ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci 25 (2) (2019) 189–213

Bifurcation analysis of a nonlinear diffusion model: Effect of evaluation period for the diffusion of a technology

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Received 3 July 2018; revised 25 November 2018; accepted 21 December 2018
Available online 19 February 2019

Abstract. A nonlinear modified form of Bass model involving the interactions of non-adopter and adopter populations has been proposed to describe the process of diffusion of a new technology in the presence of evaluation period (time delay). The basic aim is to model the diffusion of those technologies which require higher investments, and which require government subsidies for promotions in the various markets. We use government incentives and the costs in the form of external factors, as well as the internal word of mouth that considerably influence the non-adopters decisions. A qualitative analysis has been performed to determine the stability of the various equilibria. The Hopf bifurcation occurs near the positive equilibrium when the time delay passes some critical values. By applying the normal form theory and the center manifold reduction for functional differential equations, explicit formulae presenting stability properties of bifurcating periodic solutions have been computed. Moreover, the intra-specific competition has played an important role in establishing the maturity stage in the innovation diffusion model. Numerical analysis has been carried out to justify the correctness of our analytical findings.

Keywords: Delay; Stability analysis; Bifurcation; Normal form theory; Center manifold theorem

Mathematics Subject Classification: 34C23; 34D20; 37L10

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

https://doi.org/10.1016/j.ajmsc.2018.12.001

1. Introduction

To forecast social feedback to different innovations, innovation diffusion models can be utilized. Many researchers have studied the innovation diffusion models of new products [5, 7–9,12–14,16–18,20,33,34,36,37,41,44,45,48,49]. But due to the advancements of science, economics, and different techniques, the competition of products in various markets is quite common. It is an important task to study the market in which n products compete. It is hard to make an analysis of behavior of higher dimensional systems; for example, there can be complex behavior and chaotic attractors, and the analysis of the models with higher dimension is very challenging as compared to two-dimensional models [23,24,50]. Hirsch investigated the dynamical behavior of the systems having cooperative and competitive nature, and are commonly used in various applied fields such as epidemiology, economics and ecology etc. Roger's innovation adoption process could not be described by the Bass model in the realistic manner. Bass states that there is always an early impact of external as well as internal influences on the minds of the population and the maximum number of adopters can easily be reached [2]. A model representing five stages of the adoption process was given by Roger's [41], and through the behavior research, researcher's found that the stages of diffusion can be reduced to two-step, i.e., print and electronic media affects the innovative leaders to accept the new product, these leaders make their influence people to use the product [10,11,25,35]. A number of delayed models explaining the evaluation stage of a product are proposed by researchers [16,27,28,51]. Beretta et al. have also considered the evaluation stage for explaining the diffusion process [3] and applied mathematical methods to study stability switches and find periodic solutions for certain parameters. Wang et al. [51] proposed models with and without evaluation period and explained the stages of information, evaluation and the final decision. They have supposed a system which involves the information and final decision-making stages and find that the model admits a threshold below which the process is unsuccessful and above which the innovation diffusion will be successful. A mathematical study is proposed to describe the dynamics of adopters of a single product in two distinct patches [54]. The stability of an innovation diffusion model having competitive nature is explained in [50,58]. For distinct n products, the global analysis is discussed by researchers in [56]. A model for three competitive innovations in a market with nonlinear acceptance is discussed in [57]. For a variable size market, a binomial market compartmental innovation diffusion model of the entrance exit demographic processes was investigated by F. Centrone [6], Shukla et al. proved that the product diffusion process is influenced by various demographic processes and specifically explained the role of external influences, to make the equilibrium level of adopter density at higher speed [47]. Kumar et al. in [29] analyzed the effect of an evaluation period and proved that it is responsible for periodic orbits in the form of limit cycle via Hopf bifurcation in the innovation diffusion system. Wijeratne et al. in [55] discussed the modified form of Bass model by incorporating a diffusion term and a delay and observed the negative impact of initiation parameter (negative-word-of-mouth) which results in the existence of Hopf bifurcation.

Price [40] firstly suggested that the higher values of products affect the adoption of a latest technology until it is properly upgraded. Moreover, the time period between the invention of a latest technology and its final adoption is often associated with the distrust about the technical changes in the future, or the distrust developed by more technical advancements. Models of technical distrust argue that if there is a faster technological improvement, then there always is a very opportunity that the industry can regain its initial expenditure which has been used to evolve the technology. Fanelli et al. in [15] recommended another model with time delay for presenting the stages of the adoption process.

$$\frac{dP(t)}{dt} = \delta A(t) + \gamma A(t)A(t-\tau) - \left[e^{\eta(i-c)} + \alpha A(t-\tau)\right] P(t-\tau)e^{-\rho\tau},$$
$$\frac{dA(t)}{dt} = \left[e^{\eta(i-c)} + \alpha A(t-\tau)\right] P(t-\tau)e^{-\rho\tau} - \delta A(t) - \gamma A(t)A(t-\tau).$$

where the number of potential adopters and adopters are given by $P(t)$ and $A(t)$ respectively. The authors have analyzed the model with the assumption of constant population, i.e., $A(t) + P(t) = C$, (C is a constant). By applying Poincare–Lyapunov theorem, the equilibrium point A^*_τ is proved as locally stable, whereas Ballestra et al. in [1] analyzed the same model system and established the stability switches in the model, and justified the stability properties of periodic orbits. In these studies, authors have considered a fixed number of potential adopter population. It means no new individual can become the member of the system.

Motivated by the above developments in the field of innovation diffusion and extending our previous work [29,30], we proposed a more realistic nonlinear mathematical model with delay and constant input of population to explain the innovation diffusion process. We seek to model the technologies which are used for the production of renewable energy such as solar cells. These are made by using a variety of materials viz., silicon, solar dyes, solar inks, etc. Also, the latest solar cells use lenses or mirrors to concentrate sunlight onto a very small piece of high efficiency photovoltaic material. The cost of photovoltaic material is normally very high and hence the manufacturing cost of these technologies are on higher side, but incentives are significantly less. Thus, the innovation diffusion process is usually blocked at the initial stage and technology cannot take off. These factors lead us to assume the factor of government support (incentives) in the form of external influences.

It is assumed here that the members are well exposed to the technology and get awareness about the positive and negative of the new innovation (technology). We assume that the potential adopters are entering the system at the growth rate of r and the total population comprises of the individuals who have already established their manufacturing units for the technology (technology developers) and the other class comprises of those persons who are still taking their time to evaluate the technology (potential technology developers). Let $A(t)$ be defined as a class of persons who have established their technological units (adopters), and those who are still evaluating the technology defined as non-adopter and are denoted by $N(t)$ at time t. Further, if c represents the production costs of the innovation, and i is some type of government incentive needed to promote the innovative technology, so we assume that $e^{\eta(i-c)}$ is the external factor applied on potential technology adopter's, where η, c, and i are positive constants. The factor of government subsidies has helped us to decide the form of external factor by taking well into consideration its utility for the environment friendly society. Let α be the internal word of mouth of technology developers with potential technology developers, and it reflects the imitation effect (interpersonal communication). Also, let δ be the death rate (or emigration rate) of the populations and v is the rate at which the technology developers going back to potential technology developers $N(t)$, who may join later on. In the light of above facts, we state the model in the form of difference–differential equations:

$$\begin{aligned}\frac{dN(t)}{dt} &= r\,(N_0 - N) + \gamma A(t)A(t-\tau) - \left(e^{\eta(i-c)} + \alpha A(t-\tau)\right) N(t-\tau)e^{-(\delta+\varrho)\tau} \\ &\quad + vA(t) - \delta N(t),\end{aligned} \tag{1}$$

$$\frac{dA(t)}{dt} = \Big(e^{\eta(i-c)} + \alpha A(t-\tau)\Big)N(t-\tau)e^{-(\delta+\varrho)\tau} \\ -(\delta+v)A(t) - \gamma A(t)A(t-\tau), \tag{2}$$

Here τ represents the average time for potential technology adopter to make evaluation, i.e., to evolve (develop) it or not. Here, the knowledge about the technology take place at $t-\tau$ time and during the period $[t-\tau, t]$, the potential technology developers decide whether to evolve the technology or not. In other words, the potential developers are capable to examine the technology in order to make the final adoption. Moreover, the probability of the survival of a potential technology developers through the evaluation stage has been considered as $e^{-\delta\tau}$. Also, here ϱ is supposed as the rate at which the population left the evaluation stage because of the negative approach of the population about the adoption after the test period, then $e^{-\varrho\tau}$ is the portion that the people are having interest in the innovation after completing the testing period. Therefore, the probability of success through the evaluation stage, i.e., the probability that a potential technology developer who has awareness about the technology at time $t-\tau$ does not die and does not lose interest in that particular technology at time t is taken as $e^{-(\delta+\varrho)\tau}$. The factor $\big[e^{\eta(i-c)} + \alpha A(t-\tau)\big]N(t-\tau)e^{-(\delta+\varrho)\tau}$ represents the number of technology developers who move from the "non-adopter class" to the "adoption class", i.e., those who have knowledge about the technology at $t-\tau$ and make final decision to adopt the technology.

The novelty of this work lies in the fact that we have analyzed the role of intra-specific competition between the technology developers at time t and with the technology developers at time $t-\tau$, which determinants for the growth of $A(t)$. This competition coefficient is detrimental for the growth of technological units. Let γ be the contact rate of competition coefficient between the technology developers in the present time t, and those at time $t-\tau$. It is really affecting the growth of technology developers due to advertisement such that it decreases as the number of technology developers increases. The parameters supposed here are to be considered as constants with positive values.

The system (1)–(2) may be rewritten as follows:

$$\frac{dN(t)}{dt} = r\,(N_0 - N) + \gamma A(t)A(t-\tau) - \big(\kappa + \alpha A(t-\tau)\big) \\ \times\ N(t-\tau)h(\tau) + vA(t) - \delta N(t), \tag{3}$$

$$\frac{dA(t)}{dt} = \big(\kappa + \alpha A(t-\tau)\big)N(t-\tau)h(\tau) - (\delta+v)A(t) - \gamma A(t)A(t-\tau), \tag{4}$$

where $\kappa = e^{\eta(i-c)}$ and $h(\tau) = e^{-(\delta+\varrho)\tau}$, which is a function of delay (τ).

Here, the system (3)–(4) will be studied with the initial conditions

$$N(\theta) = \phi_1(\theta),\ A(\theta) = \phi_2(\theta),\ \phi_1(0) > 0,\ \phi_2(0) > 0, \tag{5}$$

where $\phi_1(\theta)$, $\phi_2(\theta)$ are continuous bounded function in the interval $[-\tau, 0]$ and $\phi_1(\theta)$, $\phi_2(\theta) \in C([-\tau, 0], \mathfrak{R}^2_+)$, the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathfrak{R}^2_+$, where

$$\mathfrak{R}^2_+ = \{(N, A) : N \geq 0, A \geq 0\}.$$

By applying the fundamental theory of functional differential equations [21], that system (3)–(4) has a unique solution $(N(t), A(t))$ justifying the initial conditions (5). It is quite convenient to prove that the solutions of the model (3)–(4) with respect to initial conditions are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The article is organized as follows. Section 2 deals with basic preliminaries such as positivity and boundedness of solutions. In Section 3, we prove the existence of the various equilibria, exclusively the existence of interior equilibrium. The stability properties of various equilibrium points, along with the global stability of non-negative equilibrium point for $\tau = 0$ have also been mentioned in this section. In Section 4, we analyzed the delayed innovation diffusion model for the existence of Hopf bifurcation around the positive equilibrium E^* with delay as bifurcation parameter. The stability properties of Hopf bifurcating periodic solutions are established by applying the technique based upon center manifold theorem and normal form theory in Section 5. Section 6 is related to the verification of analytical results with the help of numerical simulations. The key findings of our mathematical model and their importance with respect to innovation diffusion modeling has been explained in the concluding Section 7.

2. Positivity and boundedness of solutions

Setting $X = (N, A)^T \in \mathfrak{R}^2$, we may write (3)–(4) in vector form as

$$F(X) = [F_1(X), F_2(X)]^T$$

Let $\mathfrak{R}^2 = [0, +\infty)^2$ be nonnegative octant in $\mathfrak{R}^2$, where $F : C_+ \to \mathfrak{R}^2$ and $F \in C^\infty(\mathfrak{R}^2)$. Then (3)–(4) is rewritten as below:

$$\dot{X} = F(X),$$

and $X(0) = X_0 \in \mathfrak{R}^2_+$. It is convenient to check in $F(X)$ that whenever choosing $X(0) \in \mathfrak{R}^2_+$ so that $X_i = 0$ then $F_i(X)|_{X_i=0} \geq 0, (i = 1, 2)$. Now any solution of $\dot{X} = F(X)$, with $X_0 \in \mathfrak{R}^2_+$, say $X(t) = X(t, X_0)$, is such that $X(t) \in \mathfrak{R}^2_+$ for all $t > 0$ [38].

Next, we present boundedness of solutions in the system (3)–(4).

Lemma 2.1. *The solutions of the system (3)–(4) with conditions (5) in $\mathfrak{R}^2_+$ are uniformly bounded.*

Proof. Let us assume $\big(N(t), A(t)\big)$ be an arbitrary solution of the model (3)–(4).

Define $V(t) = N(t) + A(t)$, for all $t > 0$.

Thus, we shall get

$$\frac{dV}{dt} + \delta V \leq rN_0 \quad \delta > 0,$$

for any $\delta > 0$.

Now integrating the inequality and applying the theory of differential inequalities due to Birkhoff and Rota [4], we have

$$0 \leq V\big(N, A\big) \leq \frac{rN_0}{\delta} + \frac{V\big(N(0), A(0)\big)}{e^{\delta t}}.$$

When $t \to \infty$, we obtain $0 \leq V\big(N, A\big) \leq \frac{rN_0}{\delta}$.

Therefore, all the solutions of (3)–(4) that initiating at $\mathfrak{R}^2_+$ are confined in the region $\Phi = \big\{(N(t), A(t)) : 0 \leq V(t) \leq \frac{rN_0}{\delta} + \epsilon\big\}$, for small $\epsilon > 0$ and $t \to \infty$. Hence, we may

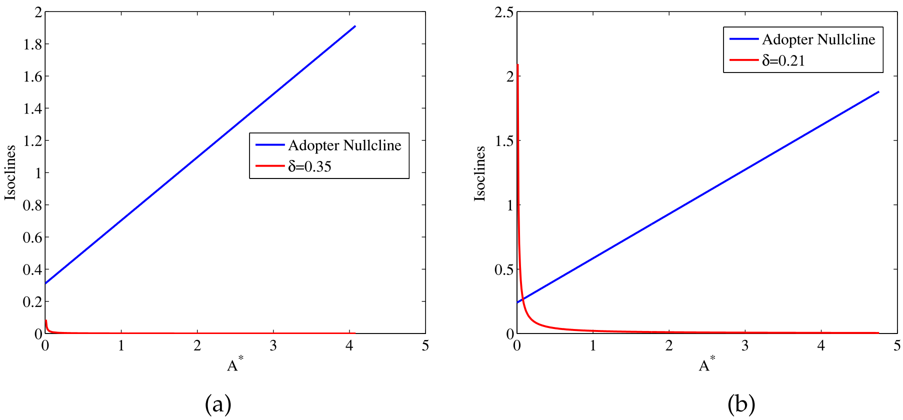


Fig. 1. Non-existence and existence of A^* for parametric values $r = 0.5245$, $N = 10$, $\gamma = 0.1$, $\kappa = 1.003$, $\alpha = 0.4253$, $v = 0.12$. (a) when $\delta \geq 0.31$ and (b) when $\delta < 0.31$.

say that all the solutions of the system (3)–(4) are positive and uniformly bounded in Φ for all positive times. $\square$

3. Stability properties of various equilibria

Regardless of the various parameter's, the model (3)–(4) maintain three appropriate equilibrium points in the closed positive quadrant of $\Re^2_+ = \{(x_1, x_2) : x_i > 0, 1 \leq i \leq 2\}$:

(i) the trivial equilibrium $E^0(0, 0)$; At $E^0(0, 0)$, the system is locally stable provided $\delta > r$, i.e., death rate of the population is more than the intrinsic growth rate of the non-adopter population and this condition is obvious.

(ii) the adopter free equilibrium $E'(N', 0)$, where $N' = \frac{rN_0}{r+\delta+\kappa h(\tau)}$. It is easy to see that $N' > 0$ and the necessary and sufficient condition for local asymptotic stability of equilibrium point E' is $v > \alpha N' h(\tau)e^{-\lambda\tau}$.

(iii) the positive equilibrium point $E^*(N^*, A^*)$, where $N^* = \frac{rN_0 - A^*\delta}{r+\delta}$ and A^* are the roots of the equation

$$\psi_1 A^{*2} + \psi_2 A^* + \psi_3 = 0,$$

such that $A^* = \frac{-\psi_2 \pm \sqrt{\psi_2^2 - 4\psi_1\psi_3}}{2\psi_1}$ is positive provided $r > \frac{1}{\alpha N_0}\left\{\kappa\delta + \frac{(r+\delta)(\delta+v)}{h(\tau)}\right\}$, where

$$\begin{cases} \psi_1 = \{\alpha(r+\delta) + \alpha\delta h(\tau)\}, \\ \psi_2 = \{(r+\delta)(\delta+v)\} + (\kappa\delta - r\alpha N_0)h(\tau), \\ \psi_3 = -r\kappa N_0 h(\tau). \end{cases}$$

It is not easy to detect the correct unique positive equilibrium in form of various parameters mandatory for more examination. By analyzing the relative locations of nullclines, we can easily dream up the non-existence and existence of positive equilibrium point (Fig. 1).

For further results, we inspect the positive equilibrium point. So, we assume here that $N^* > 0$, $A^* > 0$ for any time $t > 0$.

3.1. Characteristic equation

Linearize the system (3)–(4) around $E^*(N^*, A^*)$ presents the following set of equations,

$$\begin{cases} \dfrac{dX}{dt} = a_{11}X(t) + a_{12}X(t-\tau) + a_{13}Y(t) + a_{14}Y(t-\tau), \\ \dfrac{dY}{dt} = a_{21}X(t-\tau) + a_{22}Y(t) + a_{23}Y(t-\tau), \end{cases} \tag{6}$$

where $a_{11} = -(r+\delta)$, $a_{12} = -\big(\kappa + \alpha A^*\big)h(\tau)$, $a_{13} = \gamma A^* + v$, $a_{14} = -\alpha N^* h(\tau) + \gamma A^*$, $a_{21} = \big(\kappa + \alpha A^*\big)h(\tau)$, $a_{22} = -(\delta + v) - \gamma A^*$, and $a_{23} = \alpha N^* h(\tau) - \gamma A^*$.

The characteristic equation obtained from variational matrix J^* at E^* has the form

$$\Delta(\lambda, \tau) = \Delta_1(\lambda, \tau) + \Delta_2(\lambda, \tau)e^{-\lambda\tau} = 0, \tag{7}$$

where

$$\begin{cases} \Delta_1(\lambda, \tau) = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22}, \\ \Delta_2(\lambda, \tau) = (a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21}) - (a_{12} + a_{23})\lambda. \end{cases}$$

In the absence of evaluation period ($\tau = 0$), Eq. (7) becomes:

$$\lambda^2 - tr(J^*)\lambda + det(J^*) = 0, \tag{8}$$

Here $tr(J^*) = a_{11} + a_{22} + a_{12} + a_{23}$, and $det(J^*) = a_{11}a_{22} + a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21}$. By applying Routh–Hurwitz criterion, all the roots of Eq. (8) will have negative real parts, i.e., the positive equilibrium E^* is locally asymptotically stable (LAS) if

$$(H_0) : tr(J^*) < 0 \quad and \quad det(J^*) > 0 \quad hold.$$

Remark 1. The condition (H_0) in parametric forms implies that if $N_0 < \frac{1}{r\alpha}\big\{2\gamma(r+\delta) + \alpha\delta\big\}A^*$ is satisfied, then the model (3)–(4) remained locally stable around E^*.

Moreover, let us investigate about the global stability of $E^*(N^*, A^*)$ without any evaluation period ($\tau = 0$) in the following lemma by considering an important result from [52].

Theorem 3.1. *Suppose that the positive equilibrium point E^* exists and the condition (H_0) hold good. Then the system (3)–(4) has no non-constant periodic orbits.*

Proof. For $\tau = 0$ and (H_0) is true, it is evident that E^* is asymptotically stable. For more examination about the non-existence of periodic orbits, i.e., global stability of the equilibrium E^*, let us make an effort to apply the Dulac criteria to debar the possibility of a limit cycle. Rename the model equations as follows:

$$\Phi_1(N, A) = r\,(N_0 - N) + \gamma A^2(t) - \big(\kappa + \alpha A(t)\big)N(t) + vA(t) - \delta N(t),$$

$$\Phi_2(N, A) = \big(\kappa + \alpha A(t)\big)N(t) - (\delta + v)A(t) - \gamma A^2(t).$$

Suppose the Dulac function as

$$\Omega = \frac{1}{NA},$$

we shall have

$$\frac{\partial(\Phi_1\Omega)}{\partial N} + \frac{\partial(\Phi_2\Omega)}{\partial A} = -\frac{rN_0}{N^2A} - \frac{\gamma A + v}{N^2} - \frac{\kappa}{A^2} - \frac{\gamma}{N} < 0.$$

Applying the Dulac criteria, the system (3)–(4) with zero delay ($\tau = 0$) does not have any limit cycle in the region $\mathfrak{R}^2_+$, i.e., the system is globally stable. Hence, the result. □

4. Analysis of the Delayed Innovation Diffusion Model

Here, we are to find out the stability of the model (3)–(4) about E^* with nonzero delay ($\tau > 0$.)

Theorem 4.1. *The positive equilibrium point E^* is conditionally stable if (H_0) holds for the system (3)–(4).*

In order to examine, how delay alters the stability characteristics of the interior equilibrium E^*, we assume τ as the bifurcation parameter. To observe the instability generated by the parameter τ, consider that for the positive delay ($\tau > 0$), $\lambda = i\omega$ ($\omega > 0$, $i = \sqrt{-1}$) is an entirely imaginary solution of the exponential Eq. (7). So by using $\lambda = i\omega$ into (7) and following the framework of [43,46] to figure out real and imaginary parts, we get that

$$(a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21})cos\omega\tau - (a_{12} + a_{23})\omega sin\omega\tau = \omega^2 - a_{11}a_{22}, \tag{9}$$

$$-(a_{12} + a_{23})\omega cos\omega\tau - (a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21})sin\omega\tau = (a_{11} + a_{22})\omega. \tag{10}$$

Eliminating trigonometric functions from (9)–(10), we may obtain a fourth degree equation in ω as below:

$$\omega^4+[(a_{11}+a_{22})^2-2a_{11}a_{22}-(a_{12}+a_{23})^2]\omega^2+[a_{11}^2a_{22}^2-(a_{11}a_{23}+a_{12}a_{22}-a_{13}a_{21})^2] = 0. \tag{11}$$

Let $v = \omega^2$, then Eq. (11) becomes

$$T(v) = v^2 + T_1 v + T_2 = 0, \tag{12}$$

where

$$T_1 = (a_{11} + a_{22})^2 - 2a_{11}a_{22} - (a_{12} + a_{23})^2,$$

$$T_2 = a_{11}^2a_{22}^2 - (a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21})^2.$$

The asymptotic stability conditions are according to Routh–Hurwitz criterion for a characteristic polynomial of a two-dimensional system [32]. Therefore, all the solutions of Eq. (12) are negative if $T_1 > 0$ and $T_2 > 0$, i.e., no positive root of (11) exists. Thus, Eq. (7) will not have pure imaginary solutions. Also (H_0) ensure that the solutions of (7) have negative real parts. Using Rouche's Theorem, we observe that all solutions of (11) will have negative real parts too.

For any time delay $\tau > 0$, the positive equilibrium E^* is locally asymptotically stable. Hence, we mention the subsequent result.

Theorem 4.2. *Consider that $T_1 > 0$ and $T_2 > 0$ hold, so that the interior equilibrium point $E^* = (N^*, A^*)$ is locally asymptotically stable for any time delay $\tau > 0$.*

4.1. *Estimation for the length of delay to preserve stability*

Here, we will apply Nyquist criterion for getting the estimates on the length of τ for protecting the stability of model (3)–(4).

Let $X(t) = N(t) - \widetilde{N}^*$, $Y(t) = A(t) - \widetilde{A}^*$, and we linearize the system (3)–(4) about the equilibrium point $E^*(N^*, A^*)$.

$$\begin{cases} \dfrac{dX}{dt} = -\Big(r - \delta\Big)X - \Big(\kappa + \alpha\widetilde{A}^*\Big)h(\tau)X_\tau + (\gamma\widetilde{A}^* + v)Y + (\gamma\widetilde{A}^* - \alpha\widetilde{N}^*h(\tau))Y_\tau \ , \\ \dfrac{dY}{dt} = \Big(\kappa + \alpha\widetilde{A}^*\Big)h(\tau)X_\tau - (\delta + v + \gamma\widetilde{A}^*)Y + (\alpha\widetilde{N}^*h(\tau) - \gamma\widetilde{A}^*)Y_\tau \ . \end{cases}$$

Taking the Laplace transform, we shall get

$$\begin{aligned} sL[X] - X(0) =& -\Big(r - \delta\Big)L[u_1] - \Big(\kappa + \alpha\widetilde{A}^*\Big)h(\tau)L[u_{1\tau}] \\ &+ (\gamma\widetilde{A}^* + v)L[u_2] + (\gamma\widetilde{A}^* - \alpha\widetilde{N}^*h(\tau))L[u_{2\tau}] \ , \end{aligned} \tag{13}$$

$$\begin{aligned} sL[Y] - Y(0) =& \Big(\kappa + \alpha\widetilde{A}^*\Big)h(\tau)L[X_\tau] \\ &- (\delta + v + \gamma\widetilde{A}^*)L[Y] + (\alpha\widetilde{N}^*h(\tau) - \gamma\widetilde{A}^*)L[Y_\tau] \ , \end{aligned} \tag{14}$$

where in Eq. (13), we have

$$\begin{aligned} L[X_\tau] &= \int_0^\infty e^{-st}X(t-\tau)\,dt \\ &= \int_0^\tau e^{-st}X(t-\tau)\,dt + \int_\tau^\infty e^{-st}X(t-\tau)\,dt \end{aligned}$$

or on setting $t = t_1 + \tau$

$$\begin{aligned} L[X_\tau] &= \int_{-\tau}^0 e^{-s(t_1+\tau)}X(t_1)\text{t}_1 + \int_0^\infty e^{-s(t_1+\tau)}X(t_1)\,dt_1 \\ &= e^{-s\tau}M_1 + e^{-s\tau}L[X] \, , \end{aligned}$$

where

$$M_1 = \int_{-\tau}^0 e^{-st}X(t)\,dt.$$

Similarly, in Eq. (14) we have

$$\begin{aligned} L[Y_\tau] &= \int_{-\tau}^0 e^{-s(t_1+\tau)}Y(t_1)dt_1 + \int_0^\infty e^{-s(t_1+\tau)}Y(t_1)\,dt_1 \\ &= e^{-s\tau}M_2 + e^{-s\tau}L[Y] \, , \end{aligned}$$

where

$$M_2 = \int_{-\tau}^0 e^{-st}Y(t)\,dt.$$

Thus the system (13)–(14) can be rewritten

$$(A_1 - sI)\begin{pmatrix} L[X] \\ L[Y] \end{pmatrix} = A_2 \ ,$$

where

$$A_1 = \begin{pmatrix} -r-\delta-\big(\kappa+\alpha\widetilde{A^*}\big)h(\tau)e^{-s\tau} & (\gamma\widetilde{A^*}+\upsilon)+(\gamma\widetilde{A^*}-\alpha\widetilde{N^*}h(\tau))e^{-s\tau} \\ \big(\kappa+\alpha\widetilde{A^*}\big)h(\tau)e^{-s\tau} & -(\delta+\upsilon+\gamma\widetilde{A^*})+(\alpha\widetilde{N^*}h(\tau)-\gamma\widetilde{A^*})e^{-s\tau} \end{pmatrix},$$

$$A_2 = \begin{pmatrix} X(0)-(M_1+M_2)e^{-s\tau} \\ Y(0)-(M_1+M_2)e^{-s\tau} \end{pmatrix}.$$

The inverse Laplace of $L[X]$ and $L[Y]$ contains terms which rise exponentially with the increase of time if $L[X]$ and $L[Y]$ contain poles with the value of real parts greater than zero. For $E^*(\widetilde{N^*}, \widetilde{A^*})$ to justify the LAS (locally asymptotic stable), a necessary and sufficient condition is that all poles of $L[X]$ and $L[Y]$ have negative real parts. We will apply the Nyquist criteria, it defines that if the length of arc of a curve encircling the right half plane is X, the curve $L[X]$ will encircle the origin a number of times equal to the difference between the numbers of poles and zeros of $L[X]$ in the right half plane. This criteria is applied to X and Y.

Let

$$G(s) = s^2 + a_1 s + a_0 + (b_1 s + b_0)^{-s\tau} = 0\ ,$$

where $a_1 = -(a_{11}+a_{22})$, $a_0 = a_{11}a_{22}$, $b_1 = -(a_{12}+a_{23})$, and $b_0 = a_{11}a_{23}+a_{12}a_{22}-a_{13}a_{21}$. Note that $G(s) = 0$ is the characteristic equation of model (3)–(4) for equilibrium E^* and the zeros are the poles of $L[X]$ and $L[Y]$. The LAS properties of E^* detailed in [19] are $\Re\ G(i\upsilon_0) = 0$ and $\Im\ G(i\upsilon_0) > 0$, i.e.,

$$-\upsilon_0^2 + a_0 + b_0\cos(\upsilon_0\tau) + b_1\upsilon_0\sin(\upsilon_0\tau) = 0\ , \tag{15}$$

$$a_1\upsilon_0 + b_1\upsilon_0\cos(\upsilon_0\tau) - b_0\sin(\upsilon_0\tau) > 0\ , \tag{16}$$

where υ_0 is the smallest positive root for which $\Re\ [G(i\upsilon_0)] = 0$ and $\Im\ [G(i\upsilon_0)] > 0$. To estimate the length of delay, we need the conditions (15) and (16) that are sufficient to guarantee stability i.e. to estimate τ, we seek an upper bound υ^+ of υ_0, without any τ, such that (16) justifies for $\upsilon \in [0, \upsilon^+]$, specifically for $\upsilon = \upsilon_0$. From Eq. (15) we have

$$\upsilon_0^2 = a_0 + b_0\cos(\upsilon_0\tau) + b_1\upsilon_0\sin(\upsilon_0\tau), \tag{17}$$

such that

$$\upsilon_0^2 - |b_1|\,\upsilon_0 - |a_0| + |b_0| \le 0\ . \tag{18}$$

so if

$$\upsilon^+ = \frac{|b_1| + \sqrt{|b_1|^2 + 4(|a_0| + |b_0|)}}{2} \tag{19}$$

then $\upsilon_0 \le \upsilon^+$. From the inequality (18), we have

$$\upsilon_0^2 > \frac{b_1\upsilon_0^2}{a_1}\cos(\upsilon_0\tau) - \frac{b_0\upsilon_0}{a_1}\sin(\upsilon_0\tau) \tag{20}$$

and from Eq. (17) and inequality (20), we obtain

$$-\left(b_1\upsilon_0 + \frac{b_0\upsilon_0}{a_1}\right)\sin(\upsilon_0\tau) + \left(\frac{b_1\upsilon_0^2 - b_0}{a_1}\right)\big[\cos(\upsilon_0\tau) - 1\big] < a_0 - \frac{b_1\upsilon_0^2}{a_1} + b_0\ .$$

Using the inequalities $\sin(v_\tau) \le v_\tau$ and $1 - \cos(v_\tau) \le v^2\tau^2/2$, after simplification we get

$$\Theta_1\tau^2 + \Theta_2\tau < \Theta_3,$$

where

$$\Theta_1 = \frac{\left|b_1 v_0^2 - b_0\right|}{2\,|a_1|}(v^+)^2,\ \Theta_2 = \frac{|a_1 b_1 + b_0|}{|a_1|}v^+,\ \Theta_3 = a_0 - \frac{b_1 v_0^2}{a_1} + b_0\ ,$$

when (15) and (16) may be verified. The exact nonnegative value of τ can be given by

$$\tau_+ = \frac{1}{2\Theta_1}\left(-\Theta_2 + \sqrt{\Theta_1^2 + 4\Theta_1\Theta_3}\right), \tag{21}$$

so the Nyquist criteria holds for $0 \le \tau \le \tau_+$, and for preserving the stability, τ_+ is the proper estimation of delay. Here τ_+ is entirely depending on the different parameters of the system. Therefore, we may state the following theorem:

Theorem 4.3. *If there exists a time delay τ_+ given by Eq. (21), then for any τ such that $0 \le \tau \le \tau_+$, the equilibrium $E^*(N^*, A^*)$ is locally asymptotically stable.*

4.2. Hopf bifurcation analysis

Here, we shall conclude some criterion for Hopf bifurcation to occur in the model (3)–(4) with the use of the bifurcation parameter τ.

We give the following assumption (H_1): Eq. (12) has a minimum of one positive solution. Without loss of generality, we consider that it has two positive solutions, say v_1, v_2. Then Eq. (11) has two positive roots $\omega_i = \sqrt{v_i}$, $i = 1, 2$. Using $\lambda = i\omega_i$ in (7), and figure out the critical value of delay (τ) for which E^* will lose its stability is stated as follows:

$$\begin{aligned}\tau_i^{(k)} =& \frac{1}{\omega_{i_k}} arccos \\ &\times \left[\frac{\left\{(\omega_{i_k}^2 - a_{11}a_{22})(a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21}) - (a_{11} + a_{22})(a_{12} + a_{23})\omega_{i_k}^2\right\}}{(a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21})^2 + (a_{12} + a_{23})^2\omega_{i_k}^2}\right] \\ &+ \frac{2k\pi}{\omega_{i_k}};\ k = 0, 1, 2, \ldots;\ i = 1, 2.\end{aligned} \tag{22}$$

Let

$$\tau_0 = min\left\{\tau_i^{(0)}\right\}, i = 1, 2, \omega_0 = \omega_{i_0}.$$

Thus, at $\tau = \tau_0$, the characteristic equation (7) will have a couple of imaginary values of the type $\pm i\omega_0$.

The characteristic equation (7) can be rewritten as

$$\left(\lambda^2 + A_{22}\lambda + A_{21}\right) + \left(B_{22}\lambda + B_{21}\right)e^{-\lambda\tau} = 0, \tag{23}$$

where

$$\begin{cases} A_{22} &= -(a_{11} + a_{22}), \\ A_{21} &= a_{11}a_{22}, \\ B_{22} &= -(a_{12} + a_{23}), \\ B_{21} &= a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21}. \end{cases}$$

For the occurrence of Hopf bifurcation in the system (3)–(4), it is necessary to verify transversality condition, i.e., $Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\lambda=i\omega_0} \neq 0$, at $\tau = \tau_0$. Differentiating Equation (23) with respect to bifurcation parameter τ, we shall have:

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = -\frac{2\lambda + A_{22}}{\lambda(\lambda^2 + A_{22}\lambda + A_{21})} + \frac{B_{22}}{\lambda(B_{22}\lambda + B_{21})} - \frac{\tau}{\lambda}.$$

On solving, we get

$$\begin{aligned} Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\lambda=i\omega_0} &= Re\left\{-\frac{2\lambda + A_{22}}{\lambda(\lambda^2 + A_{22}\lambda + A_{21})}\right\}_{\lambda=i\omega_0} \\ &\quad + Re\left\{\frac{B_{22}}{\lambda(B_{22}\lambda + B_{21})}\right\}_{\lambda=i\omega_0} \\ &= \frac{A_{22}^2 + 2(\omega_0^2 - A_{21})}{A_{22}^2\omega_0^2 + (\omega_0^2 - A_{21})^2} - \frac{B_{22}^2}{B_{22}^2\omega_0^2 + B_{21}^2}. \end{aligned}$$

With the help of Eq. (11), we can obtain

$$A_{22}^2\omega_0^2 + (\omega_0^2 - A_{21})^2 = B_{22}^2\omega_0^2 + B_{21}^2.$$

Therefore,

$$\begin{aligned} Re\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\lambda=i\omega_0} &= \frac{A_{22}^2 + 2(\omega_0^2 - A_{21}) - B_{22}^2}{B_{22}^2\omega_0^2 + B_{21}^2} \\ &= \frac{T'(v^*)}{B_{22}^2\omega_0^2 + B_{21}^2}, \end{aligned}$$

where $T(v) = v^2 + T_1 v + T_2$, and $v^* = \omega_0^2$. Besides these, $Re[d\lambda/d\tau]^{-1}$ and $[dRe(\lambda)/d\tau]$ have the same sign. Hence, if $(H_2) : T'(v^*) \neq 0$ satisfies, then the transversality condition holds and the conditions for Hopf bifurcation [22] are verified for $\tau = \tau_0$, and this is the minimum positive value of $\tau_i^{(k)}$ provided by (22). Hence, as τ crosses over the critical value τ_0, the values of Eq. (7) cross the imaginary axis. The locations of the these values of (7) help us to find the stability of the zero solution of system (3)–(4). The zero solution is stable when all the values spot in the complex plane with negative real part; and unstable when real part is positive. For the examination of roots of the exponential equation (7), the subsequent lemma is helpful.

Lemma 4.4 (*Ruan and Wei [42]*)**.** *For the transcendental equation*

$$\begin{aligned} P\left(\lambda, e^{-\lambda\tau_1}, \ldots., e^{-\lambda\tau_m}\right) &= \lambda^n + p_1^{(0)}\lambda^{n-1} + \cdots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ &\quad + \left[p_1^{(1)}\lambda^{n-1} + \cdots .. + p_{n-1}^{(1)}\lambda + p_n^{(1)}\right]e^{-\lambda\tau_1} \\ &\quad + .. + \left[p_1^{(m)}\lambda^{n-1} + \cdots + p_{n-1}^{(m)}\lambda + p_n^{(m)}\right]e^{-\lambda\tau_m} \\ &= 0, \end{aligned}$$

as $(\tau_1, \tau_2, \tau_3, \ldots, \tau_m)$ *vary, the sum of orders of the zeros of*

$$P\left(\lambda, e^{-\lambda\tau_1}, \ldots., e^{-\lambda\tau_m}\right)$$

in the open right half-plane can change, and one zero appears only on or crosses the imaginary axis.

Based on the lemma 4.4, and the analysis of Hopf bifurcation, we may define the following theorem:

Theorem 4.5. *For the system (3)–(4), if the conditions* $(H_0) - (H_2)$ *are satisfied, then the nonnegative equilibrium* $E^*(N^*, A^*)$ *is LAS for* $\tau \in [0, \tau_0)$ *and becomes unstable for* $\tau > \tau_0$, *model (3)–(4) experiences Hopf bifurcation at* $E^*(N^*, A^*)$ *for* τ_0, *and*

$$\tau_0 = \frac{1}{\omega_0} arccos \times \left[\frac{\left\{(\omega_0^2 - a_{11}a_{22})(a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21}) - (a_{11} + a_{22})(a_{12} + a_{23})\omega_0^2\right\}}{(a_{11}a_{23} + a_{12}a_{22} - a_{13}a_{21})^2 + (a_{12} + a_{23})^2\omega_0^2} \right].$$

5. Stability of the Hopf bifurcating periodic solutions

Juneja et al. [26] established the stability properties of the periodic solutions by applying the results of [31]. We shall investigate the direction of Hopf bifurcation and the stability of bifurcating periodic solutions of the system (3)–(4) at $\tau = \tau_0$. More briefly, we shall employ results developed by Hassard et al. in [22], for the computation of conjugate pair of complex values of Eq. (7) on the center manifold. It is worth mentioning that Wang et al. [53] and Panja et al. [39] used the techniques to analyze a chikungunya virus infection system and phytoplankton–zooplankton–fish dynamics and harvesting model. By applying results, we shall be able to determine the direction of Hopf bifurcation, i.e., whether the bifurcation is supercritical or subcritical. By employing the Taylor series to (3)–(4) about $E^*(N^*, A^*)$, we get

$$\begin{cases} \dfrac{dX}{dt} = a_{11}X(t) + a_{12}X(t-\tau) + a_{13}Y(t) + a_{14}Y(t-\tau) + a_{15}X(t-\tau)Y(t-\tau) \\ \qquad +a_{16}Y(t)Y(t-\tau) = F_1(X, Y), \\ \dfrac{dY}{dt} = a_{21}X(t-\tau) + a_{22}Y(t) + a_{23}Y(t-\tau) + a_{24}X(t-\tau)Y(t-\tau) \\ \qquad +a_{25}Y(t)Y(t-\tau) = F_2(X, Y), \end{cases} \tag{24}$$

where $a_{11} = -(r+\delta)$, $a_{12} = -\big(\kappa + \alpha A^*\big)h(\tau)$, $a_{13} = \gamma A^* + v$, $a_{14} = -\alpha N^* h(\tau) + \gamma A^*$, $a_{15} = -\alpha h(\tau)$, $a_{16} = \gamma$, $a_{21} = \big(\kappa + \alpha A^*\big)h(\tau)$, $a_{22} = -(\delta + v) - \gamma A^*$, $a_{23} = \alpha N^* h(\tau) - \gamma A^*$, $a_{24} = \alpha h(\tau)$ and $a_{25} = -\gamma$. Suppose $\tau = \tau_0 + \mu$, $u(t) = (X(t), Y(t))^T$ and $u_t(\theta) = u(t+\theta)$, for $\theta \in [-\tau, 0]$.

Denote

$$C^k[-\tau, 0] = \{\phi \mid \phi : [-\tau, 0] \to \mathfrak{R}_+^2\},$$

where ϕ has k-order continuous derivative. Then the system (3)–(4) is equivalent to the following Functional Differential Equation :

$$\dot{u}(t) = L_\mu(u_t) + f(\mu, u_t), \tag{25}$$

with

$$L_\mu(\phi) = F_1\phi(0) + F_2\phi(-\tau),$$

and

$$f(\mu, \phi) = \begin{pmatrix} a_{15}\phi_1(-\tau)\phi_2(-\tau) + a_{16}\phi_2^2(-\tau) \\ a_{24}\phi_1(-\tau)\phi_2(-\tau) + a_{25}\phi_2^2(-\tau) \end{pmatrix}, \tag{26}$$

where

$$F_1 = \begin{pmatrix} a_{11} & a_{13} \\ 0 & a_{22} \end{pmatrix},$$

and

$$F_2 = \begin{pmatrix} a_{12} & a_{14} \\ a_{21} & a_{23} \end{pmatrix}.$$

Then L_μ is one parameter family of bounded linear operator in $C[-\tau, 0]$. By the Riesz representation theorem, there exists 2×2 matrix-valued function $\eta(\theta, \mu)$ such that

$$\eta(., \mu) : [-\tau, 0] \to \mathfrak{R}^{2\times 2}, \tag{27}$$

for $\phi \in C[-\tau, 0]$ such that

$$L_\mu = \int_{-\tau}^{0} d\eta(\theta, \mu)\phi(\theta). \tag{28}$$

We can opt for

$$\eta(\theta, \mu) = F_1\delta(\theta) + F_2\delta(\theta + \tau), \tag{29}$$

where $\delta(\theta)$ is a Dirac delta function.

For $\phi \in C'([-\tau, 0], \mathfrak{R}^{2\times 2})$, we define

$$A(\mu)\phi(\theta) = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & \theta \in [-\tau, 0), \\ \displaystyle\int_{-\tau}^{0} d\eta(\theta, \mu)\phi(\theta), & \theta = 0, \end{cases}$$

and

$$R(\mu)\phi(\theta) = \begin{cases} 0, & \theta \in [-\tau, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases}$$

Since $\frac{du_t}{d\theta} = \frac{du_t}{dt}$, the system (25) is equivalent to

$$\dot{u}(t) = A(\mu)u_t(\theta) + R(\mu)u_t(\theta). \tag{30}$$

For $\theta \in [-\tau, 0)$, (30) is just the trivial equation $\frac{du_t}{d\theta} = \frac{du_t}{dt}$; for $\theta = 0$, it is (25). For $\psi \in C^1([0, 1], (\mathfrak{R}_+^2)^*)$, the adjoint A^* of A is defined as

$$A^*(\mu)\psi(\theta) = \begin{cases} -\dfrac{d\psi}{d\theta} & if \quad \theta \in (0, \tau], \\ \displaystyle\int_{-\tau}^{0} d\eta(\theta, \mu)\psi(-\theta) & if \quad \theta = 0. \end{cases}$$

For $\phi \in [-\tau, 0)$, we define an inner bilinear form

$$\langle \psi, \phi \rangle = \bar{\psi}^T(0)\phi(0) - \int_{\theta=-\tau}^{0} \int_{\xi=0}^{\theta} \bar{\psi}^T(\xi - \theta)d\eta(\theta)\phi(\xi)d\xi, \tag{31}$$

where $\eta(\theta) = \eta(\theta, 0)$. From the above analysis, we obtain that $\pm i\omega_0$ are the eigenvalues of $A(0)$. Let $\rho(\theta)$ be the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$, then we get

$$A(0)\rho(\theta) = i\omega_0\rho(\theta).$$

Since $\pm i\omega_0$ are the eigenvalues of $A(0)$, and other eigenvalues have strictly negative real parts, $\mp i\omega_0$ are the eigenvalues of $A^*(0)$. Then we may state the following theorem

Theorem 5.1. *Let $\rho(\theta) = He^{i\omega_0\theta}$ be the eigenvector of A associated with $i\omega_0$, and $\rho^*(\theta) = DH^*e^{i\omega_0\theta}$ be the eigenvector of A^* associated with $-i\omega_0$. Then $\langle\rho^*, \rho\rangle = 1$, $\langle\rho^*, \bar{\rho}\rangle = 0$, where $H = (1, \rho_2)^T$, $H^* = (1, \rho_2^*)^T$,*

$$\rho_2 = \frac{-a_{21}e^{-i\omega_0\tau_0}}{a_{22} + a_{23}e^{-i\omega_0\tau_0} - i\omega_0},$$

$$\rho_2^* = -\frac{a_{11} + a_{12}e^{i\omega_0\tau_0} + i\omega_0}{a_{21}e^{i\omega_0\tau_0}},$$

and

$$\bar{D} = \frac{1}{1 + \rho_2\bar{\rho}_2^* + [(a_{12} + \rho_2 a_{14}) + \bar{\rho}_2^*(a_{21} + \rho_2 a_{23})]\tau_0 e^{-i\omega_0\tau_0}}.$$

By employing the phenomena of Hassard et al. [22], we find out the coordinates to express the center manifold center C_0 at $\mu = 0$, which is locally invariant, attracting two-dimensional manifold in C_0. Suppose u_t is a solution of (30) at $\mu = 0$.

Define $z(t) = \langle\rho^*, u_t\rangle$,

$$W(t, \theta) = u_t(\theta) - 2Re\{z(t)\rho(\theta)\}. \tag{32}$$

On the center manifold C_0, we get $W(t, \theta) = W(z(t), \bar{z}(t), \theta)$ where

$$W(z(t), \bar{z}(t), \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \tag{33}$$

Here z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of ρ and ρ^*. We shall consider only real solutions as W is real if u_t is real. For the solution $u_t \in C_0$ of (30), since $\mu = 0$, we have

$$\begin{aligned}\dot{z}(t) &= i\omega_0\tau_0 z + \bar{\rho}^*(0)f(0, W(z, \bar{z}, 0) + 2Re(z\rho(\theta)))\\ &= i\omega_0\tau_0 z + \bar{\rho}^*(0)f_0(z, \bar{z}),\end{aligned}$$

or

$$\dot{z}(t) = i\omega_0\tau_0 z + g(z, \bar{z}),$$

where

$$\begin{aligned}g(z, \bar{z}) &= \bar{\rho}^*(0)f_0(z, \bar{z})\\ &= g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\bar{z}^2}{2} + g_{21}(\theta)\frac{z^2\bar{z}}{2} + \cdots;\end{aligned} \tag{34}$$

From (33) and (34), we have

$$\begin{aligned}u_t(\theta) &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2}\\ &\quad + (1, \rho_2)^T e^{i\omega_0\tau_0\theta}z + (1, \bar{\rho}_2)^T e^{-i\omega_0\tau_0\theta}\bar{z} + \cdots.\end{aligned} \tag{35}$$

Now from (26) and (34), it follows that

$$g(z,\bar{z}) = \tau_0\bar{D}\Bigg[\big(\xi_{11}+\xi_{21}\bar{\rho_2^*}\big)z^2+\big(\xi_{12}+\xi_{22}\bar{\rho_2^*}\big)z\bar{z} + \big(\xi_{13}+\xi_{23}\bar{\rho_2^*}\big)\bar{z}^2+\big(\xi_{14}+\xi_{24}\bar{\rho_2^*}\big)\frac{z^2}{2}\bar{z}+\cdots\Bigg], \tag{36}$$

where

$$\begin{cases}\xi_{11} = a_{15}\rho_2 e^{-2i\omega_0\tau_0}+a_{16}\rho_2^2 e^{-2i\omega_0\tau_0},\\ \xi_{12} = 2a_{15}Re(\rho_2)+2a_{16}(\rho_2\bar{\rho}_2),\\ \xi_{13} = a_{15}\bar{\rho}_2 e^{2i\omega_0\tau_0}+a_{16}\bar{\rho}_2{}^2 e^{2i\omega_0\tau_0},\\ \xi_{14} = 2a_{15}\Bigg(W_{11}^2(-\tau_0)+W_{20}^2(-\tau_0)\\ \qquad +2\rho_2 e^{-i\omega_0\tau_0}W_{11}^1(-\tau_0)+\bar{\rho}_2 e^{i\omega_0\tau_0}W_{20}^1(-\tau_0)\Bigg)\\ \qquad +2a_{16}\Big(2\rho_2 e^{-i\omega_0\tau_0}W_{11}^2(-\tau_0)+\bar{\rho}_2 e^{i\omega_0\tau_0}W_{20}^1(-\tau_0)\Big),\end{cases}$$

and

$$\begin{cases}\xi_{21} = a_{24}\rho_2 e^{-2i\omega_0\tau_0}+a_{25}\rho_2^2 e^{-2i\omega_0\tau_0},\\ \xi_{22} = 2a_{24}Re(\rho_2)+2a_{25}(\rho_2\bar{\rho}_2),\\ \xi_{23} = a_{24}\bar{\rho}_2 e^{2i\omega_0\tau_0}+a_{25}\bar{\rho}_2{}^2 e^{2i\omega_0\tau_0},\\ \xi_{24} = 2a_{24}\Bigg(W_{11}^2(-\tau_0)+W_{20}^2(-\tau_0)\\ \qquad +2\rho_2 e^{-i\omega_0\tau_0}W_{11}^1(-\tau_0)+\bar{\rho}_2 e^{i\omega_0\tau_0}W_{20}^1(-\tau_0)\Bigg)\\ \qquad +2a_{25}\Big(2\rho_2 e^{-i\omega_0\tau_0}W_{11}^2(-\tau_0)+\bar{\rho}_2 e^{i\omega_0\tau_0}W_{20}^1(-\tau_0)\Big).\end{cases}$$

Following the same phenomena and applying the analogous computational results given in Hassard et al. [22], we find out the important quantities by comparing coefficients of (34) and (36) as below:

$$\left\{\begin{aligned} g_{20} &= 2\bar{D}\tau_0\big(\xi_{11}+\xi_{21}\bar{\rho_2^*}\big)\\ g_{11} &= \bar{D}\tau_0\big(\xi_{12}+\xi_{22}\bar{\rho_2^*}\big)\\ g_{02} &= 2\bar{D}\tau_0\big(\xi_{13}+\xi_{23}\bar{\rho_2^*}\big)\\ g_{21} &= \bar{D}\tau_0\big(\xi_{14}+\xi_{24}\bar{\rho_2^*}\big)\end{aligned}\right\} \tag{37}$$

Since W_{20} and W_{11} are in g_{21}, we need to calculate them. From (25) and (32), we have

$$\begin{aligned}\dot{W} &= \dot{u}(t)-\dot{z}\rho-\dot{\bar{z}}\bar{\rho}\\ &= \begin{cases} A(0)W-2Re\bar{\rho}^*(0)f_0\rho(\theta), & \theta\in[-\tau_0,0),\\ A(0)W-2Re\bar{\rho}^*(0)f_0\rho(0)+f_0(z,\bar{z}), & \theta=0,\end{cases}\\ &\triangleq A(0)W+H(z,\bar{z},\theta),\end{aligned} \tag{38}$$

where

$$H(z(t),\bar{z}(t),\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\bar{z}+H_{02}(\theta)\frac{\bar{z}^2}{2}+\cdots \tag{39}$$

Substituting (39) into (38) and comparing the coefficients, we get

$$\begin{aligned}(A(0)-2i\omega_0\tau_0 I)W_{20}(\theta)&=-H_{20}(\theta),\ A(0)W_{11}(\theta)\\ &=-H_{11}(\theta),\end{aligned} \tag{40}$$

From (38) and for $\theta\in[-\tau_0,0)$

$$\begin{aligned}H(z(t),\bar{z}(t),\theta)&=-\bar{\rho}^*(0)f_0\rho(\theta)-\rho^*(0)\bar{f}_0\bar{\rho}(\theta)\\ &=-g(z,\bar{z})\rho(\theta)-\bar{g}(z,\bar{z})\bar{\rho}(\theta).\end{aligned} \tag{41}$$

Using (34) in (41) and comparing the coefficients with (39), we obtain

$$H_{20}(\theta)=-g_{20}\rho(\theta)-\bar{g}_{02}\bar{\rho}(\theta), \tag{42}$$

and

$$H_{11}(\theta)=-g_{11}\rho(\theta)-\bar{g}_{11}\bar{\rho}(\theta). \tag{43}$$

From the definition of $A(0)$, (40) and (42), we obtain

$$\dot{W}_{20}(\theta)=2i\omega_0\tau_0 W_{20}(\theta)+g_{20}\rho(\theta)+\bar{g}_{02}\bar{\rho}(\theta),$$

Solving it and for $\rho(\theta)=(1,\rho_2)^T e^{i\omega_0\tau_0\theta}$, we have

$$W_{20}(\theta)=\frac{ig_{20}}{\omega_0\tau_0}\rho(0)e^{i\omega_0\tau_0\theta}+\frac{i\bar{g}_{02}}{3\omega_0\tau_0}\bar{\rho}(0)e^{-i\omega_0\tau_0\theta}+E_1e^{2i\omega_0\tau_0\theta}. \tag{44}$$

Similarly, from (40) and (43), we get

$$W_{11}(\theta)=-\frac{ig_{11}}{\omega_0\tau_0}\rho(0)e^{i\omega_0\tau_0\theta}+\frac{i\bar{g}_{11}}{\omega_0\tau_0}\bar{\rho}(0)e^{-i\omega_0\tau_0\theta}+E_2, \tag{45}$$

where $E_1=(E_1^{(1)},E_1^{(2)})^T$ and $E_2=(E_2^{(1)},E_2^{(2)})^T$ are two dimensional constant vectors, and can be determined by setting $\theta=0$ in $H(z,\bar{z},\theta)$. Again from the definition of $A(0)$ and (40), we have

$$\int_{-1}^{0}d\eta(0,\theta)W_{20}(\theta)=2i\omega_0\tau_0W_{20}(0)-H_{20}(0), \tag{46}$$

and

$$\int_{-1}^{0}d\eta(0,\theta)W_{11}(\theta)=-H_{11}(0). \tag{47}$$

From (38), we know that when $\theta=0$,

$$\begin{aligned}H(z,\bar{z},0)&=-2Re(\bar{\rho}^*(0)f_0\rho(0))+f_0(z,\bar{z})\\ &=-\bar{\rho}^*(0)f_0\rho(0)-\rho^*(0)\bar{f}_0\bar{\rho}(0)+f_0(z,\bar{z}),\end{aligned}$$

i.e.,

$$\begin{aligned}
&H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots \\
&\quad = -\rho(0)\Big\{g_{20}(\theta)\frac{z^2}{2} + g_{11}(\theta)z\bar{z} + g_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots\Big\} \\
&\qquad - \bar{\rho}(0)\Big\{\bar{g}_{20}\frac{\bar{z}^2}{2} + \bar{g}_{11}z\bar{z} + \bar{g}_{02}\frac{z^2}{2} + \cdots\cdots\Big\} + f_0(z,\bar{z}).
\end{aligned} \tag{48}$$

From (32), we have

$$\begin{aligned}
u_t(\theta) &= W(t,\theta) + 2Re\{z(t)\rho(\theta)\} \\
&= W(t,\theta) + z(t)\rho(\theta) + \bar{z}(t)\bar{\rho}(\theta) \\
&= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots,
\end{aligned}$$

We can also obtain,

$$f_0 = 2\tau_0\begin{pmatrix}\xi_{11}\\ \xi_{21}\end{pmatrix}\frac{z^2}{2} + \tau_0\begin{pmatrix}\xi_{12}\\ \xi_{22}\end{pmatrix}z\bar{z} + \cdots. \tag{49}$$

From (48) and (49), we can also have

$$H_{20}(0) = -g_{20}\rho(0) - \bar{g}_{02}\bar{\rho}(0) + 2\tau_0\begin{pmatrix}\xi_{11}\\ \xi_{21}\end{pmatrix}, \tag{50}$$

$$H_{11}(0) = -g_{11}\rho(0) - \bar{g}_{11}\bar{\rho}(0) + \tau_0\begin{pmatrix}\xi_{12}\\ \xi_{22}\end{pmatrix}. \tag{51}$$

Since $i\omega_0\tau_0$ is the eigenvalue of $A(0)$ corresponding to $\rho(0)$ and $-i\omega_0\tau_0$ is the eigen value of $A^*(0)$ corresponding to $\bar{\rho}(0)$, then

$$\Big\{i\omega_0\tau_0 I - \int_{-1}^{0} e^{i\omega_0\tau_0\theta}d\eta(\theta)\Big\}\rho(0) = 0,$$

and

$$\Big\{-i\omega_0\tau_0 I - \int_{-1}^{0} e^{-i\omega_0\tau_0\theta}d\eta(\theta)\Big\}\bar{\rho}(0) = 0.$$

Substituting (44) and (46) into (50), and after simplification, we find

$$E_1 = 2\Big(2i\omega_0\tau_0 - F_1 - F_2e^{-2i\omega_0\tau_0}\Big)^{-1}(\xi_{11},\xi_{21})^T.$$

Similarly, substituting (45) and (47) into (51), we have

$$E_2 = \big(-F_1 - F_2\big)^{-1}(\xi_{12},\xi_{22})^T.$$

Therefore, we can find $W_{20}(\theta)$, $W_{11}(\theta)$ from (44) and (45). Also, we analyze that g_{ij} in (37) can be computed by parameters and delay in the model (3)–(4). Hence, we are able to determine the quantities:

$$\begin{cases} C_1(0) = \dfrac{i\left\{g_{20}g_{11}-2|g_{11}|^2-\dfrac{|g_{02}|^2}{3}\right\}}{2\omega_0\tau_0} + \dfrac{g_{21}}{2}, \\ \beta_2 = 2Re\left\{C_1(0)\right\}, \\ \mu_2 = -\dfrac{Re\left\{C_1(0)\right\}}{Re\left\{\dfrac{d\lambda(\tau_0)}{d\tau}\right\}}, \\ T_2 = -\dfrac{\Im\left\{C_1(0)\right\}+\mu_2\Im\left\{\dfrac{d\lambda(\tau_0)}{d\tau}\right\}}{\omega_0\tau_0}. \end{cases}$$

which determine the quantities of bifurcation periodic solutions in the center manifold for τ_0. By Hassard et al. [22], we summarize the results at τ_0 as follows:

Theorem 5.2. *At the critical value of τ_0, the basic properties of the Hopf-bifurcation are stated as follows:*

(a) The sign of β_2 determines the stability of the periodic solution: the bifurcation periodic solutions are unstable (stable) if $\beta_2 > 0(< 0)$; (b) The sign of μ_2 finds the direction of bifurcation: if $\mu_2 < 0(> 0)$, then the Hopf-bifurcation is subcritical(supercritical) and the bifurcated solutions exist for $\tau > \tau_0(\tau < \tau_0)$; (c) The sign of T_2 determines the period of the periodic solutions: the period decrease (increase) if $T_2 < 0(> 0)$.

The results of Theorems 5.1 and 5.2 help us to draw the vital outcomes related to the sign of $Re(C_1(0))$. Precisely, if $Re(C_1(0)) < 0$, model (3)–(4) will have stable periodic solutions for $\tau > \tau_0$ in a τ_0-neighborhood.

6. Numerical simulation

Here, we provide some numerical simulations of the model (3)–(4) for the support of analytical achievements. We suppose the example for a set of parametric values:

$$\begin{aligned} \frac{dN(t)}{dt} &= 0.5245\,(10 - N) + 0.1A(t)A(t-\tau) - \big(1.003 + 0.4253A(t-\tau)\big) \\ &\quad \times N(t-\tau)e^{-(0.21+0.01)\tau} + 0.12A(t) - 0.21N(t), \end{aligned} \tag{52}$$

$$\begin{aligned} \frac{dA(t)}{dt} &= \big(1.003 + 0.4253A(t-\tau)\big)N(t-\tau)e^{-(0.21+0.01)\tau} - (0.21 + 0.12) \\ &\quad \times A(t) - 0.1A(t)A(t-\tau). \end{aligned} \tag{53}$$

By using Matlab software, the system (52)–(53) is integrated with initial data $N(t) = 0.1$, $A(t) = 0.1$, we get the positive equilibrium point $E^*(3.324, 13.35)$. Also, for the example of the system (52)–(53), the local asymptotic stability condition H_0 for zero delay, i.e., $tr(J^*) = -0.0129 < 0$, $det(J^*) = 0.5463 > 0$, it proves that E^* remains asymptotically stable with parameters mentioned in the example (52)–(53) (see Fig. 2).

For the innovation diffusion system with delay, we compute a positive root $\omega_0 = 0.5027$ from Eq. (11), and using it in (22), the critical value of evaluation period $\tau_0 = 1.85$ have been calculated and observe that E^* changes its stability to periodic oscillations as τ passes through this τ_0. Furthermore, we see that $(H_2) : T'(v^*) = 3.4521 \neq 0$, and the transversality

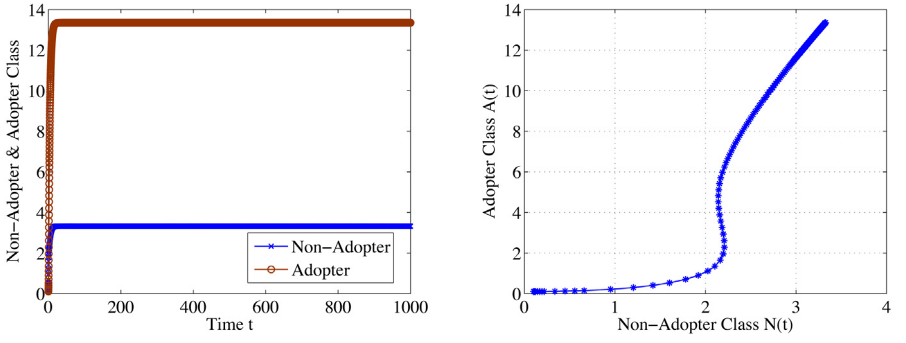


Fig. 2. Local asymptotic stability at $E^*(3.324, 13.35)$ for Non-Adopter and Adopter Class with $\tau = 0$.

condition $\left\{Re[d\lambda/d\tau]^{-1}\right\}_{\tau=\tau_0,\omega=\omega_0} = 0.0245 \neq 0$ is satisfied. It implies that the interior equilibrium point E^* remains stable for $0 \leq \tau < 1.85$ and changes its stability to instability for $\tau \geq 1.85$ (see Figs. 3 and 4). A more stable periodic solution exists for $\tau = 6$ in Fig. 5.

Also, the numerical values of stability determining quantities for periodic solutions at critical value τ_0 are given by

$$\begin{cases} C_1(0) = -0.3305 - 4.3766i; \\ \beta_2 = -0.6614; \\ \mu_2 = 0.0531; \\ T_2 = 5.0840. \end{cases}$$

By using Theorem 5.2, we summarize that the Hopf bifurcation is supercritical, the bifurcating periodic solutions exist for $\tau > \tau_0$ and solutions from E^* are asymptotically stable. The corresponding waveform and phase diagrams are plotted in Figs. 4–5.

Moreover, when the system (52)–(53) is again integrated with same initial data, and for some longer delay beyond the critical value $'\tau_0'$, the periodic solutions exist for the interval $[1.85, 12)$; and when the value of τ is considered in the interval $[12, 25)$, quasi-periodic solutions occur and is shown in Fig. 6 at $\tau = 22$, which indicate that the period of oscillations has been either increasing or decreasing, i.e., it varies continuously. It means that we are achieving the temporary stage of adoption process because the adopters are continuously changing their decisions, i.e., either they are shifting over to adopter class or to non-adopter class. Further, when τ is chosen in the interval $[25, 35)$, then their exist complex attractors around interior equilibrium point E^* arise, and it has been established in Fig. 7 at $\tau = 32$. The numerical simulation indicate that a smaller value of evaluation period helped the innovation to diffuse in the markets and it has destabilizing effects on the diffusion modeling process. This evaluation period helped the system to changing its state from a locally asymptotically stable point to a limit cycle, then to quasi-periodic solutions, and finally the existence of chaotic situations in the markets.

6.1. *Impact of intra-specific competition*

It is an important point to observe here that an increase in the coefficient of intra-specific competition γ between the existing technology developers $A(t)$ and potential technology

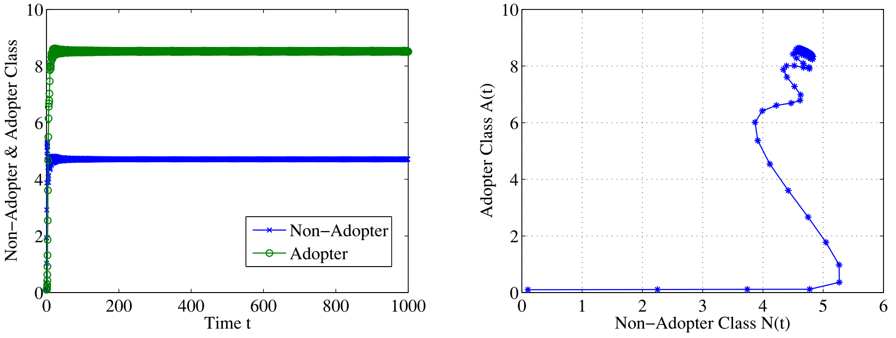
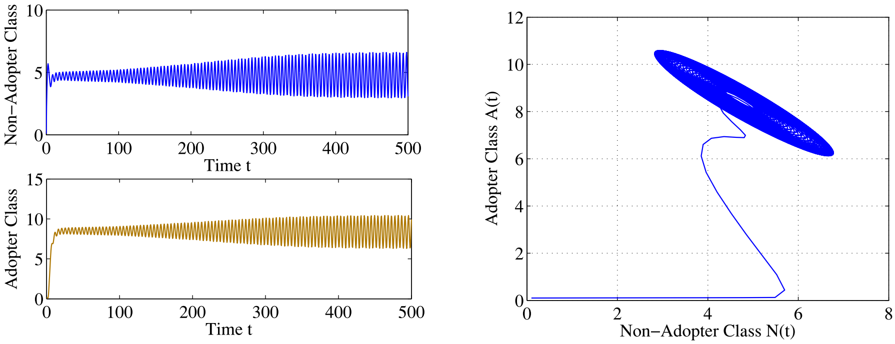
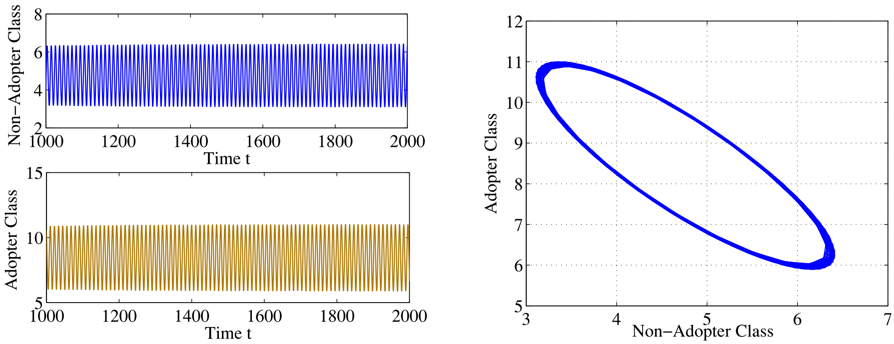


Fig. 3. Solution trajectories of system (52)–(53) are converging to $E^*(4.705, 8.517)$ at $\tau = 1.52 < 1.85 = \tau_0$.



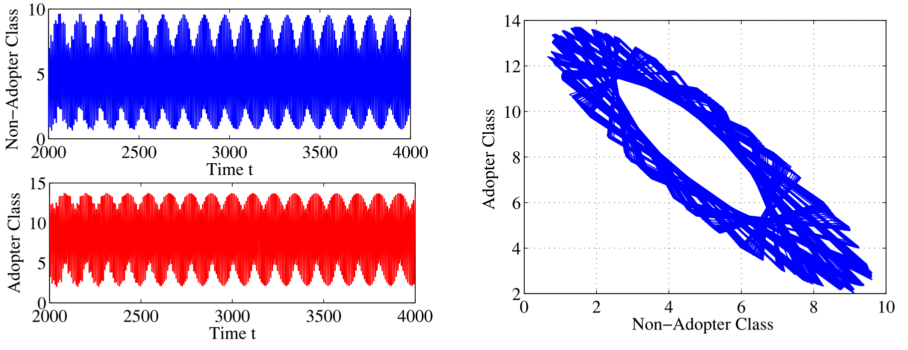
(a) (b)

Fig. 4. Hopf bifurcating periodic solutions around $E^*(4.687, 8.422)$ of the system (52)–(53) for Non-Adopter and Adopter population's at $\tau = 1.95 > 1.85 = \tau_0$.



(a) (b)

Fig. 5. A stable limit cycle at $\tau = 6$. (a) The periodic time series of Non-Adopter and Adopter populations. (b) The portrait of Non-Adopter versus Adopter population.



(a) (b)

Fig. 6. The attractive quasi-periodic solution curves around E^* at $\tau = 22$. (a) The time series of Non-Adopter and Adopter populations. (b) The plot of Non-Adopter versus Adopter population.

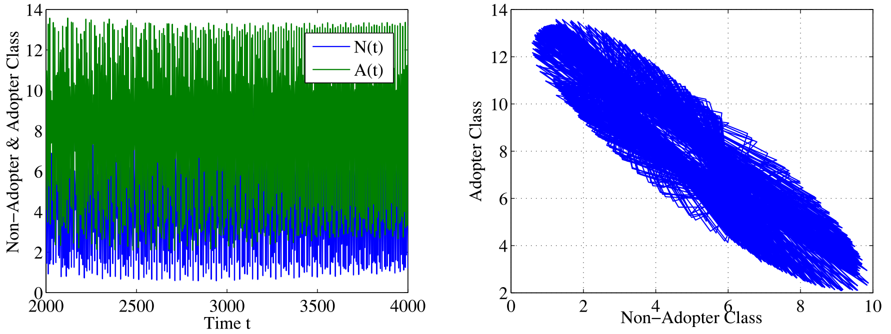


Fig. 7. System predicting Chaotic attractors of solution curves around E^* at $\tau = 32$.

developers $N(t)$ lead to maturity stage in the markets. This means that if the system (52)–(53) is integrated with the same initial data and delay $\tau = 1.95$, the periodic oscillations occur when the value of intra-specific competition γ lies in the interval $[0.01, 0.12)$ and as the value of γ is chosen in the interval $[0.12, 0.98)$, we attain stable equilibrium position in the system. This predicts that stable dynamical behavior of our proposed model has been observed by altering its states from limit cycle-to-stability, and is predicted that the adopter population $A(t)$ goes on decreasing for the intra-specific competition parameter $\gamma = 0.2, 0.5, 0.7, 0.9$ respectively (see Fig. 8). The numerical simulations revealed that the intra-specific competition between the technology developers and potential technology developers led to the mature decision making stage in the innovation diffusion system and due to this competition coefficient, we are getting maturity stage in the system.

7. Conclusion

We have established a qualitative analysis by examining the asymptotic stability of the positive equilibrium E^*, and made important observations that without any evaluation period ($\tau = 0$), the given system does not show any excitability, but crossing over a threshold limit

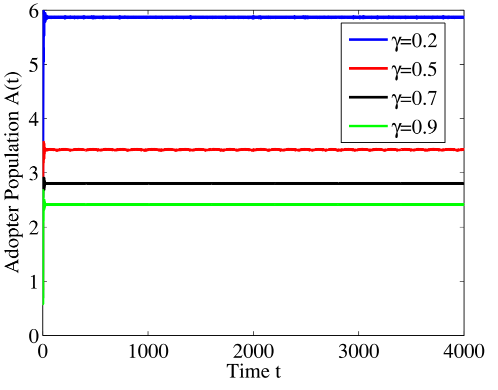


Fig. 8. Stable equilibrium position of solution curves at $\tau = 1.95$ around E^* for Non-Adopter and Adopter classes for various γ.

of evaluation period $(0 < \tau < 1.85)$, the stability exchange take place and there exist small period oscillations in the system. Numerical investigations show the oscillatory behavior for the bifurcating periodic solutions (Figs. 4–5). Hence, we have detected that evaluation period in the innovation diffusion model causes Hopf bifurcation. We have also computed the value of $Re(C_1(0)) < 0$ with the help of center manifold theorem and normal form theory. It proves that the Hopf bifurcation is supercritical and bifurcating solutions are orbitally stable. Further, as and when the potential adopters take longer evaluation period to decide about the adoption of the technology, the system exhibits irregular periodic solutions in the form of quasi-periodic attractors around E^* (see Fig. 6) and after that complex situations in the markets (see Fig. 7). The outcomes proved that the evaluation period must be required for the potential technology developers to make evaluation, and it induced instability in the innovation diffusion model by reshaping the stable equilibrium position to limit cycle and exhibits complex dynamical behavior of the innovation diffusion system (1)–(2).

Moreover, it has been observed that the intra-specific competition between the existing technology developers and technology developers in times $t-\tau$ has a big role in transforming the limit cycle to stable equilibrium position of the system (1)–(2) (see Fig. 8). This reflects that the intra-specific competition coefficient has subsided the effect of evaluation period and helped the system to arrive at stable equilibrium stage (final adoption stage). In other words, we can find the final number of adopters of the technology and it will help the marketing managers to maximize the profits. It reflects that the time delay and intra-specific competition have played a vital role for gaining better understanding of the innovation diffusion systems and interpreting diffusion patterns of a technology in theory and in practical. This research will have a great future scope, as it will help to investigate the innovation diffusion models for two or more than two technologies (innovations) with multiple delays. We leave the analysis of this type of design with complex bifurcations as the future work.

Acknowledgments

We shall be thankful to the reviewers for their valuable time spent on the reviewing the article and welcome vital remarks and suggestions for improving the current form of the work. The authors are thankful to I.K.G. Punjab Technical University, Kapurthala, Punjab, India for the support and cooperation.

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