The conditions in Corollary 37 can be simplified to the following conditions

$$\rho_{x \cdot y} = \rho_y \circ \rho_x, \tag{12}$$

$$\lambda_{x \cdot y} = \rho_{x \cdot y}, \tag{13}$$

$$\rho_x \circ \lambda_y = \rho_y \circ \lambda_x,$$

$$g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y, z)) - \lambda_y(g(x, z)) - \rho_z(g(x, y) - g(y; x)) = 0,$$

$$g(x \cdot y, z) - g(z \cdot y; x) - \rho_x(g(z, y)) - \rho_z(g(x, y)) = 0. \tag{14}$$

In the case where

$$J = A_{3,14} = \langle e_1, e_2, e_3 : e_1 \cdot e_1 = e_2,\ e_1 \cdot e_2 = e_2 \cdot e_1 = e_3 \rangle,$$

we find, by substituting $x = y = e_i$, $i = 1, 2, 3$ into Eqs. (12) and (13), that $\rho_{e_i} = 0$ and $\lambda_{e_2} = \lambda_{e_3} = 0$.

In this case, the formula (5) and (6) become

$$\delta_1 h(x, y) = \lambda_x(h(y)) - h(x \cdot y), \tag{15}$$

and

$$\delta_2 g(x, y, z) = g(x, y \cdot z) - g(y, x \cdot z) + \lambda_x(g(y, z)) - \lambda_y(g(x, z)). \tag{16}$$

Using the formula (15) for δ_1, we get

$$\delta_1 h = \begin{pmatrix} ah_1 - h_2 & ah_2 - h_3 & ah_3 \\ -h_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $h_i = h(e_i)$, $i = 1, 2, 3$, and $a = \lambda_{e_1}$. Similarly, using formula (14) and formula (16) for δ_2, we easily verify that if g is a cocycle (i.e. $\delta_2 g = 0$) and $g_{ij} = g\left(e_i, e_j\right)$, then

$$g = \begin{pmatrix} g_{11} & g_{12} & g_{22} - ag_{21} \\ g_{21} & g_{22} & 0 \\ g_{22} & 0 & 0 \end{pmatrix},$$

with $ag_{22} = 0$.

Assume first that $a \neq 0$. It follows that, in the basis above, the class $[g] \in H^2_{\lambda,\rho}\left(A_{3,14}, \mathbb{R}_0\right)$ of a cocycle g is identically zero.

In this case, we determine the extended derivation structure on $A = A_{3,14} \bigoplus \mathbb{R}_0$. By setting $\widetilde{e_i} = (e_i, 0)$ for $i = 1, 2, 3$ and $\widetilde{e}_4 = (0, 1)$, and using formula (11), we obtain

$$\widetilde{e}_1 \cdot \widetilde{e}_1 = \widetilde{e}_2$$
$$\widetilde{e}_1 \cdot \widetilde{e}_2 = \widetilde{e}_2 \cdot \widetilde{e}_1 = \widetilde{e}_3$$
$$\widetilde{e}_1 \cdot \widetilde{e}_4 = a\widetilde{e}_4,$$

and all other products are zero, where $a \neq 0$.

By setting $e_1 = \frac{1}{\alpha}\widetilde{e}_1$, $e_2 = \frac{1}{\alpha^2}\widetilde{e}_2$, $e_3 = \frac{1}{\alpha^3}\widetilde{e}_3$, and $e_4 = \widetilde{e}_4$, we get a new basis $\{e_1, e_2, e_3, e_4\}$ of A satisfying

$$e_1 \cdot e_1 = e_2$$
$$e_1 \cdot e_2 = e_2 \cdot e_1 = e_3$$
$$e_1 \cdot e_4 = e_4.$$

Note here that this derivation structure satisfies the identity (8).

On the other hand, assume that $a = 0$, that is, the extension is central. Using the classification in [9] of all four-dimensional complete left-symmetric algebras which are central extensions of $A_{3,14}$ by $\mathbb{R}_0$ we find that the derivation structure on $A = A_{3,14} \bigoplus \mathbb{R}_0$ will be given by the nonzero relations

$$e_1 \cdot e_1 = e_2, e_2 \cdot e_2 = e_3,$$
$$e_1 \cdot e_2 = e_4, e_2 \cdot e_1 = e_3 + e_4,$$
$$e_1 \cdot e_4 = e_4 \cdot e_1 = e_3.$$

Note here that this derivation structure satisfies the identity (8) (hence, Novikov) but it is not inner.

Using the same method as above, we can obtain all the derivation structures on $A = A_{3,i} \bigoplus \mathbb{R}_0$, $11 \leq i \leq 15$.

Case 2. $\dim I = 2$.

In this case, the short exact sequence (10) becomes

$$0 \to I \to A \to J \to 0$$

where I is a complete derivation algebra whose associated Lie algebra is either $\mathbb{R}^2$ or aff$(\mathbb{R})$ and J is one of the algebras $A_{2,5}$ or $A_{2,6}$ given in Lemma 5.

We deduce from Lemma 5 and Proposition 25 that I is either $A_{2,5}$, $A_{2,6}$ or N_2. In the case where

$$I = N_2 = \langle e_1, e_2 : e_1 \cdot e_2 = e_2 \rangle,$$

let $\phi : \mathbb{R}^2 \to Der\,(aff\,(\mathbb{R}))$ be a derivation of $aff\,(\mathbb{R})$. In this case, we can easily see that for any basis $\{e_3, e_4\}$ of J, and relative to a basis $\{e_1, e_2\}$ of aff$(\mathbb{R})$ satisfying $[e_1 \cdot e_2] = e_2$, we have

$$\phi\,(e_3) = \begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix} \text{ and } \phi\,(e_4) = \begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$.

Set

$$\rho_{e_3} = \begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix}, \quad \rho_{e_4} = \begin{pmatrix} \gamma_1 & \delta_1 \\ \gamma_2 & \delta_2 \end{pmatrix},$$

relative to $\{e_1, e_2\}$. From Corollary 37, we deduce that $\rho_x\,([a, b]) = 0$ and $\rho_x\,(a) \cdot b = \rho_x\,(b) \cdot \not{a}$ for all $a, b \in I$ and $x \in J$. Hence, we get that

$$\rho_{e_3} = \begin{pmatrix} 0 & 0 \\ \alpha_2 & \beta_2 \end{pmatrix}, \quad \rho_{e_4} = \begin{pmatrix} 0 & 0 \\ \gamma_2 & \delta_2 \end{pmatrix}.$$

Since $\lambda_{e_i} = \phi\,(e_i) + \rho_{e_i}$, $i = 3, 4$, we get that

$$\lambda_{e_3} = \begin{pmatrix} 0 & 0 \\ \alpha_2 + a & \beta_2 + b \end{pmatrix}, \quad \lambda_{e_4} = \begin{pmatrix} 0 & 0 \\ \gamma_2 + c & \delta_2 + d \end{pmatrix}.$$

Also, from Corollary 37, we deduce that

$$\rho_x\,(a \cdot b) = \lambda_x(b) \cdot a,$$

and

$$\lambda_x\,(a \cdot b) = \lambda_x(a) \cdot b + a \cdot \lambda_x(b) - \rho_x\,(a) \cdot b,$$

for all $a, b \in I$ and $x \in J$. This implies that $\beta_2 = \delta_2 = 0$ and $\alpha_2 + a = \gamma_2 + c = 0$. Thus,

$$\rho_{e_3} = \begin{pmatrix} 0 & 0 \\ \alpha_2 & 0 \end{pmatrix}, \ \rho_{e_4} = \begin{pmatrix} 0 & 0 \\ \gamma_2 & 0 \end{pmatrix}$$

and

$$\lambda_{e_3} = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \ \lambda_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix}.$$

If

$$J = A_{2,5} = \langle e_3, e_4 : e_3 \cdot e_3 = e_4 \rangle,$$

let $g \in L^2(J, I)$. In this case g can be expressed as a (2×2) $\mathbb{R}^2$-valued matrix $\begin{pmatrix} g_{33} & g_{34} \\ g_{43} & g_{44} \end{pmatrix}$ such that $g_{ij} = g\left(e_i, e_j\right)$, $i, j = 3, 4$. Set

$$\begin{aligned} g_{33} &= a_1 e_1 + a_2 e_2 \\ g_{34} &= b_1 e_1 + b_2 e_2 \\ g_{43} &= c_1 e_1 + c_2 e_2 \\ g_{44} &= d_1 e_1 + d_2 e_2. \end{aligned}$$

From Corollary 37 we obtain that

$$g = \begin{pmatrix} -de_1 + (\alpha_2 b - \gamma_2) e_2 & \gamma_2 b e_2 \\ 0 & 0 \end{pmatrix}$$

with $\alpha_2 d = \gamma_2 d = 0$.

For a linear map $h \in L^1(J, I)$ we get that

$$\delta_1 h(x, y) = \rho_y(h(x)) + \lambda_x(h(y)).$$

In this case, $\delta_1 h$ can be expressed as a (2×2) $\mathbb{R}^2$-valued matrix of the form

$$\delta_1 h = \begin{pmatrix} -h_{41} e_1 + (\alpha_2 h_{31} + b h_{32} - h_{42}) e_2 & (\gamma_2 h_{31} + b h_{42}) e_2 \\ (\alpha_2 h_{41} + d h_{32}) e_2 & (\gamma_2 h_{41} + d h_{42}) e_2 \end{pmatrix},$$

where $h(e_k) = h_{k1} e_1 + h_{k2} e_2$ for $k = 3, 4$. We deduce then that the class $[g] \in H^2_{\lambda,\rho}(J, I)$ of a cocycle g is identically zero.

We determine, in this case, the extended derivation algebra on A . By setting $\widetilde{e}_i = (0, e_i)$, $i = 1, 2$ and $\widetilde{e}_j = \left(e_j, 0\right)$, $j = 3, 4$ and using formula (9), we obtain

$$\begin{aligned} \widetilde{e}_1 \cdot \widetilde{e}_2 &= \widetilde{e}_2 \\ \widetilde{e}_1 \cdot \widetilde{e}_3 &= \alpha_2 \widetilde{e}_2 \\ \widetilde{e}_1 \cdot \widetilde{e}_4 &= \alpha_2 b \widetilde{e}_2 \\ \widetilde{e}_3 \cdot \widetilde{e}_2 &= b \widetilde{e}_2 \\ \widetilde{e}_3 \cdot \widetilde{e}_3 &= \widetilde{e}_4 \end{aligned}$$

and all other products are zero. By setting $e_1 = \widetilde{e}_1$, $e_2 = \widetilde{e}_2$, $e_3 = \widetilde{e}_3 - \alpha_2 \widetilde{e}_2$ and $e_4 = \widetilde{e}_4 - \alpha_2 b \widetilde{e}_2$, we see that the new basis $\{e_1, e_2, e_3, e_4\}$ of A satisfies

$$\begin{aligned} e_1 \cdot e_2 &= e_2 \\ e_3 \cdot e_3 &= e_4. \end{aligned}$$

Note here that this derivation structure is not inner but satisfies the identity (8) (hence, Novikov).

Using the same method as above, we can obtain all the derivation structures on $A = A_{2,i} \bigoplus I$, $i = 5, 6$, where I is either $A_{2,5}$, $A_{2,6}$ or N_2.

Case 3. $\dim I = 3$.

In this case, the short exact sequence (10) becomes

$$0 \to I \to A \to \mathbb{R}_0 \to 0,$$

where I is a complete derivation algebra whose associated Lie algebra is solvable of dimension 3.

Let $\sigma : \mathbb{R}_0 \to A$ be a section and set $\sigma(1) = x_\circ \in A$. Define two linear maps $\lambda, \rho \in End(I)$ by putting $\lambda(y) = x_\circ \cdot y$ and $\rho(y) = y \cdot x_\circ$, and set $e = x_\circ \cdot x_\circ$. Let $g : \mathbb{R}_0 \times \mathbb{R}_0 \to I$ be the bilinear map defined by $g(a, b) = \sigma(a) \cdot \sigma(b) - \sigma(a \cdot b)$. Since the product on $\mathbb{R}_0$ is trivial, then $g(a, b) = abe$, or equivalently $g(1, 1) = e$. Also, we can show that $\delta_2 g = 0$, i.e., $g \in Z^2_{\lambda,\rho}(\mathbb{R}_0, I)$.

In this case, we find that the extended left-symmetric product on $A = \mathbb{R}_0 \bigoplus I$ given by (9) takes the simplified form

$$(a, x) \cdot (b, y) = (0, x \cdot y + a\lambda(y) + b\rho(x) + abe), \tag{17}$$

for all $a, b \in \mathbb{R}$ and $x, y \in I$.

The conditions in Corollary 37 can be simplified to the following conditions

$$\lambda(x \cdot y) = \lambda(x) \cdot y + x \cdot \lambda(y) - \rho(x) \cdot y \tag{18}$$

$$\rho([x, y]) = x \cdot \rho(y) - y \cdot \rho(x) = 0 \tag{19}$$

$$[\lambda, \rho] + \rho^2 = R_e \tag{20}$$

$$\rho(x) \cdot y - \rho(y) \cdot x = 0 \tag{21}$$

$$\rho(x \cdot y) = \lambda(y) \cdot x \tag{22}$$

$$\rho^2 = L_e. \tag{23}$$

As we mentioned above, I is a complete derivation algebra whose associated Lie algebra is solvable of dimension 3. More precisely, I is some $A_{3,i}$ as in Lemma 6 with $11 \leq i \leq 15$, or it is one of the complete derivation algebras given in Proposition 27. To obtain all four-dimensional complete derivation algebras A appearing as extensions of I, we should consider all possibilities for I. We will present here only one case, and the other cases can be treated similarly. Therefore, without loss of generality, we will assume that

$$I = J_{3,0} = \langle e_1, e_2, e_3 : e_1 \cdot e_2 = e_3,\ e_1 \cdot e_3 = -e_2 \rangle.$$

In this case, I is a complete derivation algebra whose associated Lie algebra is $\mathcal{E}(2)$. Letting $\phi : \mathbb{R} \to Der(\mathcal{E}(2))$, it is easy to see that, relative to a basis $\{e_1, e_2, e_3\}$ of $\mathcal{E}(2)$, $\phi(1)$ can be represented by

$$\phi(1) = \begin{pmatrix} 0 & 0 & 0 \\ c & a & -b \\ d & b & a \end{pmatrix},$$

where $a, b, c, d \in \mathbb{R}$.

Setting

$$\rho = \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{pmatrix},$$

relative to $\{e_1, e_2, e_3\}$, and applying formulas (19) and (21) to e_1 and e_2, we get $\alpha_1 = \gamma_1 = 0$, $\gamma_2 = -\beta_3$, and $\gamma_3 = \beta_2$. Formula (19) when applied to e_1 and e_3 also gives $\beta_1 = 0$. Since $\phi(1) = \lambda - \rho$, we deduce that, relative to the same basis above, we have

$$\lambda = \begin{pmatrix} 0 & 0 & 0 \\ \alpha_2 + c & \beta_2 + a & -\beta_3 - b \\ \alpha_3 + d & \beta_3 + b & \beta_2 + a \end{pmatrix}.$$

Applying formula (18) to the product $e_1 \cdot e_1$, we get $\alpha_2 + c = \alpha_3 + d = 0$. Formula (22) when applied to e_1 and e_2 gives $\beta_2 = \beta_3 = 0$. Thus

$$\rho = \begin{pmatrix} 0 & 0 & 0 \\ -c & 0 & 0 \\ -d & 0 & 0 \end{pmatrix} \text{ and } \lambda = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & -b \\ 0 & b & a \end{pmatrix}.$$

Now, since $e \in I$, we set $e = se_1 + te_2 + ue_3$ for some $s, t, u \in \mathbb{R}$. Formula (20) when applied to e_1, gives that $t = -(ad - bc)$ and $u = ac - bd$. Since $\rho^2 = 0$, we deduce from formula (23) that $s = 0$. Therefore, we get $e = -(ad - bc)\, e_2 + (ac - bd.)\, e_3$.

Now, let us write down the structure of A. From the above discussion we get

$$\begin{aligned} e_1 \cdot x_\circ &= -ce_2 - de_3 \\ x_\circ \cdot e_2 &= ae_2 + be_3 \\ x_\circ \cdot e_3 &= -be_2 + ae_3 \\ x_\circ \cdot x_\circ &= -(ad - bc)\, e_2 + (ac - bd.)\, e_3. \end{aligned}$$

Therefore, relative to a basis $\{e_1, e_2, e_3, e_4\}$ of A with $e_4 = x_\circ - be_1 + de_2 - ce_3$, the left-symmetric product on A is given as follows

$$\begin{aligned} e_1 \cdot e_2 &= e_3, \quad e_1 \cdot e_3 = -e_2 \\ e_4 \cdot e_2 &= ae_2, \quad e_4 \cdot e_3 = ae_3. \end{aligned}$$

If $a = 0$, we find that the nonzero relations in A are

$$e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = -e_2.$$

Note here that this derivation structure is inner and satisfies the identity (8) (hence, A is Novikov).

If $a \neq 0$, then by setting $\widetilde{e}_i = e_i$, $i = 1, 2, 3$ and $\widetilde{e}_4 = \frac{1}{\alpha} e_4$, we see that the new basis $\{\widetilde{e}_1, \widetilde{e}_2, \widetilde{e}_3, \widetilde{e}_4\}$ of A satisfies

$$\begin{aligned} \widetilde{e}_1 \cdot \widetilde{e}_2 &= \widetilde{e}_3, \widetilde{e}_1 \cdot \widetilde{e}_3 = -\widetilde{e}_2 \\ \widetilde{e}_4 \cdot \widetilde{e}_2 &= \widetilde{e}_2, \widetilde{e}_4 \cdot \widetilde{e}_3 = \widetilde{e}_3. \end{aligned}$$

This is also an inner derivation structure which satisfies the identity (8) (hence, A is Novikov).

We can now state the main result of this section.

Theorem 38. *Let A be a complete non-commutative derivation algebra over $\mathbb{R}$ of dimension 4 that can be obtained as an extension of a non-trivial complete commutative associative algebra by a non-trivial complete derivation algebra. Then, A is isomorphic to one of the following algebras*

$N_{4,1} : e_1 \cdot e_2 = e_2,$
$N_{4,2} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2,$
$N_{4,3} : e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3,$
$N_{4,4} : e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3, e_4 \cdot e_4 = te_3, t \neq 0,$
$N_{4,5} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_2, e_4 \cdot e_4 = te_3, t \neq 0,$
$N_{4,6} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3,$
$N_{4,7} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3, e_4 \cdot e_4 = te_3, t \neq 0,$
$N_{4,8} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3,$
$N_{4,9} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_4 \cdot e_1 = e_2 + e_3,$
$N_{4,10} : e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_4 \cdot e_1 = e_2 + e_3,$
$N_{4,11} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_3 \cdot e_3 = e_4,$
$N_{4,12} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_3 \cdot e_3 = -e_4,$
$N_{4,13} : e_1 \cdot e_2 = e_2, e_3 \cdot e_3 = e_4,$
$N_{4,14} : e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3 \cdot e_1 = ae_2, e_3 \cdot e_3 = e_4, a \neq 0,$
$N_{4,15} : e_1 \cdot e_2 = e_2 + e_3, e_2 \cdot e_1 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3,$
$N_{4,16} : e_1 \cdot e_1 = te_4, e_1 \cdot e_2 = e_2 + e_3, e_2 \cdot e_1 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3, t \neq 0,$
$N_{4,17} : e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3 \cdot e_1 = 2e_4, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3,$
$H_{4,1} : e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = -e_3,$
$H_{4,2} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = -e_3,$
$H_{4,3} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = -e_3, e_2 \cdot e_4 = e_4 \cdot e_2 = e_3,$
$H_{4,4} : e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = -e_3, e_2 \cdot e_4 = e_4 \cdot e_2 = e_3,$
$H_{4,5} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = -e_3, e_2 \cdot e_2 = te_3, e_4 \cdot e_4 = e_3,$
$H_{4,6} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = -e_3,$
$H_{4,7} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = -e_3, e_2 \cdot e_2 = e_4,$
$H_{4,8} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = -e_3, e_2 \cdot e_2 = -e_4,$
$H_{4,9} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_3 + e_4, e_2 \cdot e_1 = -e_3 + e_4,$
$H_{4,10} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = te_3, e_2 \cdot e_1 = -te_3, e_2 \cdot e_2 = e_3, t > 0,$
$H_{4,11} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = te_3, e_2 \cdot e_1 = -te_3, e_2 \cdot e_2 = -e_3, t > 0,$
$H_{4,12} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = (1+t)\, e_3, e_2 \cdot e_1 = (1-t)\, e_3, e_2 \cdot e_2 = -e_4, t > 0,$
$H_{4,13} : e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = (1+t)\, e_3, e_2 \cdot e_1 = (1-t)\, e_3, t > 0,$
$H_{4,14} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_2 = te_3, e_4 \cdot e_4 = -e_3, t \geq 0,$
$H_{4,15} : e_1 \cdot e_2 = e_3, e_2 \cdot e_2 = e_1, e_4 \cdot e_4 = e_3,$
$H_{4,16} : e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = e_3,$
$H_{4,17} : e_1 \cdot e_2 = e_3, e_2 \cdot e_2 = e_1, e_2 \cdot e_4 = e_4 \cdot e_2 = e_3,$
$H_{4,18} : e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = te_3, e_2 \cdot e_2 = e_1, t \neq 1,$
$H_{4,19} : e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = te_3, e_4 \cdot e_4 = e_3, t \neq 1,$
$H_{4,20} : e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_3 + te_4, e_2 \cdot e_1 = -e_3 - te_4, e_2 \cdot e_2 = e_4,$
$H_{4,21} : e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = e_4, e_2 \cdot e_2 = e_1,$
$H_{4,22} : e_1 \cdot e_1 = e_2, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = te_3, e_4 \cdot e_1 = e_3,$
$H_{4,23} : e_1 \cdot e_1 = e_2, e_2 \cdot e_2 = e_3, e_1 \cdot e_2 = e_4, e_2 \cdot e_1 = e_3 + e_4, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3,$

$H_{4,24}: e_1 \cdot e_1 = -e_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = e_4, e_2 \cdot e_2 = e_1, e_2 \cdot e_4 = e_4 \cdot e_2 = e_3,$
$F_{4,1}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_3, e_1 \cdot e_4 = e_2,$
$F_{4,2}: e_1 \cdot e_1 = e_4 - e_2, e_1 \cdot e_2 = e_3, e_1 \cdot e_4 = e_2 + e_3, e_4 \cdot e_1 = e_3,$
$F_{4,3}: e_1 \cdot e_2 = e_3, e_2 \cdot e_2 = e_4, e_4 \cdot e_2 = e_1, e_4 \cdot e_4 = e_3,$
$F_{4,4}: e_1 \cdot e_1 = -e_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = e_4, e_2 \cdot e_2 = e_1, e_2 \cdot e_4 = te_3, e_4 \cdot e_2 = e_3, t \neq 1,$
$I_{4,1}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = -e_3,$
$I_{4,2}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = -e_3,$
$J_{4,1}: e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = -e_2,$
$J_{4,2}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = -e_2,$
$B_{4,1}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3,$
$B_{4,2}: e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3,$
$B_{4,3}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3,$
$B_{4,4}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3,$
$B_{4,5}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3,$
$B_{4,6}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3, e_3 \cdot e_1 = e_2,$
$B_{4,7}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3, e_3 \cdot e_1 = e_2,$
$B_{4,8}: e_1 \cdot e_1 = e_3 + e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3, e_3 \cdot e_1 = e_2,$
$B_{4,9}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3 + e_4, e_3 \cdot e_1 = e_2 + e_4,$
$B_{4,10}: e_1 \cdot e_2 = e_2 + e_3, e_2 \cdot e_1 = e_3, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3,$
$B_{4,11}: e_1 \cdot e_1 = te_4, e_1 \cdot e_2 = e_2 + e_3, e_2 \cdot e_1 = e_3, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3, t \neq 0,$
$B_{4,12}: e_1 \cdot e_1 = te_2 - e_3 + e_4, e_1 \cdot e_2 = e_2 + e_3, e_2 \cdot e_1 = e_3, e_1 \cdot e_3 = e_3,$
$C_{4,1}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3,$
$C_{4,2}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3,$
$C_{4,3}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3 + e_4, e_3 \cdot e_1 = e_4, e_1 \cdot e_4 = e_4 \cdot e_1 = te_2,$
$C_{4,4}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = (a+1)\,e_2 + e_3, e_3 \cdot e_1 = ae_2, a \neq 0,$
$C_{4,5}: e_1 \cdot e_1 = -(a+1)\,e_2 + te_3 + e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = (a+1)\,e_2 + e_3, e_3 \cdot e_1 = ae_2, a \neq 0,$
$D_{4,1}: e_1 \cdot e_2 = ae_2, e_1 \cdot e_3 = e_3, 0 < |a| < 1,$
$D_{4,2}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = ae_2, e_1 \cdot e_3 = e_3, 0 < |a| < 1,$
$D_{4,3}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = ae_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_2, 0 < |a| < 1,$
$D_{4,4}: e_1 \cdot e_2 = ae_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_2, 0 < |a| < 1, N + C$
$D_{4,5}: e_1 \cdot e_2 = ae_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_2 + e_3, 0 < |a| < 1,$
$D_{4,6}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = ae_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_2 + e_3, 0 < |a| < 1,$
$E_{4,1}: e_1 \cdot e_2 = e_2 - ae_3, e_1 \cdot e_3 = ae_2 + e_3, a > 0,$
$E_{4,2}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2 - ae_3, e_1 \cdot e_3 = ae_2 + e_3, a > 0,$
$G_{4,1}: e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_3,$
$G_{4,2}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_3,$
$G_{4,3}: e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = (a+1)\,e_3, e_4 \cdot e_1 = ae_3, a \neq 0,$
$G_{4,4}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = (a+1)\,e_3, e_4 \cdot e_1 = ae_3, a \neq 0,$
$G_{4,5}: e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_2 + e_3, e_4 \cdot e_1 = e_2,$
$G_{4,6}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_2 + e_3, e_4 \cdot e_1 = e_2,$
$G_{4,7}: e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_2 + (a+1)\,e_3, e_4 \cdot e_1 = e_2 + ae_3, a \neq 0,$
$G_{4,8}: e_1 \cdot e_1 = e_4, e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = e_2 + (a+1)\,e_3, e_4 \cdot e_1 = e_2 + ae_3, a \neq 0,$
$G_{4,9}: e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = (a+1)\,e_3, e_4 \cdot e_1 = ae_3, e_4 \cdot e_4 = te_3, t > 0,$
$G_{4,10}: e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_2, e_1 \cdot e_4 = ae_3, e_4 \cdot e_1 = (a+1)\,e_3, e_4 \cdot e_4 = te_3,$
$K_{4,1}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = ae_3, e_1 \cdot e_4 = be_4, 0 \neq a \neq 1, b \neq 0,$
$K_{4,2}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_3 + e_4, e_4 \cdot e_1 = e_3,$

$K_{4,3}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = ae_3, e_1 \cdot e_4 = e_3 + ae_4, e_4 \cdot e_1 = e_3, a \neq 0,$
$K_{4,4}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = ae_3, e_1 \cdot e_4 = e_3 + e_2 + ae_4, e_4 \cdot e_1 = e_2 + e_3, 0 \neq a \neq 1,$
$K_{4,5}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3, e_3 \cdot e_1 = e_2, e_1 \cdot e_4 = ae_4, 0 \neq a \neq 1,$
$K_{4,6}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3, e_3 \cdot e_1 = e_2, e_1 \cdot e_4 = e_2 + ae_4, e_4 \cdot e_1 = e_2, 0 \neq a \neq 1$
$L_{4,1}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3, e_1 \cdot e_4 = e_3 + e_4,$
$L_{4,2}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3, e_1 \cdot e_4 = ae_2 + e_3 + e_4, e_4 \cdot e_1 = ae_2, a \neq 0,$
$M_{4,1}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = ae_3, e_1 \cdot e_4 = e_3 + ae_4, a \neq 0,$
$M_{4,2}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = ae_3, e_1 \cdot e_4 = (b+1)\, e_3 + ae_4, e_4 \cdot e_1 = be_3, a, b \neq 0,$
$M_{4,3}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = ae_3, e_1 \cdot e_4 = e_2 + (b+1)\, e_3 + ae_4, e_4 \cdot e_1 = e_2 + be_3, a \neq 0,$
$M_{4,4}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_2 + e_3, e_3 \cdot e_1 = e_2, e_1 \cdot e_4 = (a+1)\, e_2 + e_4, e_4 \cdot e_1 = ae_2,$
$S_{4,1}: e_1 \cdot e_2 = ae_2, e_1 \cdot e_3 = be_3 - e_4, e_1 \cdot e_4 = e_3 + be_4, a > 0,$
$P_{4,1}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3, e_4 \cdot e_3 = e_2,$
$P_{4,2}: e_1 \cdot e_1 = te_3, e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3, e_4 \cdot e_3 = e_2, e_4 \cdot e_4 = e_3, t \geq 0,$
$P_{4,3}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3, e_4 \cdot e_3 = e_2, e_4 \cdot e_4 = e_2,$
$P_{4,4}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = e_3, e_1 \cdot e_4 = e_4 \cdot e_1 = e_3, e_4 \cdot e_3 = e_2, e_4 \cdot e_4 = e_2 + te_3, t \neq 0,$
$P_{4,5}: e_1 \cdot e_2 = e_2, e_1 \cdot e_3 = ae_2 + e_3, e_3 \cdot e_1 = ae_2, e_4 \cdot e_3 = e_2, a \neq 0,$
$U_{4,1}: e_1 \cdot e_2 = e_2, e_4 \cdot e_3 = e_3,$
$U_{4,2}: e_1 \cdot e_2 = e_2, e_4 \cdot e_3 = e_3, e_4 \cdot e_4 = e_3,$
$Z_{4,1}: e_1 \cdot e_2 = e_3, e_1 \cdot e_3 = -e_2, e_4 \cdot e_2 = e_2, e_4 \cdot e_3 = e_3.$

Remark 39. With the notation of Theorem 38, we notice that

1. All left-symmetric algebras $N_{4,i}$, $H_{4,i}$, $B_{4,i}$, $D_{4,i}$, $G_{4,i}$, $P_{4,i}$, $F_{4,1}$, $F_{4,2}$, $I_{4,1}$, $I_{4,2}$, $J_{4,1}$, $J_{4,2}$, $C_{4,1}$, $C_{4,2}$, $E_{4,1}$, $E_{4,2}$, $K_{4,1}$, $L_{4,1}$, $M_{4,1}$, $S_{4,1}$, $U_{4,1}$, $U_{4,2}$ and $Z_{4,1}$ satisfy the identity (8). Hence, they are all Novikov.
2. The algebras $N_{4,1}$, $H_{4,1}$, $I_{4,1}$ $J_{4,1}$, $B_{4,1}$, $C_{4,1}$, $D_{4,1}$, $E_{4,1}$, $G_{4,1}$, $K_{4,1}$, $L_{4,1}$, $M_{4,1}$, $S_{4,1}$, $P_{4,1}$, $U_{4,1}$, $U_{4,2}$ and $Z_{4,1}$ are inner derivation.

Corollary 40. *The associated Lie algebras of the left-symmetric algebras in Theorem 38 are:*

1. $N_{4,i}: \mathbb{R}^2 \times aff\,(\mathbb{R}): [e_1, e_2] = e_2.$
2. $H_{4,i}: \mathbb{R} \times \mathcal{H}_3: [e_1, e_2] = e_3.$
3. $F_{4,i}: [e_1, e_2] = e_3, [e_1, e_4] = e_2.$
4. $I_{4,i}: \mathbb{R} \times \mathcal{E}\,(1,1): [e_1, e_2] = e_2, [e_1, e_3] = -e_3.$
5. $J_{4,i}: \mathbb{R} \times \mathcal{E}\,(2): [e_1, e_2] = e_3, [e_1, e_3] = -e_2.$
6. $B_{4,i}: \mathbb{R} \times \mathcal{G}_{3,2}: [e_1, e_2] = e_2, [e_1, e_3] = e_3.$
7. $C_{4,i}: \mathbb{R} \times \mathcal{G}_{3,3}: [e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3.$
8. $D_{4,i}: \mathbb{R} \times \mathcal{G}_{3,4}^a: [e_1, e_2] = ae_2, [e_1, e_3] = e_3,\ 0 < |a| < 1.$
9. $E_{4,i}: \mathbb{R} \times \mathcal{G}_{3,5}^a: [e_1, e_2] = e_2 - ae_3, [e_1, e_3] = ae_2 + e_3,\ a > 0.$
10. $G_{4,i}: [e_1, e_2] = e_2, [e_1, e_4] = e_3.$
11. $K_{4,i}: [e_1, e_2] = e_2, [e_1, e_3] = ae_3, [e_1, e_4] = be_4,\ ab \neq 0.$
12. $L_{4,i}: [e_1, e_2] = e_2, [e_1, e_3] = e_2 + e_3, [e_1, e_4] = e_3 + e_4.$
13. $M_{4,i}: [e_1, e_2] = e_2, [e_1, e_3] = ae_3, [e_1, e_4] = e_3 + ae_4,\ a \neq 0.$
14. $S_{4,i}: [e_1, e_2] = ae_2, [e_1, e_3] = be_3 - e_4, [e_1, e_4] = e_3 + be_4,\ a > 0.$
15. $P_{4,i}: [e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_4, e_3] = e_2.$
16. $U_{4,i}: aff\,(\mathbb{R}) \times aff\,(\mathbb{R}): [e_1, e_2] = e_2, [e_4, e_3] = e_3.$
17. $Z_{4,1}: [e_1, e_2] = e_3, [e_1, e_3] = -e_2, [e_4, e_2] = e_2, [e_4, e_3] = e_3.$

Summarizing we have shown the following main theorem.

Theorem 41. *A four-dimensional derivation algebra over* $\mathbb{R}$ *is isomorphic either to an algebra as given in Propositions 8, 34, or Theorem 38.*

Remark 42. From Corollary 40, we find that every two-step solvable Lie algebra of dimension 4 can admit a derivation structure and we deduce that three-step solvable Lie algebras of dimension 4 do not admit any derivation structure compatible with the Lie structure.

Furthermore, up to dimension 6, we could not find any derivation structure on solvable nilpotent Lie algebras with order of solvability greater than two. In fact, we do not have any example of a Lie algebra with order of solvability greater than two which admit a derivation structure.

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