

## Characterizing spheres by an immersion in Euclidean spaces<sup>☆</sup>

SHARIEF DESHMUKH<sup>a,\*</sup>, IBRAHIM AL-DAYEL<sup>a,b</sup>

<sup>a</sup>Department of Mathematics, College of Science, King Saud University, P.O. Box-2455, Riyadh-11451, Saudi Arabia

<sup>b</sup>Department of Mathematics, College of Science, Al-Imam Muhammad Ibn Saud Islamic University, P.O. Box-65892, Riyadh-11566, Saudi Arabia

Received 4 September 2016; accepted 9 September 2016  
Available online 22 September 2016

**Abstract.** In this paper we study compact immersed orientable hypersurfaces in the Euclidean space  $R^{n+1}$  and show that suitable restrictions on the tangential and normal components of the immersion give different characterizations of the spheres.

Keywords: Hypersurface; Conformal vector field; Ricci curvature; Position vector field

2010 Mathematics Subject Classification: 53C15; 53B25

### 1. INTRODUCTION

One of the interesting questions in the geometry of hypersurfaces in a Euclidean space is to find necessary and sufficient conditions for the hypersurface to be isometric to a sphere. Let  $M$  be an immersed orientable hypersurface of the Euclidean space  $R^{n+1}$  with the immersion  $\psi : M \rightarrow R^{n+1}$ . If  $N$  is the unit normal vector field to the hypersurface, then we can express  $\psi$ , (the position vector field of points of  $M$  in  $R^{n+1}$ ), as  $\psi = \xi + \rho N$ , where  $\xi$  is vector field tangential to  $M$  and  $\rho = \langle \psi, N \rangle$  is the support function of  $M$ ,  $\langle \cdot, \cdot \rangle$  being the Euclidean metric on  $R^{n+1}$ . The immersion  $\psi$  of the hypersurface  $M$  naturally gives two vector fields:  $\xi$  and  $\nabla \rho$ , the gradient of the support function  $\rho$  with respect to the induced metric  $g$  on the hypersurface  $M$ . One naturally expects that these vector fields play a vital role in shaping the geometry of a hypersurfaces. Since Killing vector fields and conformal vector fields have

<sup>☆</sup> This work is supported by NSTIP strategic technologies program number (13-MAT1813-02) in the Kingdom of Saudi Arabia.

\* Corresponding author.

E-mail addresses: [shariefd@ksu.edu.sa](mailto:shariefd@ksu.edu.sa) (S. Deshmukh), [ibraheem96@live.com](mailto:ibraheem96@live.com) (I. Al-Dayel).

Peer review under responsibility of King Saud University.



Production and hosting by Elsevier

been used in finding characterizations of spheres (cf. [1,2,5,3]), one would like to impose the conditions on these vector fields on the hypersurface to be Killing vector fields or conformal vector fields. However, the vector fields  $\xi$  and  $\nabla\rho$  on the hypersurface  $M$  are both gradient vector fields ( $\xi = \nabla f$ , where  $f = \frac{1}{2} \|\psi\|^2$ ) and if they are Killing vector fields they will be parallel and will not yield interesting results (cf. Remark 2.1). Therefore a natural choice is to consider these vector fields to be conformal vector fields. Conformal vector fields have been used in the study of hypersurfaces of a Riemannian manifold (cf. [4,1,6,7]). In particular it has been observed that the conformal vector field on the ambient space are closely related to the totally umbilical hypersurfaces. In this paper, we wish to put constraints on these vector fields and analyze the effects of these restrictions on the geometry of the hypersurface  $M$ . It is interesting to note that the restrictions on the vector fields  $\xi, \nabla\rho$  on the compact hypersurface  $M$ , such as (i)  $\xi$  is a conformal vector field, (ii)  $\nabla\rho$  is a conformal vector field, (iii)  $\nabla\rho = \lambda\xi$ ,  $\lambda$  a constant (that is,  $\nabla\rho$  is parallel to  $\xi$ ), (iv)  $\xi \perp \nabla\rho$ , together with some suitable curvature restrictions, give respectively the characterizations of the spheres in  $R^{n+1}$  (cf. Theorems 3.1, 3.2, 4.1 and 4.2).

Trivial examples of such vector fields are the tangential and normal components of the natural embedding of the sphere  $S^n(c)$  in the Euclidean space. As another example consider the warped product  $M = (0, \infty) \times_t S^{n-1}$ , where  $t$  is the coordinate function on  $(0, \infty)$ . Then the map  $\varphi : M \rightarrow R^n - \{0\}$ ,  $\varphi(t, x) = tx$  is an isometry and satisfies

$$d\varphi \left( \frac{\partial}{\partial t} \right) = \sum u^i \frac{\partial}{\partial u^i},$$

where  $u^1, \dots, u^n$  are the Euclidean coordinates on  $R^n - \{0\}$ . The isometric embedding  $i : R^n - \{0\} \rightarrow R^{n+1}$ ,  $i(x) = (x, 0)$  gives the isometric embedding  $\psi : M \rightarrow R^{n+1}$ ,  $\psi = i \circ \varphi$  that satisfies

$$d\psi \left( t \frac{\partial}{\partial t} \right) = \sum tu^i \frac{\partial}{\partial u^i} = (\varphi(x), 0)$$

and consequently the vector field  $\xi$  on the hypersurface  $M$  given by  $\xi = t \frac{\partial}{\partial t}$  is easily seen to be a conformal vector field on  $M$  (cf. [8]).

## 2. PRELIMINARIES

Let  $M$  be an orientable compact immersed hypersurface of the Euclidean space  $R^{n+1}$ . We denote by  $\langle \cdot, \cdot \rangle$  the Euclidean metric on  $R^{n+1}$ , and by  $N, A$  and  $g$  the unit normal vector field, the shape operator and the induced metric on  $M$  respectively. If  $\psi : M \rightarrow R^{n+1}$  is the immersion, then we have  $\psi = \xi + \rho N$ , with  $\xi \in \mathfrak{X}(M)$ , where  $\mathfrak{X}(M)$  is the Lie algebra of smooth vector fields on  $M$ , and  $\rho = \langle \psi, N \rangle$  is called the support function of the hypersurface  $M$ . We denote by  $\nabla$  the covariant derivative operator with respect to the Riemannian connection on  $M$ . Taking the covariant derivative in equation  $\psi = \xi + \rho N$  and using the Gauss Weingarten formulas of the hypersurface, we get

$$\nabla_X \xi = X + \rho AX \quad \text{and} \quad \nabla \rho = -A\xi \quad X \in \mathfrak{X}(M), \tag{2.1}$$

where  $\nabla\rho$  is the gradient of  $\rho$  on the Riemannian manifold  $(M, g)$ .

The curvature tensor  $R$ , the Ricci tensor  $Ric$  and the scalar curvature  $S$  of the hypersurface  $M$  are given by

$$R(X, Y)Z = g(AY, Z)AX - g(AX, Z)AY \quad X, Y, Z \in \mathfrak{X}(M), \quad (2.2)$$

$$Ric(X, Y) = n\alpha g(AX, Y) - g(AX, AY) \quad X, Y \in \mathfrak{X}(M) \quad (2.3)$$

and

$$S = n^2\alpha^2 - \|A\|^2, \quad (2.4)$$

where  $\alpha = \frac{1}{n}tr A$  is the mean curvature of the hypersurface.

The shape operator  $A$  of the hypersurface  $M$  satisfies the Codazzi equation

$$(\nabla A)(X, Y) = (\nabla A)(Y, X) \quad X, Y \in \mathfrak{X}(M), \quad (2.5)$$

where the covariant derivative  $(\nabla A)(X, Y) = \nabla_X AY - A(\nabla_X Y)$ . Using the symmetry of the shape operator  $A$ , we see that the gradient  $\nabla\alpha$  of the mean curvature  $\alpha$  given by

$$n\nabla\alpha = \sum_{i=1}^n (\nabla A)(e_i, e_i), \quad (2.6)$$

where  $\{e_1, \dots, e_n\}$  is a local orthonormal frame on  $M$ .

Minkowski's formula for a compact hypersurface  $M$  is given

$$\int_M (1 + \rho\alpha) = 0. \quad (2.7)$$

Recall that a vector field  $u$  on a Riemannian manifold  $(M, g)$  is said to be a conformal vector field if the flow  $\{\Psi_t\}$  of  $u$  consists of conformal transformations of  $(M, g)$ . Thus  $u$  is a conformal vector field if and only if  $\mathcal{L}_u g = 2\sigma g$ , where  $\sigma$  is a smooth function on  $M$  called the potential function of the conformal vector field  $u$  and  $\mathcal{L}_u$  is the Lie derivative with respect to  $u$ . In addition if  $u = \nabla f$  for a smooth function  $f$ , then the conformal vector field  $u$  is said to be a gradient conformal vector field. If  $u$  is a gradient conformal vector field, then we have

$$\nabla_X u = \sigma X \quad X \in \mathfrak{X}(M). \quad (2.8)$$

If  $u$  is a gradient conformal vector field on a compact Riemannian manifold  $(M, g)$ , then by Eq. (2.8) it follows that the potential function  $\sigma$  satisfies

$$\int_M \sigma = 0. \quad (2.9)$$

As a trivial consequence of the Minkowski's formula we get the following result:

**Lemma 2.1.** *Let  $M$  be a compact connected orientable hypersurface of the Euclidean space  $R^{n+1}$ . Then the function  $\rho\alpha$  is a constant if and only if  $M$  is isometric to the sphere  $S^n(c)$  of constant curvature  $c$ .*

**Proof.** If  $\rho\alpha$  is a constant, then Eq. (2.7) gives

$$1 + \rho\alpha = 0. \quad (2.10)$$

Let  $\psi : M \rightarrow R^{n+1}$  be the immersion of the hypersurface. Define the smooth function  $f : M \rightarrow R$  by  $f = \frac{1}{2} \|\psi\|^2$ , which immediately gives the gradient  $\nabla f = \xi$  where  $\psi = \xi + \rho N$ . Using Eq. (2.1) to compute the divergence of the vector field  $\xi$ , we arrive at

$$\Delta f = n(1 + \rho\alpha),$$

where  $\Delta$  is the Laplace operator on  $M$ . Using Eq. (2.10) in the above equation we get  $\Delta f = 0$  on compact  $M$ . Hence  $\|\psi\|^2 = a$  constant, and we conclude that  $M$  is isometric to the  $n$ -sphere  $S^n(c)$ . The converse is trivial.

**Remark 2.1.** Recall that a conformal vector field  $u$  on a Riemannian manifold  $(M, g)$  is said to be a Killing vector field if its potential function  $\sigma = 0$ . In particular, if a Killing vector field  $u$  is a gradient of some smooth function, then using Koszul's formula we get

$$\nabla_X u = 0, \quad X \in \mathfrak{X}(M),$$

that is, a gradient Killing vector field is always parallel.

We have the following trivial characterizations of spheres in the Euclidean space.

**Proposition 2.1.** *Let  $M$  be a compact connected orientable immersed hypersurface of the Euclidean space  $R^{n+1}$  with the immersion  $\psi : M \rightarrow R^{n+1}$ . Then the tangential component  $\xi$  of  $\psi$  is a Killing vector field if and only if  $M$  is isometric to the sphere  $S^n(c)$  of constant curvature  $c$ .*

**Proof.** As in the proof of Lemma 2.1, we have  $\xi = \nabla f$ . Thus  $\xi$  is a gradient Killing vector field and consequently it is a parallel vector field and we get  $\Delta f = 0$ . This implies  $f$  is a constant and hence  $M$  is isometric to a sphere  $S^n(c)$ . The converse is trivial.

**Proposition 2.2.** *Let  $M$  be a compact connected orientable immersed hypersurface of positive Ricci curvature in the Euclidean space  $R^{n+1}$  with the immersion  $\psi : M \rightarrow R^{n+1}$ ,  $\psi = \xi + \rho N$ . Then the vector field  $\nabla \rho$  is a Killing vector field if and only if  $M$  is isometric to the sphere  $S^n(c)$  of constant curvature  $c$ .*

**Proof.** If  $\nabla \rho$  is a Killing vector field, we have  $\Delta \rho = 0$ , consequently  $\rho$  is a constant. Then Eq. (2.1) gives  $A\xi = 0$ , which together with Eq. (2.3) gives  $Ric(\xi, \xi) = 0$ . Consequently the hypothesis implies that  $\xi = \nabla f = 0$ , that is  $f$  is a constant. Hence  $M$  is isometric to the sphere  $S^n(c)$  of constant curvature  $c$ . The converse is trivial.

### 3. HYPERSURFACE WITH $\xi$ AND $\nabla \rho$ CONFORMAL VECTOR FIELDS

In this section, we use the tangential and normal components of the immersion  $\psi : M \rightarrow R^{n+1}$  of a compact and connected hypersurface  $M$  such that the vector fields  $\xi$  and  $\nabla \rho$  are conformal vector fields, to find characterizations of the spheres in  $R^{n+1}$ . Recall that, as noted in the proof of Lemma 2.1,  $\xi = \nabla f$ , where  $f = \frac{1}{2} \|\psi\|^2$ . Thus, if  $\xi$  is a conformal vector field, it will be a gradient conformal vector field. First, we prove the following.

**Theorem 3.1.** *Let  $M$  be an orientable compact connected immersed hypersurface of  $R^{n+1}$  with the immersion  $\psi : M \rightarrow R^{n+1}$ . Then the support function  $\rho$  is nowhere zero and the*

tangential component  $\xi$  of  $\psi$  is a conformal vector field with the potential function  $\sigma \neq 1$  on  $M$ , if and only if  $M$  is isometric to the sphere  $S^n(c)$ .

**Proof.** Suppose  $\rho(p) \neq 0$ ,  $p \in M$  and  $\xi$  is a conformal vector field with potential function  $\sigma \neq 1$ . As  $\xi = \nabla f$ , using Eqs. (2.1) and (2.8), we get

$$\rho AX = (\sigma - 1)X, \quad X \in \mathfrak{X}(M). \quad (3.1)$$

Also, Eq. (2.8) gives

$$\nabla_X \nabla_Y \xi = X(\sigma)Y + \sigma \nabla_X Y$$

and consequently

$$R(X, Y)\xi = X(\sigma)Y - Y(\sigma)X.$$

Using Eqs. (2.1) and (2.2), we arrive at

$$X(\rho)AY - Y(\rho)AX = X(\sigma)Y - Y(\sigma)X, \quad X, Y \in \mathfrak{X}(M).$$

The above equation together with Eq. (3.1) implies

$$(\sigma - 1)(X(\rho)Y - Y(\rho)X) = \rho(X(\sigma)Y - Y(\sigma)X),$$

which, by contraction in  $X$  gives

$$(\sigma - 1)Y(\rho) = \rho Y(\sigma), \quad Y \in \mathfrak{X}(M).$$

Thus

$$\frac{\nabla \rho}{\rho} = \frac{\nabla \sigma}{\sigma - 1}$$

since  $\rho(p) \neq 0$ ,  $\sigma(p) \neq 1$ ,  $p \in M$ .

As  $M$  is connected, the above equation implies

$$\nabla \ln\left(\frac{\sigma - 1}{\rho}\right) = 0 \quad \text{or} \quad \nabla \ln\left(\frac{1 - \sigma}{\rho}\right) = 0,$$

the first choice for  $\rho(\sigma - 1) > 0$  on  $M$  and the second for  $\rho(\sigma - 1) < 0$ . Thus, we conclude that  $\sigma - 1 = k\rho$  for a constant  $k$ . Inserting this value of  $\sigma - 1$  into Eq. (3.1), we arrive at

$$\rho(AX - kX) = 0, \quad X \in \mathfrak{X}(M). \quad (3.2)$$

Since  $M$  is connected we have either  $\rho = 0$  or  $A = kI$ . However, Minkowski's formula (2.7) rules out  $\rho = 0$ , and hence  $A = kI$ , that is,  $M$  is a totally umbilical hypersurface of  $R^{n+1}$ . Hence  $M$  is isometric to  $S^n(c)$ , where  $c = k^2 > 0$  (as  $k = 0$  will imply  $A = 0$ , that is  $M$  is totally geodesic, which is impossible as  $R^{n+1}$  does not admit compact totally geodesic hypersurfaces).

Conversely, if  $M$  is isometric to a sphere  $S^n(c)$  in  $R^{n+1}$ , then  $\xi = 0$ ,  $\sigma = 0$  and the support function  $\rho = \frac{1}{\sqrt{c}} \neq 0$ .

**Theorem 3.2.** *Let  $M$  be an orientable compact and connected immersed hypersurface of positive Ricci curvature in the Euclidean space  $R^{n+1}$  with support function  $\rho$ . If  $\nabla\rho$  is a conformal vector field with potential function  $\sigma$  and the Ricci curvature in the direction of the vector field  $\nabla\sigma$  is bounded above by the constant  $(n-1)c$ , where the constant  $c = \frac{\lambda_1}{n}$ ,  $\lambda_1$  being the first nonzero eigenvalue of the Laplace operator, then  $M$  is isometric to  $S^n(c)$ .*

**Proof.** Suppose  $\nabla\rho$  is a conformal vector field with potential function  $\sigma$ . Then

$$\nabla_X(\nabla\rho) = \sigma X, \quad X \in \mathfrak{X}(M), \quad (3.3)$$

and hence

$$R(X, Y)\nabla\rho = X(\sigma)Y - Y(\sigma)X.$$

Using Eq. (2.2) in the above equation, we get

$$g(A Y, \nabla\rho) A X - g(A X, \nabla\rho) A Y = X(\sigma)Y - Y(\sigma)X.$$

Inserting  $\nabla\rho = -A\xi$ , which is Eq. (2.1), in the above equation and contracting in  $X$ , yields

$$-n\alpha g(A^2\xi, Y) + g(A^3\xi, Y) = -(n-1)Y(\sigma),$$

that is,

$$Ric(A\xi, Y) = (n-1)Y(\sigma).$$

The above equation gives

$$Ric(\nabla\rho, \nabla\rho) = -(n-1)g(\nabla\rho, \nabla\sigma), \quad (3.4)$$

and

$$Ric(\nabla\rho, \nabla\sigma) = -(n-1)\|\nabla\sigma\|^2. \quad (3.5)$$

Note that

$$\operatorname{div}(\sigma\nabla\rho) = g(\nabla\rho, \nabla\sigma) + \sigma\Delta\rho,$$

which, together with Eq. (3.4), gives

$$\int_M Ric(\nabla\rho, \nabla\rho) = (n-1) \int_M \sigma\Delta\rho = n(n-1) \int_M \sigma^2. \quad (3.6)$$

Here we have used Eq. (3.3), which implies  $\Delta\rho = n\sigma$ .

If  $\lambda_1$  is the first nonzero eigenvalue of the Laplace operator on  $M$ , then Eq. (2.9) gives

$$\int_M \|\nabla\sigma\|^2 \geq \lambda_1 \int_M \sigma^2. \quad (3.7)$$

Thus

$$\begin{aligned} & \int_M Ric(\nabla(\sigma + c\rho), \nabla(\sigma + c\rho)) \\ &= \int_M Ric(\nabla\sigma, \nabla\sigma) + 2cRic(\nabla\sigma, \nabla\rho) + c^2 Ric(\nabla\rho, \nabla\rho). \end{aligned}$$

Using Eqs. (3.5)–(3.7) in the above equation with  $c = \frac{\lambda_1}{n}$ , and the condition in the hypothesis, we get

$$\begin{aligned} \int_M Ric(\nabla(\sigma + c\rho), \nabla(\sigma + c\rho)) &\leq \int_M \left\{ Ric(\nabla\sigma, \nabla\sigma) - 2(n-1)c \|\nabla\sigma\|^2 \right. \\ &\quad \left. + n(n-1) \frac{c^2}{\lambda_1} \|\nabla\sigma\|^2 \right\} \\ &= \int_M Ric(\nabla\sigma, \nabla\sigma) - (n-1)c \|\nabla\sigma\|^2 \leq 0. \end{aligned}$$

Since  $M$  has positive Ricci curvature, we have

$$\nabla\sigma + c\nabla\rho = 0,$$

where  $c = \frac{\lambda_1}{n}$  is a positive constant.

Using Eq. (3.3), we get

$$\nabla_X(\nabla\sigma) = -c\sigma X. \tag{3.8}$$

If  $\sigma$  is a constant, then by Eq. (2.9), we get  $\sigma = 0$ , hence  $\Delta\rho = 0$ , which implies  $\rho$  is a constant. Using Eq. (2.1), we have  $A\xi = 0$ , and consequently Eq. (2.3) gives

$$Ric(\xi, \xi) = 0.$$

Since the Ricci curvature is positive, the above equation can only be satisfied if  $\xi = 0$ . In view of Eq. (2.1) we conclude that  $A = -\frac{1}{\rho}I$  (as Minkowski’s formula does not allow  $\rho = 0$ ) and hence  $M$  is isometric to  $S^n(c)$ .

If  $\sigma$  is not a constant, then Eq. (3.8) is Obata’s equation with positive constant  $c$ , and consequently  $M$  is isometric to  $S^n(c)$ .

#### 4. HYPERSURFACES WITH CERTAIN RELATIONS IN $\xi$ AND $\nabla\rho$

In this section we study the geometry of the compact immersed hypersurfaces in the Euclidean space  $R^{n+1}$  for which the vector fields  $\xi$  and  $\nabla\rho$  are parallel or orthogonal. First we consider the case  $\nabla\rho = \lambda\xi$  for a constant  $\lambda$ . Note that this condition is equivalent to requiring that  $\xi$  is an eigenvector of the shape operator  $A$  with constant eigenvalue  $-\lambda$ , that is,  $\xi$  is a principal direction with principal curvature  $-\lambda$ .

**Theorem 4.1.** *Let  $M$  be an orientable compact and connected immersed hypersurface in the Euclidean space  $R^{n+1}$  ( $n > 1$ ) with support function  $\rho$ , and let  $\xi$  be the tangential component of the immersion. Then  $\nabla\rho = \lambda\xi$  for a constant  $\lambda \neq 0$  and the Ricci curvature  $Ric(\xi, \xi) \geq n(n-1)(\rho^2\alpha^2 - 1)$  if and only if  $M$  isometric to  $S^n(c)$ .*

**Proof.** Suppose  $\nabla\rho = \lambda\xi$  and  $Ric(\xi, \xi) \geq n(n-1)(\rho^2\alpha^2 - 1)$  holds. Let  $A_\rho X = \nabla_X \nabla\rho$ ,  $X \in \mathfrak{X}(M)$ , be the Hessian operator of the function  $\rho$ . Then using Eqs. (2.1) and  $\nabla\rho = \lambda\xi$ , we get

$$A_\rho = \lambda I + \lambda\rho A, \quad (4.1)$$

that is,

$$\Delta\rho = n\lambda(1 + \rho\alpha). \quad (4.2)$$

Bochner's formula for the smooth function  $\rho$  is

$$\int_M \left\{ Ric(\nabla\rho, \nabla\rho) + \|A_\rho\|^2 - (\Delta\rho)^2 \right\} = 0,$$

which gives

$$\int_M \left\{ \left( \lambda^2 Ric(\xi, \xi) - \left( \frac{n-1}{n} \right) (\Delta\rho)^2 \right) + \left( \|A_\rho\|^2 - \frac{1}{n} (\Delta\rho)^2 \right) \right\} = 0.$$

Inserting Eq. (4.2) in the middle term of the above equation, we have

$$\int_M \left\{ \lambda^2 Ric(\xi, \xi) - n(n-1)\lambda^2(1 + 2\rho\alpha + \rho^2\alpha^2) + \left( \|A_\rho\|^2 - \frac{1}{n} (\Delta\rho)^2 \right) \right\} = 0.$$

Using Minkowski's formula (2.7) in the above equation, we get

$$\int_M \left\{ \lambda^2 (Ric(\xi, \xi) - n(n-1)(\rho^2\alpha^2 - 1)) + \left( \|A_\rho\|^2 - \frac{1}{n} (\Delta\rho)^2 \right) \right\} = 0. \quad (4.3)$$

Note that the Schwartz inequality gives

$$\|A_\rho\|^2 \geq \frac{1}{n} (\Delta\rho)^2$$

with the equality holding if and only if  $A_\rho = \left(\frac{\Delta\rho}{n}\right)I$ . Thus, using the condition in the hypothesis and the above inequality in Eq. (4.3), we conclude that

$$A_\rho = \left(\frac{\Delta\rho}{n}\right)I,$$

which together with Eqs. (4.1) and (4.2) gives

$$\lambda\rho A = \left(\frac{\Delta\rho}{n} - \lambda\right)I = (\lambda(1 + \rho\alpha) - \lambda)I = \lambda\rho\alpha I.$$

Hence  $\rho(A - \alpha I) = 0$ , as  $\lambda \neq 0$ . On the connected  $M$ , we have either  $\rho = 0$  or  $A = \alpha I$ . However  $\rho = 0$  in Minkowski's formula (2.7) gives a contradiction. Hence  $A = \alpha I$ . In view of Codazzi equation (2.5) and  $n > 1$ , it follows that  $\alpha$  is a constant, that is,  $M$  is a totally umbilical hypersurface of  $R^{n+1}$ . Hence  $M$  is isometric to  $S^n(c)$ , with  $c = \alpha^2 > 0$ , as  $R^{n+1}$  does not admit a compact hypersurface with  $\alpha = 0$ .



Conversely, if  $M$  is isometric to  $S^n(c)$ , we have  $\xi = 0$ ,  $\alpha\rho = -1$  and  $\rho = \text{constant}$ . Thus both conditions are met.

Finally, we have the following trivial characterization of the spheres in  $R^{n+1}$  for the case that the vector fields  $\nabla\rho$  and  $\xi$  are orthogonal.

**Theorem 4.2.** *Let  $M$  be an orientable compact and connected immersed hypersurface of positive Ricci curvature and the immersion  $\psi : M \rightarrow R^{n+1}$  is expressed as  $\psi = \xi + \rho N$ . Then the vector fields  $\nabla\rho$  and  $\xi$  are orthogonal on  $M$  if and only if  $M$  isometric to  $S^n(c)$ .*

**Proof.** Suppose  $g(\nabla\rho, \xi) = 0$ . Then we have  $g(A\xi, \xi) = 0$  and consequently,  $Ric(\xi, \xi) = -\|A\xi\|^2$ . As the Ricci curvature is positive we have  $\xi = 0$  and  $A\xi = 0$ . As we have  $\nabla\rho = -A\xi = 0$ , we get  $\rho = \text{constant}$ . Moreover it follows from Minkowski's formula that  $\rho$  is a nonzero constant. Thus Eq. (2.1) gives  $A = -\frac{1}{\rho}I$ , that is,  $M$  is a totally umbilical hypersurface and hence isometric to  $S^n(c)$ . The converse is trivial.

## REFERENCES

- [1] S. Deshmukh, A note on compact hypersurfaces in a Euclidean space, C. R. Acad. Sci., Paris, I 350 (2012) 971–974.
- [2] S. Deshmukh, A note on hypersurfaces in a sphere, Monatsh. Math. 174 (2014) 413–426.
- [3] S. Deshmukh, Conformal vector fields and eigenvectors of Laplace operator, Math. Phys. Anal. Geom. 15 (2012) 163–172.
- [4] S. Deshmukh, Real hypersurfaces in nearly Kaehler 6-sphere, Mediterr. J. Math. 11 (2014) 789–797.
- [5] S. Deshmukh, F. Alsolamy, Conformal vector fields on a Riemannian manifold, Balkan J. Geom. Appl. 19 (2) (2014) 86–93.
- [6] Y. Katsurada, On a certain property of closed hypersurfaces in an Einstein space, Comment. Math. Helv. 38 (1964) 165–171.
- [7] D. Kim, Y.H. Kim, S. Park, Conformal vector fields and totally umbilic hypersurface, Bull. Korean Math. Soc. 39 (4) (2002) 671–680.
- [8] B. O'Neill, Semi-Riemannian Geometry, A.P., 1983.