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Characterization of *GCR*-lightlike warped product of indefinite Sasakian manifolds

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Abstract. In this paper we prove that there do not exist warped product ***GCR***-lightlike submanifolds in the form $M = N_{\perp} \times_k N_T$ such that $N_{\perp}$ is an anti-invariant submanifold tangent to V and N_T an invariant submanifold of $\overline{M}$, other than ***GCR***-lightlike product in an indefinite Sasakian manifold. We also obtain characterization theorems for a ***GCR***-lightlike submanifold to be locally a ***GCR***-lightlike warped product.

Keywords: ***GCR***-lightlike submanifold; ***GCR***-lightlike product; ***GCR***-lightlike warped product submanifold; Indefinite Sasakian manifold

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1. Introduction

Cauchy–Riemann (*CR*)-submanifolds of Kaehler manifolds were introduced by Bejancu [3] as a generalization of holomorphic and totally real submanifolds of Kaehler manifolds and further investigated by [4,6,7,9,10] and others. Contact *CR*-submanifolds of Sasakian manifolds were introduced by Yano and Kon [30]. They all studied the geometry of *CR*-submanifolds with positive definite metric. Therefore this geometry may not be applicable to the other branches of mathematics and physics,

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where the metric is not necessarily definite. Thus the geometry of CR-submanifolds with indefinite metric became a topic of chief discussion and Duggal and Bejancu [14] played a very crucial role in this study by introducing the notion of CR-lightlike submanifolds of indefinite Kaehler manifolds. Since there is a significant use of the contact geometry in differential equations, optics, and phase spaces of a dynamical system (see Arnol'd [1], Maclane [24], Nazaikinskii et al. [25]), Duggal and Sahin [16] introduced contact CR-lightlike submanifolds and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds. But there do not exist any inclusion relations between invariant and screen real submanifolds so a new class of submanifolds, called Generalized Cauchy–Riemann (GCR)-lightlike submanifolds of indefinite Sasakian manifolds (which is an umbrella of invariant, screen real, contact CR-lightlike submanifolds) was derived by Duggal and Sahin [17]. Rakesh et al. [22] characterized a GCR-lightlike submanifold to be a GCR-lightlike product of an indefinite Sasakian manifold. In [8], the notion of warped product manifolds was introduced by Bishop and O'Neill in 1969 and it was further studied by many mathematicians and physicists. These manifolds are generalizations of Riemannian product manifolds. This generalized product metric appears in differential geometric studies in a natural way. For instance a surface of revolution is a warped product manifold. Moreover, many important submanifolds in real and complex space forms are expressed as warped product submanifolds. In view of its physical applications many research articles have recently appeared exploring existence (or non existence) of warped product submanifolds in known spaces, [27,29]. Chen [11] introduced warped product CR-submanifolds and showed that there does not exist a warped product CR-submanifold in the form $M = N_\perp \times_\lambda N_T$ in a Kaehler manifold where $N_\perp$ is a totally real submanifold and N_T is a holomorphic submanifold of $\overline{M}$. He proved if $M = N_\perp \times_\lambda N_T$ is a warped product CR-submanifold of a Kaehler manifold $\overline{M}$ then M is a CR-product, that is, there do not exist warped product CR-submanifolds of the form $M = N_\perp \times_\lambda N_T$ other than a CR-product. Therefore he called a warped product CR-submanifold in the form $M = N_T \times_\lambda N_\perp$ a CR-warped product. Chen [11] also obtained a characterization for a CR-submanifold of Kaehler manifold to be locally a warped product submanifold. He showed that a CR-submanifold M of a Kaehler manifold $\overline{M}$ is a CR-warped product if and only if $A_{JZ}X = JX(\mu)Z$ for each $X \in \Gamma(D), Z \in \Gamma(D'), \mu$ a C^∞-function on M such that $Z\mu = 0$ for all $Z \in \Gamma(D')$.

In this paper we prove that there do not exist warped product GCR-lightlike submanifolds in the form $M = N_\perp \times_\lambda N_T$ such that $N_\perp$ is an anti-invariant submanifold tangent to V and N_T an invariant submanifold of $\overline{M}$, other than GCR-lightlike product in an indefinite Sasakian manifold. We also obtain characterization theorems for a GCR-lightlike submanifold to be locally a GCR-lightlike warped product.

2. Lightlike submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [14] by Duggal and Bejancu.

Let $(\overline{M}, \bar{g})$ be a real $(m + n)$-dimensional semi-Riemannian manifold of constant index q such that $m,n \geqslant 1, 1 \leqslant q \leqslant m + n - 1$ and (M,g) be an m-dimensional submanifold of $\overline{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle

TM of M then M is called a lightlike submanifold of $\overline{M}$. For a degenerate metric g on M

$$TM^{\perp} = \cup\{u \in T_x\overline{M} : \bar{g}(u, v) = 0, \forall v \in T_xM, x \in M\}, \tag{1}$$

is a degenerate n-dimensional subspace of $T_x\overline{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $Rad\ T_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping

$$Rad\ TM : x \in M \rightarrow Rad\ T_xM, \tag{2}$$

defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\overline{M}$ is called r-lightlike submanifold and $Rad\ TM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is

$$TM = Rad\ TM \perp S(TM), \tag{3}$$

$S(TM^{\perp})$ is a complementary vector subbundle to $Rad\ TM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\overline{M}|_M$ and to $Rad\ TM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{4}$$

$$T\overline{M}|_M = TM \oplus tr(TM) = (Rad\ TM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{5}$$

Let u be a local coordinate neighborhood of M and consider the local quasi-orthonormal fields of frames of $\overline{M}$ along M, on u as $\{\xi_1, \ldots, \xi_r, W_{r+1}, \ldots, W_n, N_1, \ldots, N_r, X_{r+1}, \ldots, X_m\}$, where $\{\xi_1, \ldots, \xi_r\}, \{N_1, \ldots, N_r\}$ are local lightlike bases of $\Gamma(Rad\ TM|_u)$, $\Gamma(ltr(TM)|_u)$ and $\{W_{r+1}, \ldots, W_n\}, \{X_{r+1}, \ldots, X_m\}$ are local orthonormal bases of $\Gamma(S(TM^{\perp})|_u)$ and $\Gamma(S(TM)|_u)$ respectively. For this quasi-orthonormal fields of frames, we have

Theorem 2.1 [14]. *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\overline{M}, \bar{g})$. Then, there exists a complementary vector bundle $ltr(TM)$ of Rad TM in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM)|_u)$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp}|_u$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \tag{6}$$

where $\{\xi_1, \ldots, \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\overline{M}$. Then, according to the decomposition (5), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM), \tag{7}$$

$$\bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \forall X \in \Gamma(TM), U \in \Gamma(tr(TM)), \tag{8}$$

where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on

$\Gamma(TM)$ which is called second fundamental form, A_U is a linear operator on M and known as shape operator.

According to (4), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively, (7) and (8) give

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{9}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{10}$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^{\perp}_X U), D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{11}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{12}$$

where $X \in \Gamma(TM)$,$N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$.

Using (4), (5), (9), (10), (11) and (12), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{13}$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \tag{14}$$

$$\bar{g}(A_N X, N') + \bar{g}(N, A_{N'} X) = 0, \tag{15}$$

for any $\xi \in \Gamma(Rad\ TM)$,$W \in \Gamma(S(TM^{\perp}))$ and $N,N' \in \Gamma(ltr(TM))$.

Next, we recall some basic definitions and results of indefinite Sasakian manifolds (see [5,23]). An odd dimensional semi-Riemannian manifold $(\overline{M}, \bar{g})$ is called an ϵ-contact metric manifold, if there is a *(1,1)* tensor field ϕ, a vector field V, called characteristic vector field and a 1-form η such that

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \epsilon\eta(X)\eta(Y), \quad \bar{g}(V, V) = \epsilon, \tag{16}$$

$$\phi^2(X) = -X + \eta(X)V, \quad \bar{g}(X, V) = \epsilon\eta(X), \tag{17}$$

$$d\eta(X, Y) = \bar{g}(X, \phi Y), \tag{18}$$

for any $X,Y \in \Gamma(TM)$, where $\epsilon = \pm 1$ then it follows that

$$\phi V = 0, \tag{19}$$

$$\eta o\phi = 0, \quad \eta(V) = 1. \tag{20}$$

Then $(\phi, V, \eta, \bar{g})$ is called an ϵ-contact metric structure of $\overline{M}$. We say that $\overline{M}$ has a normal contact structure if $N_{\phi} + d\eta \otimes V = 0$, where N_{ϕ} is Nijenhuis tensor field of ϕ. A normal ϵ-contact metric manifold is called ϵ-Sasakian manifold and for this we have

$$\bar{\nabla}_X V = \phi X \tag{21}$$

$$(\bar{\nabla}_X \phi) Y = -\bar{g}(X, Y)V + \epsilon\eta(Y)X. \tag{22}$$

3. Generalized Cauchy–Riemann (*GCR*)-lightlike submanifolds of indefinite Sasakian manifolds

Calin [12] proved that if the characteristic vector field V is tangent to $(M,g,S(TM))$ then it belongs to $S(TM)$. We assume the characteristic vector V is tangent to M throughout this paper.

Definition 3.1 [17]. Let $(M,g,S(TM))$ be a real lightlike submanifold of an indefinite Sasakian manifold $(\overline{M},\bar{g})$. Then M is called a generalized Cauchy–Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM). \tag{23}$$

(B) There exist two subbundles D_0 and $\overline{D}$ of $S(TM)$ such that

$$S(TM) = \{\phi D_2 \oplus \overline{D}\} \perp D_0 \perp V, \quad \phi(\overline{D}) = L \perp S, \tag{24}$$

where D_0 is invariant non-degenerate distribution on M,$\{V\}$ is one dimensional distribution spanned by V and L,S are vector subbundles of $ltr(TM)$ and $S(TM)^{\perp}$, respectively.

Then the tangent bundle TM of M is decomposed as

$$TM = D \oplus \overline{D} \oplus \{V\}, \quad \text{where} \quad D = Rad(TM) \oplus D_0 \oplus \phi(D_2). \tag{25}$$

Let Q,P_1 and P_2 be the projection morphisms on D, $\phi S = M_2$ and $\phi L = M_1$, respectively. Then any $X \in \Gamma(TM)$ can be written as

$$X = QX + V + P_1X + P_2X, \tag{26}$$

or

$$X = QX + V + PX, \tag{27}$$

where P is a projection morphism on $\overline{D}$. Applying ϕ to (27), we obtain

$$\phi X = fX + \omega P_1 X + \omega P_2 X, \tag{28}$$

where $fX \in \Gamma(D)$,$\omega P_1 X \in \Gamma(S)$ and $\omega P_2 X \in \Gamma(L)$, or, we can write (28) as

$$\phi X = fX + \omega X, \tag{29}$$

where fX and ωX are the tangential and transversal components of ϕX, respectively.

Similarly, for any $U \in \Gamma(tr(TM))$, we have

$$\phi U = BU + CU, \tag{30}$$

where BU and CU are the sections of TM and $tr(TM)$, respectively.

Differentiating (28) and using (9), (10), (11), (12) and (30) we have

$$D^l(X, \omega P_1 Y) = -\nabla^l_X \omega P_2 Y + \omega P_2 \nabla_X Y - h^l(X, fY) + Ch^l(X, Y), \tag{31}$$

$$D^s(X, \omega P_2 Y) = -\nabla^s_X \omega P_1 Y + \omega P_1 \nabla_X Y - h^s(X, fY) + Ch^s(X, Y), \tag{32}$$

for all $X, Y \in \Gamma(TM)$. By using the Sasakian property of $\bar{\nabla}$ with (7) and (8), we have the following lemmas.

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then we have*

$$(\nabla_X f)Y = A_{\omega Y}X + Bh(X, Y) - g(X, Y)V + \epsilon\eta(Y)X, \tag{33}$$

and

$$(\nabla^t_X \omega)Y = Ch(X, Y) - h(X, fY), \tag{34}$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \tag{35}$$

$$(\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega\nabla_X Y. \tag{36}$$

Lemma 3.3. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then we have*

$$(\nabla_X B)U = A_{CU}X - fA_U X - g(X, U)V, \tag{37}$$

and

$$(\nabla^t_X C)U = -\omega A_U X - h(X, BU), \tag{38}$$

where $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$ and

$$(\nabla_X B)U = \nabla_X BU - B\nabla^t_X U, \tag{39}$$

$$(\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U. \tag{40}$$

Theorem 3.4 [20]. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then:*

(A) The distribution $D \oplus \{V\}$ is integrable, if and only if,

$$h(X, fY) = h(Y, fX), \quad \forall \quad X, Y \in \Gamma(D \oplus \{V\}). \tag{41}$$

(B) The distribution $\overline{D}$ is integrable, if and only if,

$$A_{\phi Z}U = A_{\phi U}Z, \quad \forall \quad Z, U \in \Gamma(\overline{D}). \tag{42}$$

Theorem 3.5 [20]. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M, if and only if, $Bh(X, \phi Y) = 0$, for any $X, Y \in D \oplus \{V\}$.*

4. *GCR*-lightlike warped product

Warped product: Let B and F be two Riemannian manifolds with Riemannian metrics g_B and g_F respectively and $\lambda > 0$ a differentiable function on B. Assume the product manifold $B \times F$ with its projection $\pi{:}B \times F \rightarrow B$ and $\eta{:}B \times F \rightarrow F$. The warped product $M = B \times_{\lambda} F$ is the manifold $B \times F$ equipped with the Riemannian metric g where

$$g = g_B + \lambda^2 g_F. \tag{43}$$

If X is tangent to $M = B \times_{\lambda} F$ at (p,q) then using (43), we have

$$\|X\|^2 = \|\pi_* X\|^2 + \lambda^2(\pi(X))\|\eta_* X\|^2. \tag{44}$$

The function λ is called the warping function of the warped product. For differentiable function λ on M, the gradient $\nabla\lambda$ is defined by $g(\nabla\lambda, X) = X\lambda$, for all $X \in T(M)$.

Lemma 4.1 [8]. *Let* $M = B \times_{\lambda} F$ *be a warped product manifold. If* $X, Y \in T(B)$ *and* $U, Z \in T(F)$ *then*

$$\nabla_X Y \in T(B). \tag{45}$$

$$\nabla_X U = \nabla_U X = \frac{X\lambda}{\lambda} U. \tag{46}$$

$$\nabla_U Z = -\frac{g(U, Z)}{\lambda} \nabla\lambda. \tag{47}$$

Corollary 4.2. *On a warped product manifold* $M = B \times_{\lambda} F$ *we have*

(i) B is totally geodesic in M.
(ii) F is totally umbilical in M.

O'Neill extended the concept of Riemannian warped product to semi-Riemannian warped product in [26]. Further Beem–Ehrlich [2] used the scheme of semi-Riemannian warped products and constructed a Lorentzian warped product manifold. In [13], Duggal introduced two classes of warped product of lightlike manifolds. Later on, warped product lightlike submanifolds of semi-Riemannian manifolds are introduced by Sahin in [28] as given below.

Definition 4.3. Let (M_1, g_1) be a totally lightlike submanifold of dimension r and (M_2, g_2) be a semi-Riemannian submanifold of dimension m of a semi- Riemann manifold $(\overline{M}, \bar{g})$. Then the product manifold $M = M_1 \times_f M_2$ is said to be a warped product lightlike submanifold of $\overline{M}$ with the degenerate metric g defined by

$$g(X, Y) = g_1(\pi_* X, \pi_* Y) + (f o \pi)^2 g_2(\eta_* X, \eta_* Y),$$

for every $X, Y \in \Gamma(TM)$ and $*$ is the symbol for the tangent map. Here, $\pi*{:}M_1 \times M_2 \rightarrow M_1$ and $\eta*{:}M_1 \times M_2 \rightarrow M_2$ denote the projection maps given by $\pi(x,y) = x$ and $\eta(x,y) = y$ for $(x,y) \in \Gamma(M_1 \times M_2)$.

Definition 4.4 [15]. A lightlike submanifold (M,g) of a semi-Riemannian manifold $(\overline{M}, \bar{g})$ is said to be totally umbilical in $\overline{M}$ if there is a smooth transversal vector field $H \in \Gamma(tr(TM))$ on M, called the transversal curvature vector field of M, such that

$$h(X, Y) = Hg(X, Y), \tag{48}$$

for all $X, Y \in \Gamma(TM)$. It is easy to see that M is totally umbilical if and only if on each coordinate neighborhood u, there exist smooth vector fields $H^l \in \Gamma(ltr(TM))$ and $H^s \in \Gamma(S(TM^{\perp}))$, such that

$$h^l(X, Y) = H^l g(X, Y), \quad h^s(X, Y) = H^s g(X, Y), \quad D^l(X, W) = 0, \tag{49}$$

for any $W \in \Gamma(S(TM^{\perp}))$.

Lemma 4.5. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the distribution $\overline{D}$ defines a totally geodesic foliation in M.*

Proof. Let $X, Y \in \Gamma(\overline{D})$ then using (33) and (35), we have $f\nabla_X Y = -A_{wY}X - Bh(X,Y) + g(X,Y)V$. Let $Z \in \Gamma(D_0)$ then using (22), we obtain

$$\begin{aligned} g(f\nabla_X Y, Z) &= -g(A_{wY}X, Z) = \bar{g}(\bar{\nabla}_X \phi Y, Z) = -\bar{g}(\bar{\nabla}_X Y, \phi Z) \\ &= -\bar{g}(\bar{\nabla}_X Y, Z') = g(Y, \nabla_X Z'), \end{aligned} \tag{50}$$

where $Z' = \phi Z \in \Gamma(D_0)$. Since $X \in \Gamma(\overline{D})$ and $Z \in \Gamma(D_0)$ then using (34), (36) and the hypothesis that M is a totally umbilical GCR-lightlike submanifold, we get $w\nabla_X Z = h(X,fZ) - Ch(X,Z) = Hg(X,fZ) - CHg(X,Z) = 0$, this implies that $\nabla_X Z \in \Gamma(D)$, then (50) implies that $g(f\nabla_X Y,Z) = 0$. Hence the non degeneracy of the distribution D_0 implies that $f\nabla_X Y = 0$, this gives $\nabla_X Y \in \Gamma(\overline{D})$ for any $X, Y \in \Gamma(\overline{D})$. Hence the proof is complete. □

Theorem 4.6. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then the totally real distribution $\overline{D}$ is integrable.*

Proof. Using (33) and (35) with the above lemma, we get

$$A_{wY}X = -Bh(X, Y) + g(X, Y)V, \tag{51}$$

for any $X, Y \in \Gamma(\overline{D})$. Then using the symmetric property of h and g, we get $A_{wY}X = A_{wX}Y$, for any $X, Y \in \Gamma(\overline{D})$. Hence using the Theorem (3.4) the distribution $\overline{D}$ is integrable. □

Definition 4.7 [22]. A GCR-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is called a GCR-lightlike product if both the distributions $D \oplus \{V\}$ and $\overline{D}$ define totally geodesic foliations in M.

Let $M = N_{\perp} \times_{\lambda} N_T$ be a warped product GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Such submanifolds are always tangent to the structure vector field V. We distinguish two cases

(i) V is tangent to N_T.
(ii) V is tangent to $N_{\perp}$.

In this paper we consider the case when V is tangent to N_T.

Theorem 4.8. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. If $M = N_{\perp} \times_{\lambda} N_T$ is a warped product GCR-lightlike submanifold such that $N_{\perp}$ is an anti-invariant submanifold and N_T is an invariant submanifold of $\overline{M}$ tangent to V then it is a GCR-lightlike product.*

Proof. Since M is a totally umbilical *GCR*-lightlike submanifold of an indefinite Sasakian manifold then using the Lemma (4.5), the distribution $\overline{D}$ defines a totally geodesic foliation in M.

Let h^T and A^T be the second fundamental form and the shape operator of N_T in M, respectively. Then for any $X, Y \in \Gamma(D \oplus \{V\})$ and $Z \in \Gamma(\phi S) \subset \Gamma(\overline{D})$, we have $g(h^T(X, Y), Z) = g(\nabla_X Y, Z) = -\bar{g}(Y, \bar{\nabla}_X Z) = -g(Y, \nabla_X Z) - g(Y, h^l(X, Z)) = -g(Y, \nabla_X Z) - g(Y, H^l)g(X, Z) = -g(Y, \nabla_X Z)$. Using (46) for $M = N_{\perp} \times_{\lambda} N_T$, we get

$$g(h^T(X, Y), Z) = -(Z ln\lambda) g(X, Y). \tag{52}$$

Now, let $\hat{h}$ be the second fundamental form of N_T in $\overline{M}$ then

$$\hat{h}(X, Y) = h^T(X, Y) + h^s(X, Y) + h^l(X, Y), \tag{53}$$

for any X, Y tangent to N_T then using (52), we get

$$g(\hat{h}(X, Y), Z) = g(h^T(X, Y), Z) = -(Z ln\lambda) g(X, Y). \tag{54}$$

Since N_T is a holomorphic submanifold of $\overline{M}$ then we have $\hat{h}(X, \phi Y) = \hat{h}(\phi X, Y) = \phi \hat{h}(X, Y)$ therefore we have

$$g(\hat{h}(X, Y), Z) = -g(\hat{h}(\phi X, \phi Y), Z) = (Z ln\lambda) g(X, Y). \tag{55}$$

Adding (54) and (55) we get

$$g(\hat{h}(X, Y), Z) = 0. \tag{56}$$

Using (53) and (56), we have $g(h(X,Y),\phi Z) = g(\hat{h}(X,Y),\phi Z) - g(h^T(X,Y),\phi Z) = g(\hat{h}(X,Y),\phi Z) = -g(\phi \hat{h}(X,Y),Z) = -g(\hat{h}(X,\phi Y),Z) = 0$. Thus $g(h(X,Y),\phi Z) = 0$ implies that $h(X,Y)$ has no components in $L_1 \perp L_2$ for any $X, Y \in \Gamma(D \oplus \{V\})$. In other words, we can say that $Bh(X,Y) = 0$, for any $X, Y \in \Gamma(D \oplus \{V\})$. Therefore using the Theorem (3.5), the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M. Hence M is a *GCR*-lightlike product. Hence the proof is complete. □

After the proof of the Theorem 4.8, it is important to mention the following two theorems by Hasegawa and Mihai [18] and Khan et al. [21], respectively.

Theorem 4.9. *Let $\overline{M}$ be a $(2m + 1)$-dimensional Sasakian manifold. Then there do not exist warped product submanifolds $M = M_1 \times_{\lambda} M_2$ such that M_1 is an anti-invariant submanifold tangent to V and M_2 an invariant submanifold of $\overline{M}$.*

Theorem 4.10. *Let $\overline{M}$ be a $(2m + 1)$-dimensional Kenmotsu manifold. Then there do not exist warped product submanifolds $M = N_{\perp} \times_{\lambda} N_T$ such that N_T is an invariant submanifold tangent to V and $N_{\perp}$ is anti-invariant submanifold of $\overline{M}$.*

In this paper, Theorem (4.8) also shows that there do not exist warped product GCR-lightlike submanifolds of the form $M = N_\perp \times_\lambda N_T$ such that $N_\perp$ is an anti-invariant submanifold and N_T an invariant submanifold tangent to V of $\overline{M}$, other than a GCR-lightlike product. Thus for simplicity we call a warped product GCR-lightlike submanifold in the form $M = N_T \times_\lambda N_\perp$ such that $N_\perp$ is an anti-invariant submanifold and N_T is an invariant submanifold of $\overline{M}$ tangent to V, a GCR-lightlike warped product.

Lemma 4.11. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Let $M = N_T \times_\lambda N_\perp$ be a proper GCR-lightlike warped product of an indefinite Sasakian manifold $\overline{M}$ such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\overline{M}$ then N_T is totally geodesic in M.*

Proof. Let $X, Y \in N_T$ and $Z \in N_\perp$ then we have $g(\nabla_X Y, Z) = \bar{g}(\bar{\nabla}_X Y, Z) = -g(Y, \nabla_X Z) - g(Y, h^l(X, Z))$, using (46) we get $g(\nabla_X Y, Z) = -g(Y, h^l(X, Z))$. Since M is a totally umbilical GCR-lightlike submanifold therefore $h^l(X,Z) = h^s(X,Z) = 0$. Hence $g(\nabla_X Y, Z) = 0$ implies that N_T is totally geodesic in M. □

Theorem 4.12. *Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. If the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M then it is integrable.*

Proof. Let $X, Y \in \Gamma(D \oplus \{V\})$ then using (34) and (36), we have $h(X, fY) = Ch(X, Y) + \omega\nabla_X Y$. Since the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M therefore $\omega\nabla_X Y = 0$ and we get $h(X, fY) = Ch(X, Y)$, then taking into account that h is symmetric, we obtain $h(X, fY) = h(fX, Y)$, for all $X, Y \in \Gamma(D \oplus \{V\})$. This proves the assertion. □

5. Characterizations of GCR-lightlike warped products

For a GCR-lightlike warped product in indefinite Sasakian manifolds, we have

Lemma 5.1. *Let M be a totally umbilical GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$. Then for a GCR-lightlike warped product $M = N_T \times_\lambda N_\perp$ such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\overline{M}$, we have*

$$\bar{g}(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi M_2) = 0. \tag{57}$$

Proof. Since $\overline{M}$ is a Sasakian manifold therefore for any $X \in \Gamma(D \oplus \{V\})$ and $Z \in \Gamma(M_2)$ using (22), we have $\phi\bar{\nabla}_X Z = \bar{\nabla}_X \phi Z$. Since M is a totally umbilical therefore we have $\phi(\nabla_X Z) = -A_{wZ} X + \nabla^s_X wZ$, then taking inner product with ϕY where $Y \in \Gamma(D \oplus \{V\})$, we get $g(\phi\nabla_X Z, \phi Y) = -g(A_{wZ} X, \phi Y)$. Using (16) and (46), we obtain $g(A_{wZ} X, \phi Y) = 0$ then using (13), we get $\bar{g}(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi M_2) = 0$. Hence the proof is complete. □

Corollary 5.2. *Let $Z \in \Gamma(M_1) \subset \Gamma(\overline{D})$ then clearly $g(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi Z) = 0$ and also $g(h^l(D \oplus \{V\}, D \oplus \{V\}), \phi Z) = 0$ for any $Z \in \Gamma(\overline{D})$. Thus $g(h(D \oplus \{V\}, D \oplus \{V\}), \phi\overline{D}) = 0$, this implies that $h(D \oplus \{V\}, D \oplus \{V\})$ has no component in $L_1 \perp L_2$, that is, $Bh(D \oplus \{V\}, D \oplus \{V\}) = 0$ therefore using the Theorem (3.5) the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M.*

Next, we have the following characterizations of *GCR*-lightlike warped products.

Theorem 5.3. *A proper totally umbilical GCR-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is locally a GCR-lightlike warped product $M = N_T \times_\lambda N_\perp$ such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\overline{M}$, if and only if,*

$$A_{\phi Z}X = ((\phi X)\mu)Z, \tag{58}$$

for $X \in \Gamma(D \oplus \{V\}), Z \in \Gamma(\overline{D})$ and for some function μ on M satisfying $U\mu = 0, U \in \Gamma(\overline{D})$.

Proof. Assume that M be a proper *GCR*-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ satisfying (58). For any $X, Y \in \Gamma(D \oplus \{V\})$ and $Z \in \Gamma(M_2) \subset \Gamma(\overline{D})$, we have $g(A_{\phi Z}X, \phi Y) = g(((\phi X)\mu)Z, \phi Y) = ((\phi X)\mu)g(Z, \phi Y) = 0$, then using (13) we get $g(h^s(D \oplus \{V\}, D \oplus \{V\}), \phi M_2) = 0$. Then as done in the above corollary, the distribution $D \oplus \{V\}$ defines a totally geodesic foliation in M and consequently it is totally geodesic in M and using the Theorem (4.12) the distribution $D \oplus \{V\}$ is integrable.

Now, taking inner product of (58) with $U \in \Gamma(\overline{D})$ and using (16), (22), (46) and that M is a totally umbilical submanifold, we get $g(((\phi X)\mu)Z, U) = g(A_{\phi Z}X, U) = g(\phi Z, \nabla_X U) = g(\phi Z, \nabla_U X) = -\bar{g}(\bar{\nabla}_U \phi Z, X) + \epsilon g(U, Z)\eta(X) = g(\nabla_U Z, \phi X) + \bar{g}(h^l(U, Z), \phi X) + \epsilon g(U, Z)\eta(X)$, then using the definition of gradient $g(\nabla\mu, X) = X\mu$ we get

$$g(\nabla_U Z, \phi X) = g(\nabla\mu, \phi X)g(Z, U) - \bar{g}(h^l(U, Z), \phi X) - \epsilon g(U, Z)\eta(X). \tag{59}$$

Let h' and ∇' be the second fundamental form and the metric connection of $\overline{D}$, respectively in M then we have

$$g(h'(U, Z), \phi X) = g(\nabla_U Z - \nabla'_U Z, \phi X). \tag{60}$$

Therefore from (59) and (60), particularly for $X \in \Gamma(D_0)$, we get $g(h'(U,Z), \phi X) = g(\nabla_U Z, \phi X) = g(\nabla\mu, \phi X)g(Z, U)$ this further implies that

$$h'(U, Z) = \nabla\mu g(Z, U), \tag{61}$$

this implies that the distribution $\overline{D}$ is totally umbilical in M. Using the Theorem (4.6), the totally real distribution $\overline{D}$ is also integrable. Now, using the condition $U\mu = 0$ for $U \in \overline{D}$, we have $g(\nabla\mu, U) = U\mu = 0$. Hence from (61), we obtain that $\overline{D}$ is a totally umbilical submanifold with parallel mean curvature $\nabla\mu$, that is, each leaf of $\overline{D}$ is an extrinsic sphere in M. Hence by a result of ([19]) which say that "If the tangent bundle of a Riemannian manifold M splits into an orthogonal sum $TM = E_0 \oplus E_1$ of non trivial vector subbundles such that E_1 is spherical and its orthogonal complement E_0 is autoparallel, then the manifold M is locally isometric to a warped product $M_0 \times_\lambda M_1$", therefore we can conclude that M is locally a GCR-lightlike warped product $N_T \times_\lambda N_\perp$ of $\overline{M}$ where $\lambda = e^\mu$.

Conversely, let $X \in \Gamma(N_T)$ and $Z \in \Gamma(N_\perp)$, since $\overline{M}$ is a Sasakian manifold so we have $\bar{\nabla}_X \phi Z = \phi \bar{\nabla}_X Z$, which further becomes $-A_{\phi Z}X + \nabla^t_X \phi Z = ((\phi X)ln\lambda)Z$, comparing tangential components, wet $A_{\phi Z}X = -((\phi X)ln\lambda)Z$ for each $X \in \Gamma(D \oplus \{V\})$ and $Z \in (\overline{D})$. Since λ is a function on N_T so we also have $U(ln\lambda) = 0$ for all $U \in \Gamma(\overline{D})$. Hence the proof is complete. □

Lemma 5.4. *Let $M = N_T \times_\lambda N_\perp$ be a GCR-lightlike warped product of an indefinite Sasakian manifold such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\overline{M}$. Then*

$$(\nabla_Z f)X = fX(ln\lambda)Z. \tag{62}$$

$$(\nabla_U f)Z = g(Z, U)f(\nabla ln\lambda), \tag{63}$$

for any $U \in \Gamma(TM)$, $X \in \Gamma(N_T)$ and $Z \in \Gamma(N_\perp)$.

Proof. For any $X \in \Gamma(N_T)$ and $Z \in \Gamma(N_\perp)$, using (35) and (46), we have $(\nabla_Z f)X = \nabla_Z fX - f((\frac{Xf}{f})Z) = \nabla_Z fX - \frac{Xf}{f} fZ = \nabla_Z fX = fX(ln\lambda)Z$. Next, again using (35) we get $(\nabla_U f)Z = -f\nabla_U Z$ this implies that $(\nabla_U f)Z \in \Gamma(N_T)$, therefore for any $X \in \Gamma(D_0)$ we have

$$\begin{aligned} g((\nabla_U f)Z, X) &= -g(f\nabla_U Z, X) = g(\nabla_U Z, fX) = \bar{g}(\bar{\nabla}_U Z, fX) \\ &= -g(Z, \nabla_U fX) = -fX(ln\lambda)g(Z, U). \end{aligned}$$

Hence using the definition of gradient of λ and the non degeneracy of the distribution D_0, the result is as follows. □

Theorem 5.5. *A proper totally umbilical GCR-lightlike submanifold M of an indefinite Sasakian manifold $\overline{M}$ is locally a GCR-lightlike warped product $M = N_T \times_\lambda N_\perp$ such that N_T is an invariant submanifold tangent to V and $N_\perp$ an anti-invariant submanifold of $\overline{M}$ if*

$$(\nabla_X f)Y = (fY(\mu))PX + g(PX, PY)\phi(\nabla\mu) - g(X, Y)V + \epsilon\eta(Y)X, \tag{64}$$

for any $X, Y \in \Gamma(TM)$ and for some function μ on M satisfying $Z\mu = 0, Z \in \Gamma(\overline{D})$.

Proof. Let M be a GCR-lightlike submanifold of an indefinite Sasakian manifold $\overline{M}$ satisfying (64). Let $X, Y \in \Gamma(D \oplus \{V\})$ then (64) implies that $(\nabla_X f)Y = -g(X, Y)V + \epsilon\eta(Y)X$ then (33) gives $Bh(X, Y) = 0$. Thus $D \oplus \{V\}$ defines a totally geodesic foliation in M and consequently it is totally geodesic in M and integrable using the Theorem (4.12).

Let $X, Y \in \Gamma(\overline{D})$ then (64) gives

$$(\nabla_X f)Y = g(PX, PY)\phi(\nabla\mu) - g(X, Y)V. \tag{65}$$

Let $U \in \Gamma(D_0)$ then (65) implies that

$$g((\nabla_X f)Y, U) = g(PX, PY)g(\phi(\nabla\mu), U). \tag{66}$$

Also using (22) with (33), we have

$$g((\nabla_X f)Y, U) = g(A_{wY}X, U) = \bar{g}(\bar{\nabla}_X Y, \phi U) = g(\nabla_X Y, \phi U), \tag{67}$$

therefore from (66) and (67) we get

$$g(\nabla_X Y, \phi U) = -g(\nabla\mu, \phi U)g(X, Y). \tag{68}$$

Let h' and ∇' be the second fundamental form and the metric connection of $\overline{D}$, respectively in M then

$$g(h'(X, Y), \phi U) = g(\nabla_X Y - \nabla'_X Y, \phi U) = g(\nabla_X Y, \phi U), \tag{69}$$

therefore from (68) and (69), we get $g(h'(X,Y),\phi U) = -g(\nabla\mu,\phi U)g(X,Y)$. Then the non degeneracy of the distribution D_0 implies that

$$h'(X, Y) = -\nabla\mu g(X, Y), \tag{70}$$

this gives that the distribution $\overline{D}$ is totally umbilical in M and using the Theorem (4.6), the distribution $\overline{D}$ is integrable. Also $Z\mu = 0$ for $Z \in \Gamma(\overline{D})$, hence as in the Theorem (5.3) each leaf of $\overline{D}$ is an extrinsic sphere in M. Thus M is locally a GCR-lightlike warped product $N_T \times_\lambda N_\perp$ of $\overline{M}$ where $\lambda = e^\mu$. □

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