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Asymptotic behaviour of a suspension bridge problem

Soh Edwin Mukiawa

Department of Mathematics and Statistics, King Fahd University of Petroleum and Minerals, KFUPM-Saudi Arabia, P.O. Box 56, Dhahran 31261, Saudi Arabia

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Abstract. In this paper, we consider a fourth-order viscoelastic plate equation with infinite memory, and with partially hinged boundary conditions. We investigate the asymptotic behaviour of solutions. This present paper improves earlier results in the literature and allow an extended range of relaxation functions.

Keywords: Asymptotic behaviour; Fourth-Order; Infinite memory; Viscoelastic; Plate

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1. Introduction

In this paper, we investigate the asymptotic behaviour of solutions for the following fourth-order viscoelastic plate problem

$$\begin{cases} u_{tt} + \Delta^2 u(x,y,t) - \int_0^\infty g(s)\Delta^2 u(x,y,t-s)ds = 0, & \text{in } \Omega\times\mathbb{R}^+, \\ u(0,y,t) = u_{xx}(0,y,t) = 0, & \text{for } (y,t)\in(-\ell,\ell)\times\mathbb{R}^+, \\ u(\pi,y,t) = u_{xx}(\pi,y,t) = 0, & \text{for } (y,t)\in(-\ell,\ell)\times\mathbb{R}^+, \\ u_{yy}(x,\pm\ell,t) + \sigma u_{xx}(x,\pm\ell,t) = 0, & \text{for } (x,t)\in(0,\pi)\times\mathbb{R}^+, \\ u_{yyy}(x,\pm\ell,t) + (2-\sigma)u_{xxy}(x,\pm\ell,t) = 0, & \text{for } (x,t)\in(0,\pi)\times\mathbb{R}^+, \\ u(x,y,0) = u_0(x,y), \quad u_t(x,y,0) = u_1(x,y), & \text{in } \Omega, \end{cases} \tag{1.1}$$

E-mail address: sohedwin2013@gmail.com.
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where $\Omega = (0, \pi) \times (-\ell, \ell) \subset \mathbb{R}^2$, $0 < \sigma < \frac{1}{2}$, g is a positive and nonincreasing function and $(u_0, u_1,)$ are given data. The plate problem (1.1) describes the torsional oscillations in suspension bridges in the present of infinite viscoelastic damping. Recently, plate equations with infinite viscoelastic damping have become an active research area; see for instance the results in [5,15] and reference therein. The recent ground work of Ferrero and Gazzola [7], suggested a rectangular plate model describing the displacement of a suspension bridge in the downward direction. The plate $\Omega = (0, \pi) \times (-\ell, \ell)$ is assumed to be partially hinged on the vertical edges

$$u(0, y) = u_{xx}(0, y) = u(\pi, y) = u_{xx}(\pi, y) = 0, \quad \forall\, y \in (-\ell, \ell)$$

and free on the horizontal edges

$$u_{yy}(x, \pm\ell) + \sigma u_{xx}(x, \pm\ell) = u_{yyy}(x, \pm\ell) + (2 - \sigma)u_{xxy}(x, \pm\ell) = 0, \quad \forall\, x \in (0, \pi).$$

They established global existence and analyse several boundary value problems.

More recent established results concerning the static and dynamics of suspension bridges can be infer in [1,3,9–13,16]. In regard to suspension bridge models that investigated the effect of torsional oscillations, see for example [2,4]. For further details on suspension bridge models, we also refer the reader to [8], a newly published book on mathematical models for suspension bridges by Gazzola.

The main purpose of this present paper is to investigate the asymptotic stability of problem (1.1). The rest of this paper is organized as follows. In Section 2, we state some fundamental materials and assumptions that will enable us establish our result. In Section 3, we study the asymptotic stability of problem (1.1).

2. Functional setting and assumptions

In this section, we state some fundamental materials and assumptions. For this, we assume the relaxation function g has the following hypotheses:

$(G1)$ $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing C^1- function such that

$$g(0) > 0, \quad 1 - \int_0^\infty g(s)ds = \beta > 0. \tag{2.1}$$

$(G2)$ There exists a positive constant ξ and $1 \leq p < \frac{3}{2}$ such that

$$g'(t) \leq -\xi g^p(t), \quad \forall\, t \geq 0. \tag{2.2}$$

As in [7], we introduce the Hilbert space

$$H^2_*(\Omega) = \left\{ w \in H^2(\Omega) : w(0, y) = w(\pi, y) = 0, \quad \forall\, y \in (-\ell, \ell) \right\},$$

endowed with the inner product

$$(u, v)_{H^2_*(\Omega)} = \int_\Omega [\Delta u \Delta v + (1 - \sigma)(2u_{xy}v_{xy} - u_{xx}v_{yy} - u_{yy}v_{xx})]dxdy.$$

Lemma 2.1 (*Embedding*). *Suppose $1 < p < +\infty$. Then, for any $u \in H^2_*(\Omega)$, there exists an embedding constant $C_e = C_e(\Omega, p) > 0$ such that*

$$\|u\|_p \leq C_e \|u\|_{H^2_*(\Omega)}.$$

As in Dafermos [6], we define

$$\eta^t(x, y, s) = u(x, y, t) - u(x, y, t - s), s \geq 0. \tag{2.3}$$

Then, it is clear that

$$\eta_t^t + \eta_s^t - u_t = 0, \quad \eta^t(x, y, 0) = 0, \quad \eta^0(x, y, s) = u_0(x, y) - u(x, y, -s) := w(s). \tag{2.4}$$

Consequently, problem (1.1) transform into

$$\begin{cases} u_{tt} + \beta \Delta^2 u + \displaystyle\int_0^{+\infty} g(s)\Delta^2 \eta^t(s)ds = 0, & \text{in } \Omega \times \mathbb{R}^+, \\ \eta_t^t + \eta_s^t - u_t = 0, \end{cases} \tag{2.5}$$

with boundary conditions:

$$\begin{cases} u(0, y, t) = u_{xx}(0, y, t) = u(\pi, y, t) = u_{xx}(\pi, y, t) \\ \quad = 0, \quad \text{for } (y, t) \in (-\ell, \ell) \times \mathbb{R}^+, \\ u_{yy}(x, \pm\ell, t) + \sigma u_{xx}(x, \pm\ell, t) \\ \quad = 0, \qquad \text{for } (x, t) \in (0, \pi) \times \mathbb{R}^+, \\ u_{yyy}(x, \pm\ell, t) + (2 - \sigma)u_{xxy}(x, \pm\ell, t) \\ \quad = 0, \qquad \text{for } (x, t) \in (0, \pi) \times \mathbb{R}^+, \\ \eta^t(0, y, s) = \eta_{xx}^t(0, y, s) \\ \quad = 0, \qquad \text{for } (y, s) \in (-\ell, \ell) \times \mathbb{R}^+, \\ \eta^t(\pi, y, s) = \eta_{xx}^t(\pi, y, s) \\ \quad = 0, \qquad \text{for } (y, s) \in (-\ell, \ell) \times \mathbb{R}^+, \\ \eta_{yy}^t(x, \pm\ell, s) + \sigma \eta_{xx}^t(x, \pm\ell, s) \\ \quad = 0, \qquad \text{for } (x, s) \in (0, \pi) \times \mathbb{R}^+, \\ \eta_{yyy}^t(x, \pm\ell, s) + (2 - \sigma)\eta_{xxy}^t(x, \pm\ell, s) \\ \quad = 0, \qquad \text{for } (x, s) \in (0, \pi) \times \mathbb{R}^+, \end{cases} \tag{2.6}$$

and initial conditions:

$$\begin{cases} u(x, y, 0) = u_0(x, y), \quad u_t(x, y, 0) = u_1(x, y), & \text{in } \Omega, \\ \eta^0(x, y, s) = \eta_0(x, y, s) = u_0(x, y) - u(x, y, -s), & \text{in } \Omega \times [0, +\infty). \end{cases} \tag{2.7}$$

Theorem 2.1. *Let $(u_0, u_1) \in H_*^2(\Omega) \times L^2(\Omega)$ and $\eta_0 \in L_g^2(\mathbb{R}^+, H_*^2(\Omega))$ be given. Assume g satisfies $(G1)$ and $(G2)$. Then, problem (2.5)–(2.7) has a unique global weak solution*

$$u \in C\left([0, T), H_*^2(\Omega)\right), \quad u_t \in C\left([0, T), L^2(\Omega)\right), \quad \eta^t \in L_g^2(\mathbb{R}^+ \times \mathbb{R}^+, H_*^2(\Omega)), \tag{2.8}$$

where

$$L_g^2(\mathbb{R}^+, H_*^2(\Omega)) = \left\{ u : \mathbb{R}^+ \longrightarrow H_*^2(\Omega) / \int_0^{+\infty} g(s)\|u(s)\|_{H_*^2(\Omega)}^2 ds < +\infty \right\}.$$

Definition 2.1. The pair (u, η^t) satisfying (2.8) is called a weak solution if

$$\frac{d}{dt}\int_\Omega u_t w + \beta(u, w)_{H_*^2(\Omega)} - \int_0^{+\infty} g(s)\big(\eta^t(s), w\big)_{H_*^2(\Omega)} ds = 0, \quad \forall w \in H_*^2(\Omega).$$

Proof of Theorem 2.1. This result can be obtained by the Galerkin method or the linear semigroup method by repeating the same steps as in [10]. □

3. Asymptotic behaviour of solutions

In this section, we investigate the asymptotic behaviour of the energy functional associated to problem (2.5)–(2.7). The energy functional of problem (2.5)–(2.7) is defined by

$$E(t)=\frac{1}{2}\|u_t\|^2_{L^2(\Omega)}+\frac{\beta}{2}\|u\|^2_{H^2_*(\Omega)}+\frac{1}{2}(g\diamond\eta^t)(t), \tag{3.1}$$

where

$$(g\diamond\eta^t)(t)=\int_0^{+\infty}g(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds.$$

We first state and prove several important lemmas.

Lemma 3.1. *The energy functional (3.1) satisfies, along the solution of problem (2.5)–(2.7)*

$$E'(t)=\frac{1}{2}(g'\diamond\eta^t)(t)\le 0,\ \forall\, t\ge 0. \tag{3.2}$$

Proof. Multiply $(2.5)_1$ by u_t and integrate over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u_t\|^2_{L^2(\Omega)}+\frac{\beta}{2}\frac{d}{dt}\|u\|^2_{H^2_*(\Omega)}+\int_0^{\infty}g(s)\big(\eta^t(s),u_t(t)\big)_{H^2_*(\Omega)}ds=0. \tag{3.3}$$

Using $(2.5)_2$, we have

$$\begin{aligned}
&\int_0^{+\infty}g(s)\big(\eta^t(s),u_t(t)\big)_{H^2_*(\Omega)}ds=\int_0^{+\infty}g(s)\big(\eta^t(s),\eta^t_t(s)+\eta^t_s(s)\big)_{H^2_*(\Omega)}ds\\
&=\int_0^{+\infty}g(s)\big(\eta^t(s),\eta^t_t(s)\big)_{H^2_*(\Omega)}ds+\int_0^{+\infty}g(s)\big(\eta^t(s),\eta^t_s(s)\big)_{H^2_*(\Omega)}ds\\
&=\frac{1}{2}\frac{d}{dt}\left(\int_0^{+\infty}g(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds\right)+\frac{1}{2}\int_0^{+\infty}g(s)\frac{d}{ds}\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds\\
&=\frac{1}{2}\frac{d}{dt}(g\diamond\eta^t)(t)+\frac{1}{2}\left[g(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}|^{+\infty}_{s=0}-\int_0^{+\infty}g'(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds\right]\\
&=\frac{1}{2}\frac{d}{dt}(g\diamond\eta^t)(t)-\frac{1}{2}(g'\diamond\eta^t)(t).
\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) and making use of $(G2)$, we obtain

$$E'(t)=\frac{1}{2}(g'\diamond\eta^t)(t)\le 0,\quad \forall\, t\ge 0. \qquad \square$$

Lemma 3.2. *Let (u,η^t) be the solution of problem (2.5)–(2.7) and $1<p<\frac{3}{2}$. Then, there exists a constant $C>0$ such that*

$$(g\diamond\eta^t)(t)\le C\left((g^p\diamond\eta^t)(t)\right)^{\frac{1}{2p-1}}. \tag{3.5}$$

Proof. By using Hölder's inequality, we have for any $q>1$

$$\begin{aligned}
&\int_0^{+\infty}g(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds=\int_0^{+\infty}g^{\frac{1}{2q}}(s)\|\eta^t(s)\|^{\frac{2}{q}}_{H^2_*(\Omega)}g^{\frac{2q-1}{2q}}(s)\|\eta^t(s)\|^{\frac{2q-2}{q}}_{H^2_*(\Omega)}ds\\
&\le\left(\int_0^{+\infty}g^{\frac{1}{2}}(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds\right)^{\frac{1}{q}}\left(\int_0^{+\infty}g^{\frac{2q-1}{2q-2}}(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds\right)^{\frac{q-1}{q}}.
\end{aligned}$$

Using the fact that $\|u(t)\|^2_{H^2_*(\Omega)} \le \frac{2}{\beta}E(t) \le \frac{2}{\beta}E(0)$, $|a-b|^2 \le 2(|a|^2+|b|^2)$ and assumption $(G2)$, we have from $(2.5)_2$ that

$$\begin{aligned}
&\int_0^{+\infty} g^{\frac{1}{2}}(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds = \int_0^{+\infty} g^{\frac{1}{2}}(s)\|u(t)-u(t-s)\|^2_{H^2_*(\Omega)}ds\\
&\le \frac{8E(0)}{\beta}\int_0^{+\infty} g^{\frac{1}{2}-p}(s)g^p(s)ds \le -\frac{8E(0)}{\beta\xi}\int_0^{+\infty} g^{\frac{1}{2}-p}(s)g'(s)ds\\
&= -\frac{16E(0)}{\beta\xi(3-2p)}\int_0^{+\infty}\left(g^{\frac{3-2p}{2}}(s)\right)'ds = \frac{16E(0)}{\beta\xi(3-2p)}g^{\frac{3-2p}{2}}(0) < +\infty,
\end{aligned}$$

since $p < \frac{3}{2}$. Thus choosing $q = \frac{2p-1}{2p-2}$, we obtain the result, where

$$C = \left(\frac{16E(0)}{\beta\xi(3-2p)}g^{\frac{3-2p}{2}}(0)\right)^{\frac{2p-2}{2p-1}}.$$

This completes the proof. □

Lemma 3.3. *Let (u,η^t) be the solution of problem (2.5)–(2.7). Then, the following inequality holds*

$$\left(\int_0^{+\infty} g(s)\eta^t(s)ds\right)^2 \le C_e(1-\beta)(g\diamond\eta^t)(t).$$

Proof. Exploiting the Cauchy–Schwarz' inequality and Lemma 2.1, we have

$$\begin{aligned}
\left(\int_0^{+\infty} g(s)\eta^t(s)ds\right)^2 &= \left(\int_0^{+\infty}\sqrt{g(s)}\sqrt{g(s)}\eta^t(s)ds\right)^2\\
&\le \left(\int_0^{+\infty} g(s)ds\right)\left(\int_0^{+\infty} g(s)|\eta^t(s)|^2ds\right)\\
&\le C_e(1-\beta)(g\diamond\eta^t)(t).
\end{aligned}$$

Next, we define the Lyapunov functional

$$F(t) = E(t) + n_1 I_1(t) + n_2 I_2(t), \tag{3.6}$$

where n_1 and n_2 are positive constants to be determined later and

$$I_1(t) = \int_\Omega uu_t, \quad I_2(t) = -\int_\Omega u_t\int_0^{+\infty} g(s)\eta^t(s)ds.$$

Lemma 3.4. *For n_1 and n_2 small enough, there exist $\alpha_1 > 0$ and $\alpha_2 > 0$ such that*

$$\alpha_1 E(t) \le F(t) \le \alpha_2 E(t). \tag{3.7}$$

Proof. On one hand, using Young's inequality, Lemmas 2.1 and 3.3, we have

$$\begin{aligned}
F(t) \le\ & E(t) + \frac{n_1}{2}\|u_t\|^2_{L^2(\Omega)} + \frac{n_1 C_e}{2}\|u\|^2_{H^2_*(\Omega)}\\
&+ \frac{n_2}{2}\|u_t\|^2_{L^2(\Omega)} + \frac{n_2 C_e(1-\beta)}{2}\int_0^{+\infty} g(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds\\
&\le \alpha_2 E(t).
\end{aligned}\tag{3.8}$$

On the other hand, we have

$$\begin{aligned} F(t) \geq\ & E(t) - \frac{n_1}{2}\|u_t\|^2_{L^2(\Omega)} - \frac{n_1 C_e}{2}\|u\|^2_{H^2_*(\Omega)} \\ & - \frac{n_2}{2}\|u_t\|^2_{L^2(\Omega)} - \frac{n_2 C_e(1-\beta)}{2}\int_0^{+\infty} g(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds \\ & = \left(\frac{1}{2} - \frac{(n_1+n_2)}{2}\right)\|u_t\|^2_{L^2(\Omega)} + \left(\frac{1}{2} - \frac{n_1 C_e}{2}\right)\|u\|^2_{H^2_*(\Omega)} \\ & + \left(\frac{1}{2} - \frac{n_2 C_e(1-\beta)}{2}\right)\int_0^{+\infty} g(s)\|\eta^t(s)\|^2_{H^2_*(\Omega)}ds. \end{aligned}$$

We choose n_1 and n_2 small enough so that

$$\left(\frac{1}{2} - \frac{(n_1+n_2)}{2}\right),\ \left(\frac{1}{2} - \frac{n_1 C_e}{2}\right),\ \left(\frac{1}{2} - \frac{n_2 C_e(1-\beta)}{2}\right) > 0$$

and obtain

$$F(t) \geq \alpha_1 E(t). \tag{3.9}$$

Combining (3.8) and (3.9), we obtain the result. □

Lemma 3.5. *Let (u, η^t) be the solution of problem (2.5)–(2.7). Then, under the assumptions $(G1)$ and $(G2)$, the functional*

$$I_1(t) = \int_\Omega u u_t$$

satisfies

$$I_1'(t) \leq \|u_t\|^2_{L^2(\Omega)} - \frac{\beta}{2}\|u\|^2_{H^2_*(\Omega)} + \frac{1}{2\beta}\left(\int_0^{+\infty} g^{2-p}(s)ds\right)(g^p \diamond \eta^t)(t). \tag{3.10}$$

Proof. Exploiting $(2.5)_1$, direct differentiation gives

$$I_1'(t) = \|u_t\|^2_{L^2(\Omega)} - \beta\|u\|^2_{H^2_*(\Omega)} - \int_0^{+\infty} g(s)\big(\eta^t(s), u(t)\big)_{H^2_*(\Omega)}ds. \tag{3.11}$$

Now, by using Cauchy–Schwarz' and Hölder's inequalities, we estimate the term

$$J_1 = -\int_0^{+\infty} g(s)\big(\eta^t(s), u(t)\big)_{H^2_*(\Omega)}ds$$

for some $\delta_1 > 0$ as follows

$$\begin{aligned}
J_1 &\le \int_0^{+\infty} g(s)\|\eta^t(s)\|_{H_*^2(\Omega)}\|u(t)\|_{H_*^2(\Omega)}ds \\
&\le \frac{\delta_1}{2}\|u(t)\|^2_{H_*^2(\Omega)} + \frac{1}{2\delta_1}\left(\int_0^{+\infty} g(s)\|\eta^t(s)\|_{H_*^2(\Omega)}ds\right)^2 \\
&\le \frac{\delta_1}{2}\|u(t)\|^2_{H_*^2(\Omega)} \\
&+ \frac{1}{2\delta_1}\left[\left(\int_0^{+\infty}\left(g^{1-\frac{p}{2}}(s)\right)^2 ds\right)^{\frac{1}{2}}\left(\int_0^{+\infty}\left(g^{\frac{p}{2}}(s)\|\eta^t(s)\|_{H_*^2(\Omega)}\right)^2 ds\right)^{\frac{1}{2}}\right]^2 \\
&\le \frac{\delta_1}{2}\|u(t)\|^2_{H_*^2(\Omega)} + \frac{1}{2\delta_1}\left(\int_0^{+\infty} g^{2-p}(s)ds\right)(g^p \diamond \eta^t)(t). \qquad (3.12)
\end{aligned}$$

Substituting (3.12) into (3.11), we arrive at

$$I_1'(t) \le \|u_t\|^2_{L^2(\Omega)} - \left(\beta - \frac{\delta_1}{2}\right)\|u\|^2_{H_*^2(\Omega)} + \frac{1}{2\delta_1}\left(\int_0^{+\infty} g^{2-p}(s)ds\right)(g^p \diamond \eta^t)(t).$$

We then choose $\delta_1 = \beta$ and obtain the result. □

Lemma 3.6. *Let (u, η^t) be the solution of problem (2.5)–(2.7). Then, under the assumptions $(G1)$ and $(G2)$, the functional*

$$I_2(t) = -\int_\Omega u_t \int_0^{+\infty} g(s)\eta^t(s)ds$$

satisfies

$$\begin{aligned}
I_2'(t) &\le (\delta - (1-\beta))\|u_t\|^2_{L^2(\Omega)} + \beta^2\delta_2\|u(t)\|^2_{H_*^2(\Omega)} - \frac{C_e g(0)}{4\delta}(g' \diamond \eta^t)(t) \\
&+ \left(1 + \frac{1}{4\delta_2}\right)\left(\int_0^{+\infty} g^{2-p}(s)ds\right)(g^p \diamond \eta^t)(t), \qquad (3.13)
\end{aligned}$$

for some positive constants δ and δ_2 to be specified later.

Proof. Direct differentiation and using $(2.5)_1$ leads to

$$\begin{aligned}
I_2'(t) &= \beta\left(u(t), \int_0^{+\infty} g(s)\eta^t(s)ds\right)_{H_*^2(\Omega)} \\
&+ \int_0^{+\infty} g(s)\left(\eta^t(s), \int_0^{+\infty} g(s)\eta^t(s)ds\right)_{H_*^2(\Omega)} ds \\
&- \int_\Omega u_t \int_0^{+\infty} g(s)\eta_t^t(s)ds. \qquad (3.14)
\end{aligned}$$

We make use of the Cauchy–Schwarz' and Hölder's inequalities to estimate the terms in the right-hand side of (3.14). For the term

$$J_2 = \beta\left(u(t), \int_0^{+\infty} g(s)\eta^t(s)ds\right)_{H_*^2(\Omega)},$$

we have for some $\delta_2 > 0$

$$\begin{aligned} J_2 &\leq \beta \|u(t)\|_{H_*^2(\Omega)} \int_0^{+\infty} g(s)\|\eta^t(s)\|_{H_*^2(\Omega)} ds \\ &\leq \beta^2 \delta_2 \|u(t)\|^2_{H_*^2(\Omega)} + \frac{1}{4\delta_2}\left(\int_0^{+\infty} g(s)\|\eta^t(s)\|_{H_*^2(\Omega)} ds\right)^2 \\ &\leq \beta^2 \delta_2 \|u(t)\|^2_{H_*^2(\Omega)} + \frac{1}{4\delta_2}\left(\int_0^{+\infty} g^{2-p}(s) ds\right)(g^p \diamond \eta^t)(t). \end{aligned} \tag{3.15}$$

For the term

$$J_3 = \int_0^{+\infty} g(s)\left(\eta^t(s), \int_0^{+\infty} g(s)\eta^t(s)ds\right)_{H_*^2(\Omega)} ds,$$

we get

$$\begin{aligned} J_3 &\leq \left(\int_0^{+\infty} g(s)\|\eta^t(s)\|_{H_*^2(\Omega)} ds\right)^2 \\ &\leq \left(\int_0^{+\infty} g^{2-p}(s) ds\right)(g^p \diamond \eta^t)(t). \end{aligned} \tag{3.16}$$

For the term

$$J_4 = -\int_\Omega u_t \int_0^{+\infty} g(s)\eta^t_t(s)ds,$$

we obtain from $(2.5)_2$ and integration by part that

$$\begin{aligned} J_4 &= -\int_\Omega u_t \int_0^{+\infty} g(s)\left(u_t(t) - \eta^t_s(s)\right) ds \\ &= -\left(\int_0^{+\infty} g(s)ds\right)\|u_t\|^2_{L^2(\Omega)} + \int_\Omega u_t \int_0^{+\infty} g(s)\eta^t_s(s)ds \\ &= -(1-\beta)\|u_t\|^2_{L^2(\Omega)} + \int_\Omega u_t \left[g(s)\eta^t(s)|_{s=0}^{+\infty} - \int_0^{+\infty} g'(s)\eta^t(s)ds\right] \\ &= -(1-\beta)\|u_t\|^2_{L^2(\Omega)} - \int_\Omega u_t \int_0^{+\infty} g'(s)\eta^t(s)ds \\ &\leq (\delta - (1-\beta))\|u_t\|^2_{L^2(\Omega)} + \frac{1}{4\delta}\int_\Omega \left(\int_0^{+\infty} g'(s)\eta^t(s)ds\right)^2 \\ &\leq (\delta - (1-\beta))\|u_t\|^2_{L^2(\Omega)} - \frac{g(0)}{4\delta}\int_\Omega \int_0^{+\infty} g'(s)|\eta^t(s)|^2 ds \\ &\leq (\delta - (1-\beta))\|u_t\|^2_{L^2(\Omega)} - \frac{C_e g(0)}{4\delta}(g' \diamond \eta^t)(t). \end{aligned} \tag{3.17}$$

By substituting (3.15)–(3.17) into (3.14), we obtain the result. □

Now, we state our main stability result.

Theorem 3.1. *Let $(u_0, u_1, \eta_0) \in H_*^2(\Omega) \times L^2(\Omega) \times L_g^2(\mathbb{R}_+, H_*^2(\Omega))$ be given. Assume g satisfies $(G1)$ and $(G2)$. Then, there exist $K, \gamma > 0$ such that the energy functional (3.1) satisfies, for all $t \geq 0$*

$$E(t) \leq K e^{-\gamma t}, \quad \text{if } p = 1, \tag{3.18}$$

$$E(t) \leq \frac{K}{(1+t)^{\frac{1}{2(p-1)}}}, \text{ if } p > 1. \tag{3.19}$$

Proof. By using Lemmas 3.1, 3.5 and 3.6, we obtain

$$\begin{aligned} F'(t) &= E'(t) + n_1 I_1'(t) + n_2 I_2'(t) \\ &\leq n_1 \left[\|u_t\|^2_{L^2(\Omega)} - \frac{\beta}{2}\|u\|^2_{H^2_*(\Omega)} + \frac{1}{2\beta}\left(\int_0^{+\infty} g^{2-p}(s)ds\right)(g^p \diamond \eta^t)(t)\right] \\ &+ n_2 \left[(\delta - (1-\beta))\|u_t\|^2_{L^2(\Omega)} + \beta^2\delta_2\|u(t)\|^2_{H^2_*(\Omega)} - \frac{C_e g(0)}{4\delta_2}(g' \diamond \eta^t)(t)\right] \\ &+ n_2\left(1 + \frac{1}{4\delta_2}\right)\left(\int_0^{+\infty} g^{2-p}(s)ds\right)(g^p \diamond \eta^t)(t) + \frac{1}{2}(g' \diamond \eta^t)(t). \end{aligned}$$

Exploiting $g'(t) \leq -\xi g^p(t)$, $\forall\, t \geq 0$ and choosing $\delta_2 = \frac{\delta}{\beta} > 0$, we obtain

$$\begin{aligned} F'(t) \leq &- [n_2((1-\beta)-\delta) - n_1]\,\|u_t\|^2_{L^2(\Omega)} - \left[\frac{n_1\beta}{2} - n_2\delta\right]\|u(t)\|^2_{H^2_*(\Omega)} \\ &- \left[\xi\left(\frac{1}{2} - \frac{n_2 C_e \beta g(0)}{4\delta}\right) - \left(\frac{n_1}{2\beta} + n_2\left(1 + \frac{\beta^2}{4\delta}\right)\right)\int_0^{+\infty} g^{2-p}(s)ds\right] \\ &\times (g^p \diamond \eta^t)(t). \end{aligned}$$

Again, from $(G2)$, we have $g'(s) \leq -\xi g^p(s)$. This implies $g^p(s) \leq -\frac{1}{\xi}g'(s)$. Thus, we get

$$\int_0^{+\infty} g^{2-p}(s)ds = \int_0^{+\infty} g^{2(1-p)}(s) g^p(s)ds \leq -\frac{1}{\xi}\int_0^{+\infty} g^{2(1-p)}(s)g'(s)ds.$$

Simple integration leads to

$$\int_0^{+\infty} g^{2-p}(s)ds \leq -\frac{1}{\xi(3-2p)} g^{3-2p}(s)|_{s=0}^{+\infty} = \frac{1}{\xi(3-2p)} g^{3-2p}(0) < +\infty,$$

since $p < \frac{3}{2}$. Next, we choose $\delta > 0$ such that

$$\begin{cases} (1-\beta) - \delta > \dfrac{1}{2}(1-\beta) \\ \dfrac{2\delta}{\beta} < \dfrac{1}{2}(1-\beta). \end{cases} \tag{3.20}$$

From (3.20), we obtain that for any n_1 and n_2 satisfying

$$\frac{(1-\beta)n_2}{4} < n_1 < \frac{(1-\beta)n_2}{2} \tag{3.21}$$

will make

$$\gamma_1 = n_2((1-\beta)-\delta) - n_1 > 0, \quad \gamma_2 = \frac{n_1\beta}{2} - n_2\delta > 0.$$

Next, we choose n_1 and n_2 small enough such that (3.21) and (3.7) remains valid and further, we make

$$\gamma_3 = \xi\left(\frac{1}{2} - \frac{n_2 C_e \beta^2 g(0)}{4\delta}\right) - \left(\frac{n_1}{2\beta} + n_2\left(1 + \frac{\beta^2}{4\delta}\right)\right)\int_0^{+\infty} g^{2-p}(s)ds > 0.$$

Thus, we obtain

$$F'(t) \leq -\gamma_4 \left(\|u_t\|^2_{L^2(\Omega)} + \|u(t)\|^2_{H^2_*(\Omega)} + (g^p \diamond \eta^t)(t) \right), \tag{3.22}$$

where $\gamma_4 = \min\{\gamma_1, \gamma_2, \gamma_3\}$. To complete the proof, we have two cases

Case of $p = 1$

In this case, we have

$$F'(t) \leq -\gamma_4 \left(\|u_t\|^2_{L^2(\Omega)} + \|u(t)\|^2_{H^2_*(\Omega)} + (g \diamond \eta^t)(t) \right) \leq -2\gamma_4 E(t).$$

Exploiting Lemma 3.4, we obtain

$$F'(t) \leq -\gamma F(t), \quad \forall t \geq 0. \tag{3.23}$$

Integrating (3.23) over $(0, t)$, we get

$$F(t) \leq K e^{-\gamma t}, \quad \forall t \geq 0.$$

Recalling Lemma 3.4 again, we obtain

$$E(t) \leq K e^{-\gamma t}, \quad \forall t \geq 0.$$

Case of $p > 1$

Now, recalling (3.1) and Lemma 3.2, we have for some $\lambda > 1$ to be specified later that

$$\begin{aligned} E^\lambda(t) &= \left(\frac{1}{2}\|u_t\|^2_{L^2(\Omega)} + \frac{\beta}{2}\|u\|^2_{H^2_*(\Omega)} + \frac{1}{2}(g \diamond \eta^t)(t) \right)^\lambda \\ &\leq C \left(\|u_t\|^{2\lambda}_{L^2(\Omega)} + \|u\|^{2\lambda}_{H^2_*(\Omega)} + \left((g \diamond \eta^t)(t)\right)^\lambda \right) \\ &\leq C^\lambda E^{\lambda-1}(0) \left(\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{H^2_*(\Omega)} \right) + C \left((g^p \diamond \eta^t)(t)\right)^{\frac{\lambda}{2p-1}}. \end{aligned}$$

Choosing $\lambda = 2p - 1$, we obtain

$$E^{2p-1}(t) \leq C \left(\|u_t\|^2_{L^2(\Omega)} + \|u\|^2_{H^2_*(\Omega)} + (g^p \diamond \eta^t)(t) \right).$$

Thus, it follows from (3.22) that

$$F'(t) \leq -\gamma E^{2p-1}(t), \quad \text{for some } \gamma > 0.$$

By using Lemma 3.4 again, we arrive at

$$F'(t) \leq -\gamma \alpha_2^{2p-1} F^{2p-1}(t). \tag{3.24}$$

Simple integration over $(0, t)$ gives

$$\frac{F^{2(1-p)}(t) - F^{2(1-p)}(0)}{2(1-p)} \leq -\gamma \alpha_2^{2p-1} t.$$

We have that $1 < p$, so $2(1 - p) < 0$. Thus, we obtain

$$F^{2(1-p)}(t) \geq 2(p-1)\gamma \alpha_2^{2p-1} t + F^{2(1-p)}(0).$$

This implies

$$F^{2(p-1)}(t) \leq \frac{1}{2(p-1)\gamma \alpha_2^{2p-1} t + F^{2(1-p)}(0)} \leq \frac{1}{K_0(1+t)},$$

where $K_0 = \min\{2(p-1)\gamma\alpha_2^{2p-1}, F^{2(1-p)}(0)\}$. Thus, we get

$$F(t) \leq \left(\frac{1}{K_0(1+t)}\right)^{\frac{1}{2(p-1)}}, \ \forall t \geq 0. \tag{3.25}$$

Recalling Lemma 3.4 again, we get

$$E(t) \leq \frac{K}{(1+t)^{\frac{1}{2(p-1)}}}, \ \forall t \geq 0. \tag{3.26}$$

This completes the proof. □

Example 3.2. Let

$$g_1(t) = e^{-bt}, \ b > 1.$$

Then, it is easy to see that g_1 satisfies $(G1)$ and $(G2)$. Thus, we obtain from (3.18) that

$$E(t) \leq Ke^{-\gamma t}, \ \forall t \geq 0.$$

Example 3.3. Let

$$g_2(t) = \frac{a}{(1+t)^v}, \ v > 2,$$

where $a > 0$ is chosen appropriately so that

$$\int_0^\infty g_2(t)dt < 1.$$

Direct computations gives

$$g_2'(t) = -\frac{av}{(1+t)^{v+1}} = -\frac{v}{a^{\frac{1}{v}}}\left(\frac{a}{(1+t)^v}\right)^{\frac{v+1}{v}} = -\xi g_2^p(t), \tag{3.27}$$

where $\xi = \frac{v}{a^{\frac{1}{v}}} > 0$, $p = \frac{v+1}{v} < \frac{3}{2}$. Hence, we get from (3.19) that

$$E(t) \leq \frac{K}{(1+t)^{\frac{1}{2(p-1)}}} = \frac{K}{(1+t)^{\frac{v}{2}}}, \quad \forall t \geq 0.$$

Remark 3.1. The result in (3.19) shows that, in the past history case, the decay rate is slower compared to the finite memory case, see for instance the result of Messaoudi and Tatar [14] and references therein, where

$$E(t) \leq \frac{K}{(1+t)^{\frac{1}{(p-1)}}}, \ \forall t \geq 0. \tag{3.28}$$

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