



ORIGINAL ARTICLE

Asymptotics of extremal polynomials off the unit circle

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Abstract We investigate the asymptotic behaviour of L_p extremal polynomials for $p > 0$ on the unit circle plus a denumerable set of mass points, with only Szegő's condition imposed on the absolute part of the measure.

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1. Introduction

This paper deals with the asymptotic behavior of L_p extremal polynomials ($0 < p < \infty$) outside the unit circle under only Szegő's condition. More precisely, we highlight some results regarding the asymptotics of this class of polynomials. The study of the asymptotics of L_p extremal polynomials is of great interest in the theory of general orthogonal and extremal polynomials. It is a generalization of the widely known problem of the asymptotic of polynomials orthonormal with

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respect to a positive measure with an infinite compact support. The results on the extremal polynomials over the circle have applications to stochastic processes, Toeplitz operators, random matrix theory scattering theory, rational approximation, eigenvalue problems, Fourier expansion, best ratio approximation. . .

Until now most of the results on the asymptotic theory of orthogonal polynomials, have concentrated on orthogonal polynomials for which the measure of orthogonality is perturbed by an infinite set of mass points outside the segment [9] or the unit circle [8] and the only requirement on the absolute part of the measure is Szegő's condition. Lastly research on this subject has been mainly due to Bello Hernandez and Minguez Cenicerros [1], Kaliaguine [3], Khaldi [4,5], Li and Pan [6], Lubinsky and Saff [7], Nazarov et al. [8], Peherstorfer and Yudiskii [9], where the case of measure supported on the segment, circle, curve and arc has been hardly touched. In this sense, our main result is related to prove the asymptotic of L_p extremal polynomials for $p > 0$, on the unit circle plus a denumerable set of mass points, with only Szegő's condition imposed on the absolute part of the measure. Up to now, the only known results in this direction were obtained by Peherstorfer and Yudiskii [9], their results concerned the orthogonal polynomials ($p = 2$) on the segment plus a denumerable set of mass points, and by Nazarov et al. [8] concerning the circle. This problem is known to be interesting and difficult to analyze and the results of this paper are new and can be considered as a contribution to the evolution of this field. This paper is organized as follows: We give in Section 2 some basic definitions and notations to be able to state our results, we also recall the definition of Hardy space, Szegő function, and expose the extremal problems in Hardy spaces. Our main results, namely Theorem 3.1 and Corollary 3.1 are proved in Section 3.

2. Preliminaries and extremal problems

We define the $L_p(\sigma)$ extremal polynomials associated to the measure σ as the monic polynomials $T(z) = z^n + \dots$ that minimize the $L_p(\sigma)$ norm in the set of monic polynomials of degree n :

$$\|T_{n,p}\|_{L_p(\sigma)} := \min_{Q \in P_{n-1}} \|z^n + Q\|_{L_p(\sigma)} = m_{n,p}(\sigma),$$

where

$$\|f\|_{L_p(\sigma)} := \left(\int |f(\xi)|^p d\sigma(\xi) \right)^{1/p},$$

σ is a finite positive Borel measure with an infinite compact support in the complex plane and P_n is the set of polynomials of degree n . For $p = 2$, the $L_2(\sigma)$ extremal polynomials are exactly the orthogonal polynomials associated to the Borel measure σ .

The main result of the present paper is the study of the $L_p(\sigma)$ extremal polynomials $T_{n,p}(z)$ outside the unit circle Γ , where the measure σ has a decomposition of the form

$$\sigma = \frac{\mu}{2\pi} + \gamma$$

μ is a measure supported on the unit circle $\Gamma := \{z \in \mathbb{C} : |z| = 1\}$ and is absolutely continuous with respect to the Lebesgue measure $d\theta$ on $[-\pi, +\pi]$, that is:

$$d\mu(\theta) = \rho(\theta) d\theta, \rho \geq 0, \rho \in L^1([-\pi, +\pi], d\theta), \quad (1)$$

and γ is a point measure supported on $\{z_k\}_{k=1}^{\infty}$, ($|z_k| > 1$), that is:

$$\gamma = \sum_{k=1}^{\infty} A_k \delta(z - z_k), A_k > 0, \sum_{k=1}^{\infty} A_k < +\infty. \quad (2)$$

Next, we give some materials that will be used in the study of the asymptotics of extremal polynomials. Denote $G = \{z \in \mathbb{C} : |z| > 1\} \cup \{\infty\}$. We say that $f \in H^p(G)$ if f is analytic in G and $\int_{C_r} |f(z)|^p |dz| \leq C, 1 < r \leq 2, C_r = \{z \in \mathbb{C} : |z| = r\}$, where C is a constant independent of r . If the Radon–Nikodim derivative ρ of the measure μ satisfies Szegő's condition:

$$\int_{-\pi}^{+\pi} \log(\rho(\theta)) d\theta > -\infty \quad (3)$$

then the associate Szegő function D :

$$D(w) = \exp \left\{ -\frac{1}{2p\pi} \int_{-\pi}^{+\pi} \log(\rho(t)) \frac{w + e^{it}}{w - e^{it}} dt \right\}; (|w| > 1),$$

has the following properties:

$$D \in H^p(G), D(w) \neq 0, D(\infty) > 0, |D(e^{i\theta})|^{-p} = \rho(\theta), (a.e. \text{ on } [-\pi, +\pi]).$$

Let f be an analytic function in G . We say that $f \in H^p(G, \rho)$ if $f/D \in H^p(G)$.

For $1 \leq p \leq \infty, H^p(G, \rho)$ is a Banach space and each function f from $H^p(G, \rho)$ has limit values on Γ . The norm in the Hardy space $H^p(G, \rho)$ is defined by

$$\|f\|_{H^p(\Omega, \rho)}^p := \lim_{R \rightarrow 1^+} \frac{1}{2\pi R} \int_{C_R} \frac{|f(z)|^p}{|D(z)|^p} |dz| = \frac{1}{2\pi} \int_{-\pi}^{+\pi} |f(e^{i\theta})|^p \rho(\theta) d\theta,$$

where $C_R = \{z \in G : |z| = R\}$.

For $0 < p < 1, H^p(G, \rho)$ as defined above is a quasi-Banach space. We give an important property of the Hardy space $H^p(G, \rho)$:

Lemma 2.1 [3]. *If $f \in H^p(G, \rho)$, then for every compact set $K \subset G$, there exists a constant C_K such that:*

$$\sup_K |f(z)| \leq C_K \|f\|_{H^p(\Omega, \rho)}.$$

Let K be a compact set, $\alpha_{n,1}, \dots, \alpha_{n,k} \in \mathbb{C} \setminus K$ be given points and F_n be the set of functions of the form

$$\pi_n(z) = \frac{b_{n,0}z^n + b_{n,1}z^{n-1} + \dots + b_{n,n}}{(z - \alpha_{n,1})(z - \alpha_{n,2}) \dots (z - \alpha_{n,n})}.$$

Let f be a continuous function on K and $r_n(f)$ be the best approximation to $f(z)$ on K in the class F_n in the sense of Chebyshev, that is,

$$\|f - r_n(f)\|_\infty = \min_{\pi_n \in F_n} \|f - \pi_n\|_\infty,$$

where

$$\|f\|_\infty = \max_{z \in K} |f(z)|.$$

Theorem 2.1 [10]. *Let the points $\alpha_{n,k}$ satisfy $|\alpha_{n,k}| > 1$. A necessary and sufficient condition to have*

$$\lim_{n \rightarrow \infty} r_n(f)(z) = f(z),$$

uniformly in $|z| \leq 1$, for every such function f analytic in $|z| \leq 1$, is that

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} \left(1 - \frac{1}{\alpha_{n,k}}\right) = +\infty.$$

2.1. Extremal problems in the $H^p(G, \rho)$ spaces

For $0 < p < \infty$, we denote by $\mu(\rho)$ and $\mu^\infty(\rho)$ the extremal values of the following problems:

$$\mu(\rho) := \inf\{\|\varphi\|_{H^p(\Omega, \rho)}^p : \varphi \in H_p(\Omega, \rho), \varphi(\infty) = 1\}. \quad (4)$$

$$\mu^\infty(\rho) := \inf\{\varphi_{H^p(\Omega, \rho)}^p : \varphi \in H_p(\Omega, \rho), \varphi(\infty) = 1, \varphi(z_k) = 0, k = 1, 2, \dots\}. \quad (5)$$

We denote by φ^* and ψ the extremal functions of problems (4) and (5) respectively. It is proved in Khaldi [4] that $\varphi^*(z) = D(z)/D(\infty)$ and $\psi(z) = B(z)D(z)/D(\infty)$ where

$$B(z) = \prod_{k=1}^{\infty} \frac{z - z_k}{z\bar{z}_k - 1} \frac{|z_k|^2}{z_k}, \quad (6)$$

is the Blaschke product and the extremal values are connected by

$$\mu^\infty(\rho) = \|\psi\|_{H^p(\Omega, \rho)}^p = \left(\prod_{k=1}^{\infty} |z_k|\right)^p \mu(\rho) = \left(\frac{\prod_{k=1}^{\infty} |z_k|}{D(\infty)}\right)^p. \quad (7)$$

3. Asymptotics of extremal polynomial

Definition 3.1. A measure $\sigma = \frac{\mu}{2\pi} + \sum_{k=1}^{\infty} A_k \delta(z - z_k)$ is said to belong to the class A , if the absolutely continuous part μ satisfies the Szegő condition (3) and the discrete part satisfies the condition:

$$\left(\sum_{k=1}^{\infty} |z_k| - 1 \right) < \infty. \quad (8)$$

Remark 3.1. Condition (8) is natural as it guarantees the convergence of the Blaschke product (6).

Now, we state our main result

Theorem 3.1. Let a measure $\sigma = \frac{\mu}{2\pi} + \sum_{k=1}^{\infty} A_k \delta(z - z_k)$ such that $\sigma \in A$ then we have

$$\lim_{n \rightarrow \infty} m_{n,p}(\sigma) = (\mu^\infty(\rho))^{\frac{1}{p}}.$$

Proof. Following the same ideas as in [8], we show that $\liminf_{n \rightarrow \infty} m_{n,p}(\sigma) \geq (\mu^\infty(\rho))^{1/p}$. Let $Q_{n,p}(z, \sigma) = T_{n,p}(z, \sigma)/m_{n,p}(\sigma)$ be the normalized extremal polynomial with respect to measure σ and $B_\ell(z) = \prod_{k=1}^{\ell} \frac{z - z_k}{z \bar{z}_k - 1} \frac{|z_k|^2}{z_k}$ be the finite Blaschke product with zeros z_1, z_2, \dots, z_ℓ . Consider the integral

$$\int_{\mathbb{T}} \frac{Q_{n,p}(t, \sigma)}{t^n B_\ell(t) D(t)} dm(t),$$

where dm denotes the Lebesgue measure on \mathbb{T} . Using Hölder inequality, the fact that $\|Q_{n,p}(z_k, \sigma)\|_{L_p(\sigma)}^p = 1$ and $|B_\ell(z)| = \prod_{k=1}^{\ell} |z_k|$, $t = e^{i\theta}$, we obtain

$$\begin{aligned} \left| \int_{\mathbb{T}} \frac{Q_{n,p}(t, \sigma)}{t^n B_\ell(t) D(t)} dm(t) \right| &\leq \frac{1}{\prod_{k=1}^{\ell} |z_k|} \int_{\mathbb{T}} \left| \frac{Q_{n,p}(t, \sigma)}{t^n D(t)} \right| dm(t) \\ &\leq \frac{1}{\prod_{k=1}^{\ell} |z_k|} \left(\int_{\mathbb{T}} \frac{|Q_{n,p}(t, \sigma)|^p}{|D(t)|^p} dm(t) \right)^{1/p} \leq \frac{1}{\prod_{k=1}^{\ell} |z_k|} \|Q_{n,p}(z_k, \sigma)\|_{L_p(\sigma)}^p = \frac{1}{\prod_{k=1}^{\ell} |z_k|} \end{aligned} \quad (9)$$

On the other hand, applying the residue theorem to this integral and the fact that $B_\ell(\infty) = 1$, it yields

$$\int_{\mathbb{T}} \frac{Q_{n,p}(t, \sigma)}{t^n B_\ell(t) D(t)} dm(t) = \frac{1}{m_{n,p}(\sigma) D(\infty)} + \sum_{k=1}^{\ell} \frac{Q_{n,p}(z_k, \sigma)}{z_k^{n+1} B'_\ell(z_k) D(z_k)}, \quad (10)$$

Now, since $A_k |Q_{n,p}(z_k, \sigma)|^p \leq \|Q_{n,p}(z_k, \sigma)\|_{L_p(\sigma)}^p = 1$, then $|Q_{n,p}(z_k, \sigma)| \leq A_k^{-1/p}$ for any n . From this and the fact that $|z_k| > 1$, we get

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\ell} \frac{Q_{n,p}(z_k, \sigma)}{z_k^{n+1} B'_\ell(z_k) D(z_k)} = 0. \quad (11)$$

Combining (10) and (11) we obtain for all ℓ

$$\int_{\mathbb{T}} \frac{Q_{n,p}(t, \sigma)}{t^n B_\ell(t) D(t)} dm(t) = \frac{1}{m_{n,p}(\sigma) D(\infty)} + \varepsilon_n$$

where $\varepsilon_n \rightarrow 0$ as $n \rightarrow 0$. Taking into account (9) and (7), we see that

$$\liminf_{n \rightarrow \infty} m_{n,p}(\sigma) \geq \frac{\pi_{k=1}^\infty |z_k|}{d(\infty)} \geq (\pi_{k=1}^\infty |z_k|) (\mu(\rho))^{\frac{1}{p}} = (\mu^\infty(\rho))^{\frac{1}{p}}. \quad (12)$$

Now let us prove that $\limsup_{n \rightarrow \infty} m_{n,p}(\sigma) \leq (\mu^\infty(\rho))^{1/p}$. Let $T_{n,p}(z, \alpha)$ be the monic extremal polynomial with respect to absolute continuous part α of the measure σ . First, since the function $\varphi_b(z) = B_\ell \frac{T_{n,p}(z, \alpha)}{z^n}$ is analytic in $\overline{G} = \{z \in \mathbb{C} : |z| \geq 1\} \cup \{\infty\}$, then from Theorem 2.1, there exists a sequence $\frac{P_{m_n}(z)}{z^{m_n}}$ such that

$$\lim_{n \rightarrow \infty} \left(\sup_{z \in \overline{G}} \left| \frac{P_{m_n}(z)}{z^{m_n}} - \varphi_b(z) \right| \right) = 0,$$

and the convergence is uniform in \overline{G} . In particular, there is convergence for $z_\infty = \infty$. We have $\lim_{n \rightarrow \infty} \frac{P_{m_n}(z_\infty)}{z_\infty^{m_n}} = 1$ since $\varphi_n(\infty) = 1$.

Second, from the extremality of $T_{n,p}(z, \alpha)$ we can easily see that the sequence $\{m_{n,p}(\rho) = \|T_{n,p}(z, \alpha)\|_{L_p(\mu/2\pi)}\}_{n=1}^\infty$ is decreasing, indeed

$$\|T_{n+1,p}(z, \alpha)\|_{L_p(\frac{\mu}{2\pi})} \leq \left(\int_{\Gamma} |t T_{n,p}(t, \alpha)|^p \rho(t) dm(t) \right)^{\frac{1}{p}} = \|T_{n,p}(z, \alpha)\|_{L_p(\frac{\mu}{2\pi})},$$

so, we have

$$\int_{\Gamma} |t^{m_n} T_{n,p}(t, \alpha)|^p \rho(t) dm(t) = \int_{\Gamma} |T_{n,p}(t, \alpha)|^p \rho(t) dm(t) \leq \int_{\Gamma} \rho(t) dm(t)$$

We deduce from this inequality that there exists a constant C independent of n such that

$$\max \{ |z^{m_n} T_{n,p}(z, \alpha)|^p : |z| \leq 2 \} \leq C.$$

On the other hand, Geronimus [2] has proved that

$$\lim_{n \rightarrow \infty} \int_{\Gamma} |T_{n,p}(t, \alpha)|^p \rho(t) dm(t) = (\mu(\rho))^{\frac{1}{p}}.$$

Hence

$$\begin{aligned}
& \text{Lim}_{n \rightarrow \infty} \left\{ \int_{\Gamma} |p_{m_n}(t)|^p \rho(t) \, dm(t) + \sum_{k=1}^{\infty} A_k |p_{m_n}(z_k)|^p \right\} \\
&= \text{Lim}_{n \rightarrow \infty} \left\{ \int_{\Gamma} \left| \frac{p_{m_n}(t)}{t^{m_n}} \right|^p \rho(t) \, dm(t) + \sum_{k=1}^{\infty} A_k |z_k^{m_n}|^p \left| \frac{p_{m_n}(z_k)}{z_k^{m_n}} \right|^p \right\} \\
&= \text{Lim}_{n \rightarrow \infty} \left\{ \int_{\Gamma} |\varphi_n(t)|^p \rho(t) \, dm(t) + \sum_{k=1}^{\infty} A_k |z_k^{m_n}(\varphi_n(z_k))|^p \right\} \\
&\leq \text{Lim}_{n \rightarrow \infty} \left\{ \int_{\Gamma} |B_{\ell}(t)|^p |T_{n,p}(t, \alpha)|^p \rho(t) \, dm(t) + C \sum_{k=\ell+1}^{\infty} A_k \right\} \\
&= \left(\prod_{k=1}^{\ell} |z_k| \right)^p \mu(\rho) + \delta_{\ell}
\end{aligned} \tag{13}$$

where $\delta_{\ell} \rightarrow 0$ as $\ell \rightarrow \infty$, therefore, for all ℓ

$$\limsup_{n \rightarrow \infty} (m_{n,p}(\sigma))^p \leq \left(\prod_{k=1}^{\ell} |z_k| \right)^p \mu(\rho) + \delta_{\ell}. \tag{14}$$

Passing to the limit as $\ell \rightarrow \infty$ in (14) then using (7), it results

$$\limsup_{n \rightarrow \infty} m_{n,p}(\sigma) \leq \left(\prod_{k=1}^{\infty} |z_k| \right) (\mu(\rho))^{\frac{1}{p}} = (\mu^{\infty}(\rho))^{\frac{1}{p}}. \tag{15}$$

Hence, from this and (12) the Theorem is completely proved. \square

As a consequence we have the following

Corollary 3.1. *If $0 < p < \infty$, $\sigma \in A$, then we have*

- (i) $\lim_{n \rightarrow \infty} \left\| \frac{T_{n,p}}{z^n} - \psi^{\infty} \right\|_{H^p(\Omega, \rho)} = 0$.
- (ii) $\frac{T_{n,p}(z)}{z^n} = \psi^{\infty}(z) + \varepsilon_n(z)$, $\varepsilon_n(z) \rightarrow 0$ uniformly on the compact sets of G .

To prove this Corollary, we need the following Lemma

Lemma 3.1. *Let f_n be a sequence of analytic functions in the usual Hardy space $HP(U)$, ($U = \{w \in \mathbb{C}, |w| < 1\}$), $f(e^{i\theta})$ be a limit values of f_n on Γ such that $f_n(0) \rightarrow 1, f_n(w_k) \rightarrow 0$, when $n \rightarrow \infty, (w_k \in U, k = 1, 2, \dots)$. If*

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{2\pi} \int_{-\pi}^{+\pi} |f_n(e^{i\theta})|^p \, d\theta \right\} = \prod_{k=1}^{\infty} \frac{1}{|w_k|^p},$$

then

$$\lim_{n \rightarrow \infty} \left\{ \frac{1}{2\pi} \int_{-\pi}^{+\pi} |f_n(e^{i\theta}) - b(e^{i\theta})|^p d\theta \right\} = 0,$$

where

$$b(w) = \prod_{k=1}^{\infty} \frac{w - w_k}{w\bar{w}_k - 1} \frac{\bar{w}_k}{|w_k|^2}.$$

Proof. We prove (i) in two steps.

First step: For $0 < p < 1$, we apply Lemma 3.1 to the sequence $\frac{T_{n,p}(z)}{z^n} / \varphi^*(z)$ with $w = 1/z$.

Second step: For $1 \leq p < \infty$, we obtain (i) by proceeding as in [3].

To prove (ii) we apply Lemma 2.1, for the function

$$\varepsilon_n(z) = \frac{T_{n,p}(z)}{z^n} - \psi^\infty(z),$$

which belongs to $H^p(G, \rho)$, then for all compact $K \subset G$, we have

$$\sup_{z \in K} \left| \frac{T_{n,p}(z)}{z^n} - \psi^\infty(z) \right| = \sup_{z \in K} |\varepsilon_n(z)| \leq C_K \|\varepsilon_n\|_{H^p(\Omega, \rho)} \xrightarrow{n \rightarrow \infty} 0.$$

This achieves the proof of Corollary 3.1. \square

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References

- [1] M. Bello Hernandez, J. Minguez Cenicerros, Strong asymptotic behavior for extremal polynomials with respect to varying measures on the unit circle, *J Approx Theory* 125 (2003) 131–144.
- [2] L. Geronimus, On some extremal problem in $L_\sigma^{(p)}$ spaces, *Math Sbornik* 31 (1952) 3–23.
- [3] V.A. Kaliaguine, On asymptotics of L_p extremal polynomials on a complex curve ($0 < p < \infty$), *J Approx Theory* 74 (1993) 226–236.
- [4] R. Khaldi, Strong asymptotics for L_p extremal polynomials off a complex curve, *J Appl Math* 5 (2004) 371–378.
- [5] R. Khaldi, Szegő asymptotics of extremal polynomials on the segment $[-1, +1]$: the case of a measure with finite discrete part, *Georgian Math J* 14 (4) (2007) 673–680.
- [6] X. Li, K. Pan, Asymptotics of L_p extremal polynomials on the unit circle, *J Approx Theory* 67 (1991) 270–283.
- [7] D.S. Lubinsky, E.B. Saff, Strong asymptotics for L_p extremal polynomials ($1 < p \leq \infty$) associated with weights on $[-1, 1]$, *Lecture Notes Math* 1287 (1987) 83–104.
- [8] F. Nazarov, A. Volberg, P. Yuditskii, Asymptotics of orthogonal polynomials via the Koosis theorem, *Math Res Lett* 13 (5-6) (2006) 975–983.

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- [9] F. Peherstorfer, P. Yudiskii, Asymptotics of orthogonal polynomials in the presence of a denumerable set of mass points, *Proc Am Math Soc* 129 (2001) 3213–3220.
- [10] J.L. Walsh, *Interpolation and Approximation by Rational Functions in the Complex Domain*, vol. XX, 3rd ed., Amer. Math. Soc. Colloquium Publications, Amer. Math. Society, Providence 6, RI, 1960.