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Characterization of self-adjoint domains for differential operators with interior singular points

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Abstract. We characterize the self-adjoint domains of general even order linear ordinary differential operators which have finite interior singular points in terms of real-parameter solutions of the differential equation. For the purpose we constructed a direct sum space. By the theory of direct sum space and the decomposition of the corresponding maximal domain, we give this complete and analytic characterization in terms of limit-circle solutions. This is for endpoints which are regular or singular and for arbitrary deficiency index.

Keywords: Differential operators; Interior singular points; Deficiency index; Self-adjoint domains; Real-parameter solutions

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1. Introduction

W.N. Everitt and A. Zettl in [3] developed a theory of self-adjoint realizations of Sturm–Liouville problems on two intervals in the direct sum of Hilbert spaces associated with these intervals, for solving the Sturm–Liouville eigenvalue problems with interior singular points. In 1988, A.M. Krall and A. Zettl in [7] generalized the method given by Coddington [1], which obtains the characterization of self-adjoint domains by describing the boundary conditions of the domain of a conjugate differential operator, and obtains the characterization of self-adjoint domains for Sturm–Liouville differential operators with interior singular points.

As noted in [3], a simple way of getting self-adjoint operators in a direct sum Hilbert space is to take the direct sum of self-adjoint operators from each of the separate Hilbert spaces. However, there are many self-adjoint operators which are not merely the sum of self-adjoint

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operators from each of the separate intervals. These "new" self-adjoint operators involve interactions between the two intervals. Therefore in [3] the authors develop a "two-interval" theory. In particular, they characterized self-adjoint extensions of the minimal operator in the direct sum space in terms of boundary conditions. This theory was extended in [4] to higher order equations and any number of intervals, finite or infinite.

As in the case with no interior singular point the GKN characterization (see [2]) depends on maximal domain vectors. These vectors depend on the coefficients of each differential equation and this dependence is implicit and complicated. In [10] Wang, Sun and Zettl give an explicit characterization of all self-adjoint domains for singular problems in terms of the LC solutions for real λ when one endpoint a is regular and the other b is singular. Under the assumption that the differential equation $My = \lambda wy$ has d linearly independent solutions in H for some real λ, for $m = 2d-2k$, they constructed solutions $u_1, \dots, u_m$ and $u_{m+1}, \dots, u_d$ of the equation all lying in H such that the solutions u_j for $j > m$ do not contribute to the boundary conditions at the singular endpoint b and the solutions $u_1, \dots, u_m$ do contribute. Thus, in analogy with the celebrated Weyl limit-point (LP) and limit-circle (LC) cases for second order i.e. Sturm–Liouville problems, we say that the solutions $u_1, \dots, u_m$ are of LC type at b and $u_{m+1}, \dots, u_d$ are of LP type at b. Following [10], Hao, Wang, Sun and Zettl give a new characterization by dividing (a, b) into two intervals (a, c) and (c, b) for some $c \in (a, b)$ and using the LC solutions on each interval constructed in [10] when a and b are singular in [6]. In [9], Suo and Wang extend the characterization in [6] to two-interval case when one or two or three or four endpoints of two interval $(a_1, b_1) \cup (a_2, b_2)$ are regular and illustrate the interactions between the regular points and singular points with some examples.

In this paper we extend the characterization in [9] to the case when the differential operators which have finite interior singular points. For the purpose we firstly construct a direct sum space $H = \sum_{r=1}^{q} \oplus L^2((a_r, b_r), w_r)$ and give the corresponding notations and basic facts for direct sum space differential operators. On each internal (a_r, b_r), we choose a point c_r, $r = 1, 2, \dots, q$. Then we apply the construction in [10] on the interval (a_r, c_r) to obtain LC solutions u_{r1}, …, u_{rm_r}, and we apply the construction in [10] on the interval (c_r, b_r) to obtain LC solutions v_{r1}, …, v_{rn_r}. Using the LC solutions u_{r1}, …, u_{rm_r} and v_{r1}, …, v_{rn_r} for the left endpoint a_r and the right endpoint b_r of each of the q intervals $[a_r, b_r]$, $r = 1, 2, \dots, q$, we give the characterizations of all self-adjoint domains for singular symmetric operators with $q-1$ interior singular points or equivalently, all self-adjoint restrictions of the singular maximal operators in direct sum space in terms of the LC solutions of the $2r$ endpoints. These extensions yield "new" self-adjoint operators which are not merely direct sums of self-adjoint operators from the subintervals but involve interactions between the subintervals. These interactions are the interactions between singular endpoints. These will be illustrated in Section 4 with several examples.

2. Notations and preliminaries

Consider the even order symmetric differential expression

$$M = \sum_{j=0}^{n} p_j(x) D^j$$

over interval $I = (a, b)$, $-\infty < a < b < \infty$, where $p_j(x)$, $j = 0, 1, \dots, n$, are real-valued functions with some smooth and integrable conditions. We assume that there exists $q-1 (1 \le q < +\infty)$ singular points of M in I.

Without loss of generality, assume that the interval I is decomposed into a set of subintervals,

$$I_r = (a_r, b_r), \quad r = 1, \ldots, q,$$

where $a_1 = a$, $a_2 = b_1$, …, $a_q = b_{q-1}$ and $b_q = b$. In addition, a_r, b_r are singular endpoints with deficiency indices (m_{2r-1}, m_{2r-1}) and (m_{2r}, m_{2r}), respectively.

In general, we assume that $I_r = (a_r, b_r),\ r = 1, \ldots, q$, are a set of intervals on the real axis. An nth-order symmetric differential expression M_r is defined on every I_r for any r, and we provide M with deficiency indices (m_{2r-1}, m_{2r-1}) and (m_{2r}, m_{2r}) at a_r and b_r, respectively, and there is not any singular point in (a_r, b_r).

Let $M = (M_1, \ldots, M_q)$.

In this paper we only consider even order equations with real coefficients. However, in the following we summarize some basic facts about general quasi-differential equations of even and odd order and real or complex coefficients for the convenience of the reader.

Let $I = (a, b)$ be an interval with $-\infty \le a < b \le \infty$ and n be a positive integer (even or odd) and let

$$\begin{aligned} Z_n(I) := \{&Q = (q_{rs})_{r,s=1}^n, q_{r,r+1} \neq 0 \ a.e. \text{ on } I, q_{r,r+1}^{-1} \in L_{loc}(I), 1 \le r \le n-1, \\ &q_{rs} = 0 \ a.e. \text{ on } I, 2 \le r+1 < s \le n, q_{rs} \in L_{loc}(I), \\ &s \neq r+1, 1 \le r \le n-1\}. \end{aligned} \tag{1}$$

Let $Q \in Z_n(I)$. We define

$$V_0 := \{y : I \to \mathbb{C}, y \text{ is measurable}\} \tag{2}$$

and

$$y^{[0]} := y \ \ (y \in V_0). \tag{3}$$

Inductively, for $r = 1, \ldots, n$, we define

$$V_r := \{y \in V_{r-1} : y^{[r-1]} \in AC_{loc}(I)\}, \tag{4}$$

$$y^{[r]} = q_{r,r+1}^{-1}\Big(y^{[r-1]'} - \sum_{s=1}^{r} q_{rs} y^{[s-1]}\Big) \quad (y \in V_r), \tag{5}$$

where $q_{n,n+1} := 1$, and $AC_{loc}(I)$ denotes the set of complex valued functions which are absolutely continuous on all compact subintervals of I. Finally we set

$$My = M_Q y := i^n y^{[n]} \quad (y \in V_n). \tag{6}$$

The expression $M = M_Q$ is called the quasi-differential expression associated with Q. For V_n we also use the notations $V(M)$ and $D(Q)$. The function $y^{[r]}(0 \le r \le n)$ is called the rth quasi-derivative of y. Since the quasi-derivative depends on Q, we sometimes write $y_Q^{[r]}$ instead of $y^{[r]}$.

Remark 2.1. The operator $M : D(Q) \to L_{loc}(I)$ is linear.

Let $Z_n(I, \mathbb{R})$ denote the matrices $Q \in Z_n(I)$ which have real valued components.

Definition 2.2. Let $Q \in Z_n(I,\mathbb{R})$ and let $M = M_Q$ be defined as above. Assume that

$$Q = -E^{-1}Q^*E, \quad \text{where } E = ((-1)^r \delta_{r,n+1-s})_{r,s=1}^n. \tag{7}$$

Then $M = M_Q$ is called a symmetric differential expression.

Definition 2.3. Assume $Q \in Z_n(I,\mathbb{R})$ satisfies (7) and let $M = M_Q$ be the associated symmetric expression. Let $w \in L_{loc}(I)$ be positive a.e. on I. Define

$$\begin{aligned} &D_{max} = \{y \in L^2(I,w) : y \in D(Q), w^{-1}My \in L^2(I,w)\}, \\ &S_{max}y = w^{-1}My, y \in D_{max}. \\ &S_{min} = S^*_{max}, D_{min} = D(S_{min}). \end{aligned} \tag{8}$$

Lemma 2.4 (*Lagrange Identity*)**.** *Assume $Q \in Z_n(I,\mathbb{R})$, $n = 2k$, satisfies (7) and let $M = M_Q$ be the corresponding differential expression. Then for any $y, z \in D(Q)$ we have*

$$\overline{z}My - y\overline{Mz} = [y,z]', \tag{9}$$

where

$$[y,z] = (i)^k \sum_{r=0}^{n-1} (-1)^{n+1-r} \overline{z}^{[n-r-1]} y^{[r]} = (i)^k (Z^*EY), \tag{10}$$

$$Y = \begin{pmatrix} y \\ y^{[1]} \\ \vdots \\ y^{[n-1]} \end{pmatrix}, \qquad Z = \begin{pmatrix} z \\ z^{[1]} \\ \vdots \\ z^{[n-1]} \end{pmatrix}. \tag{11}$$

Definition 2.5 (*Regular Endpoints*)**.** Let $Q \in Z_n(I,\mathbb{R})$, $I = (a,b)$. The expression $M = M_Q$ is said to be regular at a if for some c, $a < c < b$, we have

$$q^{-1}_{r,r+1} \in L(a,c), r = 1,\ldots,n-1; \quad q_{rs} \in L(a,c), 1 \le r,s \le n, s \ne r+1.$$

Similarly the endpoint b is regular if for some c, $a < c < b$, we have

$$q^{-1}_{r,r+1} \in L(c,b), r = 1,\ldots,n-1; \quad q_{rs} \in L(c,b), 1 \le r,s \le n, s \ne r+1.$$

Note that, from (1) it follows that if the above hold for some $c \in I$ then they hold for any $c \in I$. We say M is regular on I, or just M is regular, if M is regular at both endpoints.

Theorem 2.6 (*GKN Theorem*)**.** *Let $S_{\min}$ be the minimal operator in H and let d be the deficiency index of $S_{\min}$. A linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exist vectors $w_1, w_2, \ldots, w_d$ in $D_{\max}$ satisfying the following conditions:*

(i) *$w_1, w_2, \ldots, w_d$ are linearly independent modulo $D_{\min}$;*
(ii) *$[w_i, w_j](b) - [w_i, w_j](a) = 0$, $i, j = 1, \ldots, d$;*
(iii) *$D(S) = \{y \in D_{\max} : [y, w_i](b) - [y, w_i](a) = 0, i = 1, \ldots, d\}$.*

Now we revert to the our case. Let $w_r \in L_{loc}(I_r),\ r = 1, \ldots, q$, *the basic space we considered is*

$$H = \sum_{r=1}^{q} \oplus L^2(I_r, w_r), \quad w_r > 0.$$

We define the inner product in H *as*

$$\langle y, z \rangle = \sum_{r=1}^{q} \langle y_r, z_r \rangle_r = \sum_{r=1}^{q} \int_{a_r}^{b_r} y_r \overline{z}_r w_r dx,$$

where $y = (y_1, \ldots, y_q)$, $z = (z_1, \ldots, z_q) \in H$. *Then* H *is a weighted Hilbert space with this inner product.*

Define the maximal operator $S_{\max}$ *generated by* M *in* H *as follows:*

$$D_{\max} = D(S_{\max}) = \sum_{r=1}^{q} \oplus D(S_{r\max}),$$

$S_{\max} y = (w_1^{-1} M_1 y_1, \ldots, w_q^{-1} M_q y_q),\ y = (y_1, \ldots, y_q) \in D_{\max}$.

Define the minimal operator $S_{\min}$ *generated by* M *in* H *as follows:*

$$D_{\min} = D(S_{\min}) = \sum_{r=1}^{q} \oplus D(S_{r\min}),$$

$S_{\min} y = (w_1^{-1} M_1 y_1, \ldots, w_q^{-1} M_q y_q),\ y = (y_1, \ldots, y_q) \in D_{\min}$.

For any $y = (y_1, \ldots, y_q)$ *and* $z = (z_1, \ldots, z_q) \in D_{\max}$, *all of the limits* $[y_r, z_r]_r(a_r) = \lim_{x \to a_r} [y_r, z_r]_r(x)$ *and* $[y_r, z_r]_r(b_r) = \lim_{x \to b_r} [y_r, z_r]_r(x)$ *exist and*

$$[y, z] = \sum_{r=1}^{q} [y_r, z_r]_r \Big|_{a_r}^{b_r}, \tag{12}$$

where

$$[y_r, z_r]_r = (-1)^k (Z_r^* E Y_r), \quad Y_r = \begin{pmatrix} y_r \\ y_r^{[1]} \\ \vdots \\ y_r^{[n-1]} \end{pmatrix}, Z_r = \begin{pmatrix} z_r \\ z_r^{[1]} \\ \vdots \\ z_r^{[n-1]} \end{pmatrix}. \tag{13}$$

Lemma 2.7. *The minimal operator* $S_{\min}$ *is a closed, symmetric, densely defined operator in the Hilbert space* H *with deficiency index* $d = \sum_{r=1}^{q} d_r$. *Here* d_r *is the deficiency indices of* $S_{r\min}$ $(r = 1, \ldots, q)$.

Proof. See [3]. □

Definition 2.8. Assume that $a_r \le \alpha_r < \beta_r \le b_r$ and $S_{r\min}$ are defined on (α_r, β_r) as above. Then the deficiency indexes d_r of $S_{r\min}$ are the number of linearly independent solutions of

$$M_r y = i w_r y \text{ on } (\alpha_r, \beta_r), \quad i = \sqrt{-1},\ r = 1, \ldots, q,$$

which lie in $L^2((\alpha_r, \beta_r), w_r)$.

Lemma 2.9. *Let $a_r \le \alpha_r < \beta_r \le b_r$. The number d_r of linearly independent solutions of*

$$M_r y = \lambda_r w_r y \quad \text{on } (\alpha_r, \beta_r) \tag{14}$$

lying in $L^2((\alpha_r, \beta_r), w_r)$ is independent of $\lambda_r \in \mathbb{C}$, provided $\mathrm{Im}\lambda_r \neq 0$. If one endpoint of (α_r, β_r) is regular and the other is singular, then the inequalities

$$k \le d_r \le 2k = n \tag{15}$$

hold. For $\lambda = \lambda_r \in \mathbb{R}$, the number of linearly independent solutions of (14) lying in $L^2((\alpha_r, \beta_r), w_r)$ is less than or equal to d_r.

Let $c_r \in (a_r, b_r) = I_r, r = 1, \ldots, q$. If d_{r1} is the deficiency index on (a_r, c_r), d_{r2} is the deficiency index on (c_r, b_r) and d_r is the deficiency index on (a_r, b_r), then

$$d_r = d_{r1} + d_{r2} - n, \quad r = 1, \ldots, q. \tag{16}$$

Proof. Cf. Lemma 4.3 in [9]. □

Lemma 2.10. *Suppose M_r are regular at c_r. Then for any $y = (y_1, \ldots, y_q) \in D_{\max}$ the limits*

$$y_r^{[j]}(c_r) = \lim_{t \to c_r} y^{[j]}(t), \quad r = 1, \ldots, q$$

exist and are finite, $j = 0, 1, \ldots, n-1$. In particular this holds at any regular endpoint and at each interior point of I_r. At an endpoint the limit is the appropriate one sided limit.

Proof. This follows from Lemma 3 in [10]. □

Lemma 2.11 (*Naimark Patching Lemma*). *Let $Q_r \in Z_{n(r)}(I_r, \mathbb{R})$ and assume that M_r are regular on I_r. Let $\alpha_{rs}, \beta_{rs} \in \mathbb{C}$, $s = 0, \ldots, n-1$, $r = 1, \ldots, q$. Then there is a function $y = (y_1, \ldots, y_q) \in D_{\max}$ such that*

$$y^{[s]}(a_r) = \alpha_{rs}, \qquad y^{[s]}(b_r) = \beta_{rs} \quad (s = 0, \ldots, n-1,\ r = 1, \ldots, q)$$

and

$$\alpha_{r+1,s} = \beta_{rs} \quad (s = 0, \ldots, n-1,\ r = 1, \ldots, q-1).$$

Proof. This follows from Lemma 4 in [10]. □

Corollary 2.12. *Let $\alpha_r < \beta_r \in I_r$, $\alpha_{rs}, \beta_{rs} \in \mathbb{C}$, $s = 0, 1, \ldots, n-1$, $r = 1, \ldots, q$. Then there is a function $y = (y_1, \ldots, y_q) \in D_{\max}$ such that y_r has compact support in I_r and satisfies:*

$$y^{[s]}(\alpha_r) = \alpha_{rs}, \qquad y^{[s]}(\beta_r) = \beta_{rs} \quad (s = 0, \ldots, n-1,\ r = 1, \ldots, q).$$

Proof. This follows from Corollary 4 in [10]. □

Lemma 2.13. *The minimal domain $D_{\min}$ consists of all functions $y = (y_1, \dots, y_q) \in D_{\max}$ which satisfy the following conditions:*

$$[y_r, z_r]_r(a_r) = [y_r, z_r]_r(b_r) = 0, \quad r = 1, \dots, q$$

for all $y = (y_1, \dots, y_q) \in D_{\max}$.

Proof. It can be obtained directly by the closedness of $D_{\min}$, Lemma 2.9, and the Calkin theory of extensions of symmetric operators in Hilbert spaces. □

3. Characterization of all Self-Adjoint Domains for Problems

In this section we still assume that M_{Q_r} are generated by $Q_r \in Z_{n(r)}(I, \mathbb{R})$ satisfying (7), $n = 2k$, $k > 1$. We give the decomposition of the maximal domain and the characterization of all selfadjoint extensions of the minimal operator with $q - 1 (1 \leq q < +\infty)$ interior singular points.

Theorem 3.1. *Let M_r be a symmetric differential expression on (a_r, b_r) and let $c_r \in (a_r, b_r)$. Consider the equations*

$$M_r y = \lambda_r w_r y, \quad r = 1, \dots, q. \tag{17}$$

Let d_{r1} denote the deficiency index of (17) on (a_r, c_r) and d_{r2} the deficiency index of (17) on (c_r, b_r). Assume that for some $\lambda = \lambda_{r1} \in \mathbb{R}$, Eq. (17) has d_{r1} linearly independent solutions on (a_r, c_r) which lie in $L^2((a_r, c_r), w_r)$ and that for some $\lambda = \lambda_{r2} \in \mathbb{R}$, Eq. (17) has d_{r2} linearly independent solutions on (c_r, b_r) which lie in $L^2((c_r, b_r), w_r)$, $r = 1, \dots, q$. Then

1. *There exist d_{r1} linearly independent real-valued solutions $u_{r1}, \dots, u_{rd_{r1}}$ on (a_r, c_r) which lie in $L^2((a_r, c_r), w_r)$.*

2. *There exist d_{r2} linearly independent real-valued solutions $v_{r1}, \dots, v_{rd_{r2}}$ on (c_r, b_r) which lie in $L^2((c_r, b_r), w_r)$.*

3. *For $m_r = 2d_{r1} - 2k$, the solutions $u_{r1}, \dots, u_{rd_{r1}}$ on (a_r, c_r) can be ordered such that the $m_r \times m_r$ matrix $U_r = ([u_{ri}, u_{rj}]_r(c_r))$, $1 \leq i, j \leq m_r$ is given by*

$$U_r = (-1)^{k+1} E_{m_r}.$$

4. *For $n_r = 2d_{r2} - 2k$, the solutions $v_{r1}, \dots, v_{rd_{r2}}$ on (c_r, b_r) can be ordered such that the $n_r \times n_r$ matrix $V_r = ([v_{ri}, v_{rj}]_r(c_r))$, $1 \leq i, j \leq n_r$ is given by*

$$V_r = (-1)^{k+1} E_{n_r}.$$

5. *For every $y = (y_1, \dots, y_q) \in D_{\max}$ we have*

$$[y_r, u_{rj}]_r(a_r) = 0, \quad \textit{for } j = m_r + 1, \dots, d_{r1}.$$

6. *For every $y = (y_1, \dots, y_q) \in D_{\max}$ we have*

$$[y_r, v_{rj}]_r(b_r) = 0, \quad \textit{for } j = n_r + 1, \dots, d_{r2}.$$

7. *For $1 \leq i \leq j \leq d_{r1}$, we have*

$$[u_{ri}, u_{rj}]_r(a_r) = [u_{ri}, u_{rj}]_r(c_r).$$

8. *For* $1 \leq i \leq j \leq d_{r2}$*, we have*

$$[v_{ri}, v_{rj}]_r(b_r) = [v_{ri}, v_{rj}]_r(c_r).$$

9. *The solutions* $u_{r1}, \ldots, u_{rd_{r1}}$ *can be extended to* (a_r, b_r) *such that the extended functions, also denoted by* $u_{r1}, \ldots, u_{rd_{r1}}$*, satisfy* $u_{rj} \in D_{rmax}(a_r, b_r)$ *and* u_{rj} *is identically zero in a left neighborhood of* b_r*,* $j = 1, \ldots, d_{r1}$*.*

10. *The solutions* $v_{r1}, \ldots, v_{rd_{r2}}$ *can be extended to* (a_r, b_r) *such that the extended functions, also denoted by* $v_{r1}, \ldots, v_{rd_{r2}}$*, satisfy* $v_{rj} \in D_{rmax}(a_r, b_r)$ *and* v_{rj} *is identically zero in a left neighborhood of* a_r*,* $j = 1, \ldots, d_{r2}$*.*

Proof. Parts 1 and 2 follow from the fact that the real and imaginary parts of a complex solution are real solutions. Parts 3–6 follow from Corollary 6 in [10]. Parts 7 and 8 follow from Corollary 2 in [6]. By Lemma 2.11 the solutions $u_{r1}, \ldots, u_{rd_{r1}}$ can be patched at c_r to obtain maximal domain functions in $D_{r\max}(a_r, b_r)$. By another application of Lemma 4 in [6] these extended functions can be modified to be identically zero in a left neighborhood of b_r. This establishes 9 and 10 follows similar. □

Remark 3.2. We say that the solutions $u_{r,m_r+1}, \ldots, u_{rd_{r1}}$, $v_{r,n_r+1}, \ldots, v_{rd_{r2}}$ are of LP type at a_r and b_r, respectively. Since the Lagrange brackets in the conditions 5 and 6 of Theorem 3.1 are zero for all maximal domain functions y the LP solutions play no role in the determination of the self-adjoint boundary conditions. Nevertheless, the LP solutions play an important role in the study of the continuous spectrum (see [8]) and in the approximation of singular problems with regular ones.

Next we give the decomposition of the maximal domain and the characterization of all self-adjoint domains.

Theorem 3.3. *Let the hypotheses and notation of Theorem 3.1 hold. Then we have*

$$D_{r\max}(a_r, b_r) = D_{r\min}(a_r, b_r) \dot{+} span\{u_{r1}, \ldots, u_{rm_r}\} \dot{+} span\{v_{r1}, \ldots, v_{rn_r}\},$$
$$r = 1, \ldots, q.$$

Proof. By Von Neumann's formula, $D_{r\max}(a_r, b_r)/D_{r\min}(a_r, b_r) \leq 2d_r$ since $2d_r = 2(d_{r1} + d_{r2} - n) = m_r + n_r$. From Theorem 3.1 parts 7, 10 and the observation that the matrices U_r and V_r are nonsingular it follows that $u_{r1}, \ldots, u_{rm_r}$, $v_{r1}, \ldots, v_{rn_r}$ are linearly independent mod$(D_{r\min}(a_r, b_r))$ and therefore dim$(D_{r\max}(a_r, b_r))/D_{r\min}(a_r, b_r) \geq 2d_r$, $r = 1, \ldots, q$. Thus the proof is completed. □

Corollary 3.4. *If* $y \in D_{\max}$*, then* y *has a unique representation*

$$y = \widehat{y} + (\widetilde{y}_1, \ldots, \widetilde{y}_q),$$

where $\widehat{y} \in D_{\min}$*, and*

$$\widetilde{y}_r = \sum_{j=1}^{m_r} a_{rj} u_{rj} + \sum_{j=1}^{n_r} b_{rj} v_{rj}, \quad r = 1, \ldots, q.$$

Based on Theorems 3.1 and 3.3 we characterize all self-adjoint extensions of the minimal operator with interior singular points or, equivalently, all self-adjoint restrictions of the maximal operator with interior singular points in terms of real-valued solutions of Eq. (17) for real λ_r. The next theorem is our main result.

Theorem 3.5 (*Main Theorem*)**.** *Let the hypotheses and notation of Theorem 3.1 hold. Let $d_r = d_{r1} + d_{r2} - n$, $r = 1, \dots, q$. Then d_r is the deficiency index of Eq. (17) on (a_r, b_r) and $d = \sum_{r=1}^q d_r$. A linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exist complex $d \times m_r$ matrix A_r ($r = 1, \dots, q$) and complex $d \times n_r$ matrix B_r ($r = 1, \dots, q$) such that the following three conditions hold:*

1. *The rank$(A_1, \dots, A_q, B_1, \dots, B_q) = d$;*
2. $\sum_{r=1}^q (A_r E_{m_r} A_r^* - B_r E_{n_r} B_r^*) = 0$;
3.

$$D(S) = \left\{ y = (y_1, \dots, y_q) \in D_{\max} : \vphantom{\int} \right.$$

$$\left. \sum_{r=1}^{q} A_r \begin{pmatrix} [y_r, u_{r1}]_r(a_r) \\ \vdots \\ [y_r, u_{rm_r}]_r(a_r) \end{pmatrix} + \sum_{r=1}^{q} B_r \begin{pmatrix} [y_r, v_{r1}]_r(b_r) \\ \vdots \\ [y_r, v_{rn_r}]_r(b_r) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}. \tag{18}$$

In condition 2, E_j is the symmetric matrix (7) of order j.

Proof. Sufficiency. Let the matrices A_r, B_r ($r = 1, \dots, q$) satisfy the conditions 1 and 2 of Theorem 3.5. We show that $D(S)$ defined by condition 3 is the domain of a self-adjoint extension S of $S_{\min}$.

Let

$$A_r = -(\overline{a}_{ij}^r)_{d \times mr}, B_r = -(\overline{b}_{ij}^r)_{d \times nr},$$
$$w_{ri} = \sum_{j=1}^{m_r} a_{ij}^r u_{rj} + \sum_{j=1}^{n_r} b_{ij}^r v_{rj}, \quad r = 1, \dots, q,\ i = 1, \dots, d.$$

Then for $y = (y_1, \dots, y_q) \in D_{\max}$ we have

$$-A_r \begin{pmatrix} [y_r, u_{r1}]_r(a_r) \\ \vdots \\ [y_r, u_{rm_r}]_r(a_r) \end{pmatrix} = \begin{pmatrix} \left[y_r, \sum_{j=1}^{m_r} a_{1j}^r u_{rj} \right]_r (a_r) \\ \vdots \\ \left[y_r, \sum_{j=1}^{m_r} a_{dj}^r u_{rj} \right]_r (a_r) \end{pmatrix} = \begin{pmatrix} [y_r, w_{r1}]_r(a_r) \\ \vdots \\ [y_r, w_{rd}]_r(a_r) \end{pmatrix},$$

$$B_r \begin{pmatrix} [y_r, v_{r1}]_r(b_r) \\ \vdots \\ [y_r, v_{rn_r}]_r(b_r) \end{pmatrix} = \begin{pmatrix} \left[y_r, \sum_{j=1}^{n_r} b_{1j}^r v_{rj} \right]_r (b_r) \\ \vdots \\ \left[y_r, \sum_{j=1}^{n_r} b_{dj}^r v_{rj} \right]_r (b_r) \end{pmatrix} = \begin{pmatrix} [y_r, w_{r1}]_r(b_r) \\ \vdots \\ [y_r, w_{rd}]_r(b_r) \end{pmatrix}.$$

Therefore the boundary condition 3 of Theorem 3.5 becomes the boundary condition (iii) of Theorem 2.6, i.e.,

$$\sum_{r=1}^{q}([y_r, w_{ri}]_r(b_r) - [y_r, w_{ri}]_r(a_r)) = 0, \quad i = 1, \dots, d.$$

Next we prove that $w_i = (w_{1i}, \dots, w_{qi})$, $i = 1, \dots, d$, satisfy the conditions (i) and (ii) of Theorem 2.6.

If the condition (i) is not true, then there exist constants $c_1, \dots, c_d$, not all zero, such that

$$\gamma = \sum_{i=1}^{d} c_i w_i \in D_{\min},$$

i.e.,

$$\gamma_r = \sum_{i=1}^{d} c_i w_{ri} \in D_{r\,\min}, \quad r = 1, \dots, q.$$

By Lemma 2.11 we have $[\gamma_r, y_r]_r(a_r) = [\gamma_r, y_r]_r(b_r) = 0$, $r = 1, \dots, q$, for any $y \in D_{\max}$. Using the notation U_r from Theorem 3.1,

$$\begin{aligned}(0, \dots, 0) &= \left(\left[\sum_{j=1}^{d} c_j w_{rj}, u_{r1}\right]_r (a_r), \dots, \left[\sum_{j=1}^{d} c_j w_{rj}, u_{rm_r}\right]_r (a_r)\right) \\ &= (c_1, \dots, c_d)(a_{ij}^r)_{d\times m_r} U_r.\end{aligned}$$

Since U_r is nonsingular, we have $(\overline{c}_1, \dots, \overline{c}_d)A_r = 0$. Similarly, we have $(\overline{c}_1, \dots, \overline{c}_d)B_r = 0$. Hence

$$(\overline{c}_1, \dots, \overline{c}_d)(A_1, \dots, A_q, B_1, \dots, B_q) = 0.$$

This contradicts the fact that $\operatorname{rank}(A_1, \dots, A_q, B_1, \dots, B_q) = d$.

Next we show that (ii) holds. We have

$$[w_{ri}, w_{rj}]_r(a_r) = \left[\sum_{l=1}^{m_r} a_{il}^r u_{rl}, \sum_{s=1}^{m_r} a_{js}^r u_{rs}\right]_r (a_r) = \sum_{l=1}^{m_r}\sum_{s=1}^{m_r} a_{il}^r \overline{a}_{js}^r [u_{rl}, u_{rs}]_r(a_r).$$

From Theorem 3.1 we obtain

$$([w_{ri}, w_{rj}]_r(a_r))_{d\times d}^T = A_r U_r^T A_r^* = (-1)^k A_r E_{m_r} A_r^*, \quad r = 1, \dots, q.$$

Similarly,

$$([w_{ri}, w_{rj}]_r(b_r))_{d\times d}^T = (-1)^k B_r E_{n_r} B_r^*, \quad r = 1, \dots, q.$$

$$\begin{aligned}&\left(\sum_{r=1}^{q}[w_{ri}, w_{rj}]_r(b_r) - \sum_{r=1}^{q}[w_{ri}, w_{rj}]_r(a_r)\right)^T \\ &\quad = (-1)^k \sum_{r=1}^{q}(B_r E_{n_r} B_r^* - A_r E_{m_r} A_r^*) = 0.\end{aligned}$$

From above, by Theorem 2.6, we can get the conclusion that $D(S)$ is a self-adjoint domain.

Necessity. Let $D(S)$ be the domain of a self-adjoint extension S of $S_{\min}$. Then there exist $w_1 = (w_{11}, \dots, w_{1q}), \dots, w_d = (w_{d1}, \dots, w_{dq}) \in D_{\max}$ satisfying the conditions (i), (ii), (iii) of Theorem 2.6. By Corollary 3.4, each w_{ri} can be uniquely written as:

$$w_{ri} = \widehat{y}_{ri} + \sum_{j=1}^{m_r} a_{ij}^r u_{rj} + \sum_{j=1}^{n_r} b_{ij}^r v_{rj}, \tag{19}$$

where $\widehat{y}_{ri} \in D_{r\min}, a_{ij}^r, b_{ij}^r \in \mathbb{C}, r = 1, \dots, q.$

Let

$$A_r = -(\overline{a}_{ij}^r)_{d\times m_r}, \qquad B_r = (\overline{b}_{ij}^r)_{d\times n_r}, \quad r = 1, \dots, q.$$

Then

$$\begin{pmatrix} [y_r, w_{r1}]_r(a_r) \\ \vdots \\ [y_r, w_{rd}]_r(a_r) \end{pmatrix} = \begin{pmatrix} \left[y_r, \sum_{j=1}^{m_r} a_{1j}^r u_{rj}\right]_r (a_r) \\ \vdots \\ \left[y_r, \sum_{j=1}^{m_r} a_{dj}^r u_{rj}\right]_r (a_r) \end{pmatrix} = -A_r \begin{pmatrix} [y_r, u_{r1}]_r(a_r) \\ \vdots \\ [y_r, u_{rm_r}]_r(a_r) \end{pmatrix},$$

$$\begin{pmatrix} [y_r, w_{r1}]_r(b_r) \\ \vdots \\ [y_r, w_{rd}]_r(b_r) \end{pmatrix} = \begin{pmatrix} \left[y_r, \sum_{j=1}^{n_r} b_{1j}^r v_{rj}\right]_r (b_r) \\ \vdots \\ \left[y_r, \sum_{j=1}^{n_r} b_{dj}^r v_{rj}\right]_r (b_r) \end{pmatrix} = B_r \begin{pmatrix} [y_r, v_{r1}]_r(b_r) \\ \vdots \\ [y_r, v_{rn_r}]_r(b_r) \end{pmatrix}.$$

Hence the boundary condition (iii) of Theorem 2.6 is equivalent to part 3 of Theorem 3.5.

Next we show that A_r, B_r $(r = 1, \dots, q)$ satisfy the condition 1 of Theorem 3.5.

Clearly $\operatorname{rank}(A_1, \dots, A_r, B_1, \dots, B_r) \le d$. If $\operatorname{rank}(A_1, \dots, A_r, B_1, \dots, B_r) < d$, then there exist constants $h_1, \dots, h_d$, not all zero, such that

$$(h_1, \dots, h_d)(A_1, \dots, A_r, B_1, \dots, B_r) = 0. \tag{20}$$

Let $g = \sum_{i=1}^d \overline{h}_i w_i$, then from (19), we get

$$g_r = \sum_{i=1}^d \overline{h}_i \widehat{y}_{ri} + \sum_{i=1}^d \sum_{j=1}^{m_r} \overline{h}_i a_{ij}^r u_{rj} + \sum_{i=1}^d \sum_{j=1}^{n_r} \overline{h}_i b_{ij}^r v_{rj}. \tag{21}$$

By (20), we know $(h_1, \dots, h_d)A_r = (h_1, \dots, h_d)B_r = 0$. Thus let $g = \sum_{i=1}^d \overline{h}_i w_i$, then from (21), we get

$$g_r = \sum_{i=1}^d \overline{h}_i \widehat{y}_{ri}.$$

So we have $g_r \in D_{r\,\min}$, $= 1,\dots,q$, i.e., $g \in D_{\min}$. This contradicts the fact that the functions $w_1,\dots,w_d$ are linearly independent modulo $D_{\min}$. Therefore $\operatorname{rank}(A_1,\dots,A_r, B_1,\dots,B_r) = d$.

It remains to prove that A_r, B_r $(r = 1,\dots,q)$ satisfy the condition 2 of Theorem 3.5.

From (19), we can have

$$[w_{ri}, w_{rj}]_r(a_r) = \left[\sum_{k=1}^{m_r} a^r_{ik} u_{rk}, \sum_{s=1}^{m_r} a^r_{js} u_{rs}\right]_r (a_r) = \sum_{k=1}^{m_r}\sum_{s=1}^{m_r} a^r_{ik}\overline{a}^r_{js}[u_{rk}, u_{rs}]_r(a_r),$$
$$i, j = 1,\dots,d.$$

So it follows from Theorem 3.1, we can obtain

$$([w_{ri}, w_{rj}]_r(a_r))^T_{d\times d} = A_r U^T_r A^*_r = (-1)^k A_r E_{m_r} A^*_r.$$

Similarly, we have

$$([w_{ri}, w_{rj}]_r(b_r))^T_{d\times d} = B_r U^T_r B^*_r = (-1)^k B_r E_{n_r} B^*_r.$$

Thus condition (ii) of Theorem 2.6 is transform into

$$A_r E_{m_r} A^*_r = B_r E_{n_r} B^*_r, \quad r = 1,\dots,q,$$

i.e.,

$$\sum_{r=1}^{q} A_r E_{m_r} A^*_r = \sum_{r=1}^{q} B_r E_{n_r} B^*_r. \quad \square$$

Remark 3.6 (*LC and LP Solutions*). Note that for $\lambda = \lambda_{r1}$ there are d_{r1} linearly independent real solutions on (a_r, c_r) which can be ordered such that the first $u_{r1},\dots,u_{rm_r}$ with $m_r = 2d_{r1} - 2k$ contribute to the self-adjoint boundary conditions (18) and $u_{r,m_r+1},\dots,u_{rd_{r1}}$ make no contribute to the boundary conditions (18). By conclusion 5 of Theorem 3.1, $[y_r, u_{rj}]_r(a_r) = 0$ for every $y_r \in D_{r\,\max}$, $j = m_r + 1,\dots,d_{r1}$. If $u_{r1},\dots,u_{rd_{r1}}$ is completed to a full basis $u_{r1},\dots,u_{rd_{r1}},\dots,u_{rn}$ of solutions of Eq. (17) on (a_r, c_r), then no nontrivial linear combination of $u_{r,d_{r1}+1},\dots,u_{rn}$ is in the Hilbert space $L^2((a_r, c_r), w_r)$ and thus these solutions play no role in the formulation of the self-adjoint boundary conditions. For this reason we call $u_{r1},\dots,u_{rm_r}$ LC solutions at a_r and $u_{r,m_r+1},\dots,u_{rd_{r1}}$ LP solutions at a_r. Similarly, we call $v_{r1},\dots,v_{rn_r}$ LC solutions at b_r and $v_{r,n_r+1},\dots,v_{rd_{r2}}$ LP solutions at b_r, $r = 1,\dots,q$.

In Theorem 3.5 it is assumed that endpoints $a = a_1$ and $b = b_q$ are singular. It can be specialized to known results when one or two endpoints are regular. Here we state several cases for the convenience of the reader.

Theorem 3.7. *Let the hypotheses and notation of Theorem 3.1 hold and assume that a and b are regular. Then $d = d_{12} + \sum_{r=2}^{q-1}(d_{r1} + d_{r2} - n) + d_{q1}$. The solutions $v_{12},\dots,v_{1d_{12}}$ and $u_{q1},\dots,u_{qd_{q1}}$ can be extended to solutions on (a, b_1) and (a_q, b) such that $v_{12},\dots,v_{1d_{12}} \in L^2((a, b_1), w_1)$ and $u_{q1},\dots,u_{qd_{q1}} \in L^2((a_q, b), w_q)$, respectively. Let $m_r = 2d_{r1} - n$*

$(r=2,\dots,q)$ *and* $n_r = 2d_{r2} - n$ $(r=1,\dots,q-1)$. *A linear submanifold* $D(S)$ *of* $D_{\max}$ *is the domain of a self-adjoint extension* S *of* $S_{\min}$ *if and only if there exist two complex* $d\times n$ *matrices* A_1 *and* B_q*, complex* $d\times m_r$ *matrix* A_r $(r=1,\dots,q-1)$ *and complex* $d\times n_r$ *matrix* B_r $(r=2,\dots,q)$ *such that the following three conditions hold:*

1. *The* $\operatorname{rank}(A_1,\dots,A_q,B_1,\dots,B_q)=d$*;*
2. $A_1E_nA_1^* + \sum_{r=2}^{q} A_rE_{m_r}A_r^* - \sum_{r=1}^{q-1} B_rE_{n_r}B_r^* - B_qE_nB_q^* = 0$*;*
3.

$$D(S) = \Bigg\{ y=(y_1,\dots,y_q)\in D_{\max}:$$

$$A_1\begin{pmatrix} y_1(a)\\ \vdots \\ y_1^{[n-1]}(a)\end{pmatrix} + \sum_{r=2}^{q} A_r \begin{pmatrix} [y_r,u_{r1}]_r(a_r)\\ \vdots \\ [y_r,u_{rm_r}]_r(a_r)\end{pmatrix} + \sum_{r=1}^{q-1} B_r\begin{pmatrix} [y_r,v_{r1}]_r(b_r)\\ \vdots \\ [y_r,v_{rn_r}]_r(b_r)\end{pmatrix}$$

$$+B_q\begin{pmatrix} y_q(b)\\ \vdots \\ y_q^{[n-1]}(b)\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}\Bigg\}.$$

Proof. We show that the solutions $v_{12},\dots,v_{1d_{12}}$ lying in $L^2((c_1,b_1),w_1)$ for some $c_1\in(a,b_1)$ can be extended to real-valued solutions on (a,b_1) which lie in $L^2((a,b_1),w_1)$. Determine solutions z_j on (a,c_1) with the initial conditions: $z_j^{[s]}(c_1) = v_{1j}^{[s]}(c_1), s = 0,\dots,n-1$ and rename these $z_j = v_{1j}$ to obtain solutions v_{1j} on (a,b_1) for $j=1,\dots,d_{12}$. Since a is a regular endpoint, these extended v_{1j} are bounded on (a,c_1) and therefore the extended v_{1j} are in $L^2((a,b_1),w_1)$. Similarly, the solutions $u_{q1},\dots,u_{qd_{q1}}$ lying in $L^2((a_q,c_q),w_q)$ for some $c_q\in(a_q,b)$ can be extended to real-valued solutions on (a_q,b) which lie in $L^2((a_q,b),w_q)$. Now this theorem can follow from Theorem 4.14 in [9]. □

Remark 3.8. Obviously, in Theorem 3.7, the condition $q\geq 2$ is necessary.

Remark 3.9. In the minimal deficiency case $d_{12}=\frac{n}{2}$, $m_2=m_3=\cdots=m_q=0$, $n_1=n_2=\cdots=n_{q-1}=0$, $d_{q1}=\frac{n}{2}$, the terms involving $A_2,\dots,A_q$ and $B_1,\dots,B_{q-1}$ disappear and Theorem 3.5 reduces to the self-adjoint boundary conditions at the regular endpoints a and b:

$$A_1\begin{pmatrix} y_1(a)\\ \vdots \\ y_1^{[n-1]}(a)\end{pmatrix} + B_q\begin{pmatrix} y_q(b)\\ \vdots \\ y_q^{[n-1]}(b)\end{pmatrix} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}.$$

where the $n\times n$ complex matrices A_1 and B_q satisfy $\operatorname{rank}(A_1,B_q)=n$ and $A_1E_nA_1^* = B_qE_nB_q^*$. In this case there are no conditions required or allowed at the singular interior points.

Theorem 3.10. *Let the hypotheses and notation of Theorem 3.1 hold and assume that* a *and* b *are regular and there is not any singular point in* (a,b)*, i.e.,* $q=1$*. Then* $d=n$ *and a* *linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exists a complex $n \times n$ matrix A and a complex $n \times n$ matrix B such that the following three conditions hold:*

1. *The rank$(A, B) = n$;*
2. *$AE_nA^* = BE_nB^*$;*
3.

$$D(S) = \left\{ y \in D_{\max} : \right.$$

$$\left. A \begin{pmatrix} y(a) \\ \vdots \\ y^{[n-1]}(a) \end{pmatrix} + B \begin{pmatrix} y(b) \\ \vdots \\ y^{[n-1]}(b) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

Proof. Cf. the proof of Theorem 4.3.2 in [5]. □

Theorem 3.11. *Let the hypotheses and notation of Theorem 3.1 hold and assume that a is regular. Then $d = d_{12} + \sum_{r=2}^{q}(d_{r1} + d_{r2} - n)$. The solutions $v_{12}, \dots, v_{1d_{12}}$ can be extended to solutions on (a, b_1) such that $v_{12}, \dots, v_{1d_{12}} \in L^2((a, b_1), w_1)$. Let $m_r = 2d_{r1} - n$ $(r = 2, \dots, q)$ and $n_r = 2d_{r2} - n$ $(r = 1, \dots, q)$. A linear submanifold $D(S)$ of $D_{\max}$ is the domain of a self-adjoint extension S of $S_{\min}$ if and only if there exists a complex $d \times n$ matrices A_1, complex $d \times m_r$ matrix A_r $(r = 1, \dots, q-1)$ and complex $d \times n_r$ matrix B_r $(r = 1, \dots, q)$ such that the following three conditions hold:*

1. *The rank$(A_1, \dots, A_q, B_1, \dots, B_q) = d$;*
2. *$A_1E_nA_1^* + \sum_{r=2}^{q} A_rE_{m_r}A_r^* - \sum_{r=1}^{q} B_rE_{n_r}B_r^* = 0$;*
3.

$$D(S) = \left\{ y = (y_1, \dots, y_q) \in D_{\max} : \right.$$

$$A_1 \begin{pmatrix} y_1(a) \\ \vdots \\ y_1^{[n-1]}(a) \end{pmatrix} + \sum_{r=2}^{q} A_r \begin{pmatrix} [y_r, u_{r1}]_r(a_r) \\ \vdots \\ [y_r, u_{rm_r}]_r(a_r) \end{pmatrix}$$

$$\left. + \sum_{r=1}^{q} B_r \begin{pmatrix} [y_r, v_{r1}]_r(b_r) \\ \vdots \\ [y_r, v_{rn_r}]_r(b_r) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

Proof. It is similar to the proof of Theorem 3.5. □

Remark 3.12. Similarity to Remark 3.9, in the minimal deficiency case $d_{12} = \frac{n}{2}$, $m_2 = m_3 = \cdots = m_q = 0$, $n_1 = n_2 = \cdots = n_q = 0$, the terms involving $A_2, \dots, A_q$ and $B_1, \dots, B_q$ disappear and Theorem 3.5 reduces to the self-adjoint boundary conditions at the regular endpoint a:

$$A_1 \begin{pmatrix} y_1(a) \\ \vdots \\ y_1^{[n-1]}(a) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

4. Examples

In this section, we will give a number of examples to illustrate the self-adjoint boundary conditions given by Theorem 3.5. These examples include interactions between the singular endpoints and interior singular points. Here we give some examples for

$$n = 4, \quad q = 2, \; 3 \le d \le 8.$$

Similar examples can easily be constructed for all higher order cases $n = 2k, k > 2$ and more singular interior points cases $q \ge 3$.

Example 1. If $d_{11} = 3$, $d_{12} = 3$, $d_{21} = 3$, $d_{22} = 2$, then $d_1 = d_{11} + d_{12} - 4 = 2$, $d_2 = d_{21} + d_{22} - 4 = 1$, $d = d_1 + d_2 = 3$ and $m_1 = 2d_{11} - 4 = 2$, $m_2 = 2d_{12} - 4 = 2$, $n_1 = 2d_{21} - 4 = 2$, $n_2 = 2d_{22} - 4 = 0$.

Let

$$A_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ C_1 & C_2 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 1 & 0 \\ h_2 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -1 & h_1 \\ 0 & -1 \\ 0 & 0 \end{pmatrix},$$

where $C_1, C_2, h_1, h_2 \in \mathbb{R}$ and $C_1^2 + C_2^2 \neq 0$. Then $\text{Rank}(A_1, A_2, B_1) = 3$ and from a straightforward computation, it follows that

$$A_1 E_2 A_1^* + A_2 E_2 A_2^* - B_1 E_2 B_1^* = 0, \qquad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore, we obtain the following self-adjoint boundary conditions:

$$\begin{aligned} &C_1[y_1, u_{11}]_1(a_1) + C_2[y_1, u_{12}]_1(a_1) = 0, \\ &[y_1, v_{11}]_1(b_1) = [y_2, u_{21}]_2(a_2) - h_1[y_2, u_{22}]_2(a_2), \\ &[y_1, v_{12}]_1(b_1) = [y_2, u_{22}]_2(a_2) - h_2[y_1, v_{11}]_1(b_1). \end{aligned}$$

Here we have one general separated singular condition at a_1 and two singular jump conditions, these singular conditions involving the Lagrange bracket.

Similarity to the method of Example 1, we can get self-adjoint boundary conditions of the other cases for $d \ge 4$. So we only list out some conclusions. The concrete processes would be omitted.

Example 2. In this example we have 4 conditions, all of them involving interactions between singular endpoints i.e. interactions between Lagrange brackets. Let $d_{11} = 3$, $d_{12} = 3$,

$d_{21} = 3$, $d_{22} = 3$. Then $d = d_1 + d_2 = 4$ and $m_1 = 2d_{11} - 4 = 2$, $m_2 = 2d_{12} - 4 = 2$, $n_1 = 2d_{21} - 4 = 2$, $n_2 = 2d_{22} - 4 = 2$.

$$
\begin{aligned}
&[y_1, u_{11}]_1(a_1) = [y_2, v_{21}]_2(b_2) - h_1[y_2, v_{22}]_2(b_2),\\
&[y_1, u_{12}]_1(a_1) = [y_2, v_{22}]_2(b_2) - h_2[y_1, u_{11}]_1(a_1),\\
&[y_1, v_{11}]_1(b_1) = -[y_1, u_{14}]_1(a_1) - h_3[y_1, v_{12}]_1(b_1),\\
&[y_1, v_{12}]_1(b_1) = [y_2, u_{22}]_2(a_2) - h_4[y_1, v_{11}]_1(b_1),
\end{aligned}
$$

where $h_i \in \mathbb{R}, i = 1, 2, 3, 4$.

Example 3. Assume $d_{11} = 4$, $d_{12} = 3$, $d_{21} = 3$, $d_{22} = 3$. Then $d = d_1 + d_2 = 5$ and $m_1 = 2d_{11} - 4 = 4$, $m_2 = 2d_{12} - 4 = 2$, $n_1 = 2d_{21} - 4 = 2$, $n_2 = 2d_{22} - 4 = 2$. Note that here we have one general separated singular boundary condition at b_2 and four singular coupled singular jump conditions.

$$
\begin{aligned}
&[y_1, u_{12}]_1(a_1) = -[y_2, u_{22}]_2(a_2) - h_1[y_1, u_{13}]_1(a_1),\\
&[y_1, u_{13}]_1(a_1) = [y_2, u_{21}]_2(a_2) - h_2[y_2, u_{22}]_2(a_2),\\
&[y_1, v_{11}]_1(b_1) = [y_2, u_{21}]_2(a_2) - h_3[y_2, u_{22}]_2(a_2),\\
&[y_1, v_{12}]_1(b_1) = [y_1, u_{11}]_1(a_1) - h_4[y_1, u_{14}]_1(a_1),\\
&C_1[y_2, v_{21}]_2(b_2) + C_2[y_2, v_{22}]_2(b_2) = 0,
\end{aligned}
$$

where $C_i, h_j \in \mathbb{R}$, $i = 1, 2$, $j = 1, 2, 3, 4$ and $C_1^2 + C_2^2 \neq 0$.

Example 4. In this example we have 6 conditions: $d = 6$. There are four nonreal singular boundary conditions and two singular coupled jump conditions. Assume $d_{11} = 4$, $d_{12} = 3$, $d_{21} = 3$, $d_{22} = 4$. Then $m_1 = 2d_{11} - 4 = 4$, $m_2 = 2d_{12} - 4 = 2$, $n_1 = 2d_{21} - 4 = 2$, $n_2 = 2d_{22} - 4 = 4$.

$$
\begin{aligned}
&[y_1, u_{11}]_1(a_1) + i[y_1, u_{12}]_1(a_1) = 0, \qquad [y_1, u_{13}]_1(a_1) - i[y_1, u_{14}]_1(a_1) = 0,\\
&[y_2, v_{21}]_2(b_2) + i[y_2, v_{22}]_2(b_2) = 0, \qquad [y_2, v_{23}]_2(b_2) - i[y_2, v_{24}]_2(b_2) = 0,\\
&[y_1, v_{11}]_1(b_1) = [y_2, u_{21}]_2(a_2) - h_1[y_1, v_{12}]_1(b_1), \quad h_1 \in \mathbb{R},\\
&[y_1, v_{12}]_1(b_1) = [y_2, u_{22}]_2(a_2) - h_2[y_2, u_{21}]_2(a_2), \quad h_2 \in \mathbb{R}.
\end{aligned}
$$

Example 5. This example features six nonreal singular boundary conditions and one general separated singular boundary condition at b_2. Let $d_{11} = 4$, $d_{12} = 4$, $d_{21} = 4$, $d_{22} = 3$. Then $d = d_1 + d_2 = 7$ and $m_1 = 2d_{11} - 4 = 4$, $m_2 = 2d_{12} - 4 = 4$, $n_1 = 2d_{21} - 4 = 4$, $n_2 = 2d_{22} - 4 = 2$.

$$
\begin{aligned}
&[y_1, u_{11}]_1(a_1) + i[y_1, u_{12}]_1(a_1) = 0, \qquad [y_1, u_{13}]_1(a_1) - i[y_1, u_{14}]_1(a_1) = 0,\\
&[y_1, v_{11}]_1(b_1) + i[y_1, v_{12}]_1(b_1) = 0, \qquad [y_1, v_{13}]_1(b_1) - i[y_1, v_{14}]_1(b_1) = 0,\\
&[y_2, u_{21}]_2(a_2) + i[y_2, u_{22}]_2(a_2) = 0, \qquad [y_2, u_{23}]_2(a_2) - i[y_2, u_{24}]_2(a_2) = 0,\\
&C_1[y_2, v_{21}]_2(b_2) + C_2[y_2, v_{22}]_2(b_2) = 0, \qquad C_1, C_2 \in \mathbb{R},\ C_1^2 + C_2^2 \neq 0.
\end{aligned}
$$

Example 6. This example features nonreal singular boundary condition at all four endpoints. Assume $d_{11} = 4$, $d_{12} = 4$, $d_{21} = 4$, $d_{22} = 4$. Then $d = d_1 + d_2 = 8$ and $m_1 = 2d_{11} - 4 = 4$,

$m_2 = 2d_{12} - 4 = 4, n_1 = 2d_{21} - 4 = 4, n_2 = 2d_{22} - 4 = 4.$

$$\begin{aligned}
&[y_1, u_{11}]_1(a_1) + i[y_1, u_{12}]_1(a_1) = 0, \qquad &[y_1, u_{13}]_1(a_1) - i[y_1, u_{14}]_1(a_1) = 0,\\
&[y_1, v_{11}]_1(b_1) + i[y_1, v_{12}]_1(b_1) = 0, \qquad &[y_1, v_{13}]_1(b_1) - i[y_1, v_{14}]_1(b_1) = 0,\\
&[y_2, u_{21}]_2(a_2) + i[y_2, u_{22}]_2(a_2) = 0, \qquad &[y_2, u_{23}]_2(a_2) - i[y_2, u_{24}]_2(a_2) = 0,\\
&[y_2, v_{21}]_2(b_2) + i[y_2, v_{22}]_2(b_2) = 0, \qquad &[y_2, v_{23}]_2(b_2) - i[y_2, v_{24}]_2(b_2) = 0.
\end{aligned}$$

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