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Approximating fixed points of nearly asymptotically nonexpansive mappings in CAT(k) spaces

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Abstract. In this paper we approximate common fixed points of nearly asymptotically nonexpansive mappings under modified SP-iteration process in the setting of CAT(k) spaces and establish strong and Δ-convergence theorems. Our results generalize and improve the corresponding known results of the existing literature.

Keywords: Δ-convergence; Modified SP-iteration process; Nearly asymptotically nonexpansive mapping; Common fixed point; CAT(k) space

Mathematics Subject Classification: 47H09; 47H10

1. Introduction

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [8] as an important generalization of the class of nonexpansive mappings. They proved that if K is a nonempty closed and bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive self-mapping of K has a fixed point. There are many papers dealing with the approximation of fixed points of asymptotically nonexpansive mappings and asymptotically quasi-nonexpansive mappings in uniformly convex Banach spaces, using modified Mann, Ishikawa and three-step iteration processes (see, [8,16,23,24,26,27,29–34]).

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The concept of Δ-convergence in general metric spaces was introduced by Lim [15]. Kirk [13] proved the existence of fixed points of nonexpansive mappings in CAT(0) spaces. Kirk and Panyanak [14] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. Dhompongsa and Panyanak [6] proved some results by using Mann and Ishikawa iteration process involving one mapping. After that Khan and Abbas [12] studied the approximation of common fixed point by the Ishikawa-type iteration process involving two mappings in CAT(0) spaces.

The aim of this paper is to establish strong and Δ-convergence of modified SP-iteration process for nearly asymptotically nonexpansive mappings in CAT(k) spaces with $k > 0$. Our results extend and improve the corresponding results of Abbas et al. [1], Dhompongsa and Panyanak [6], Khan and Abbas [12], Phuengrattana and Suantai [21], Thiainwan [33] and many other results of this direction. For more details one can be referred to [3,28–30].

2. Preliminaries

This section contains preliminary notions, basic definitions and relevant well known results which are required to prove the main results.

Let $F(T) = \{x \in K : Tx = x\}$ denotes the set of fixed points of mapping T. We begin with the following definitions:

Definition 1. Let K be a nonempty subset of a metric space (X, d). Then the mapping $T : K \to K$ is said to be

(1) nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$;
(2) asymptotically nonexpansive if there exists a sequence $\{t_n\} \subset [0, \infty)$, with $\lim_{n\to\infty} t_n = 0$, such that $d(T^n x, T^n y) \leq (1 + t_n)d(x, y)$ for all $x, y \in K$ and $n \geq 1$;
(3) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{t_n\} \subset [0, \infty)$, with $\lim_{n\to\infty} t_n = 0$, such that $d(T^n x, p) \leq (1 + t_n)d(x, p)$ for all $x \in K$, $p \in F(T)$ and $n \geq 1$;
(4) uniformly L-Lipschitzian if there exists a constant $L > 0$ such that $d(T^n x, T^n y) \leq Ld(x, y)$ for all $x, y \in K$ and $n \geq 1$;
(5) semi-compact if for a sequence $\{x_n\}$ in K with $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to p \in K$ as $k \to \infty$;
(6) a sequence $\{x_n\}$ in K is called an approximating fixed point sequence for T (AFPS, in short) if $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.

The class of nearly Lipschitzian mappings was introduced by Sahu [22]. Actually it is an important generalization of the class of Lipschitzian mappings.

Definition 2. Let K be a nonempty subset of a metric space (X, d). Fix a sequence $\{s_n\} \subset [0, \infty)$ with $\lim_{n\to\infty} s_n = 0$. A mapping $T : K \to K$ is said to be nearly Lipschitzian with respect to $\{s_n\}$ if for all $n \geq 1$, there exists a constant $k_n \geq 0$ such that

$$d(T^n x, T^n y) \leq k_n[d(x, y) + s_n] \quad \text{for all } x, y \in K.$$

The infimum of the constants k_n for which the above inequality holds, is denoted by $\eta(T^n)$ and called nearly Lipschitz constant. Notice that

$$\eta(T^n) = \sup\left\{\frac{d(T^n x, T^n y)}{d(x, y) + s_n} : x, y \in K,\ x \neq y\right\}.$$

A nearly Lipschitzian mapping T with sequence $\{s_n, \eta(T^n)\}$ is said to be

(1) nearly nonexpansive if $\eta(T^n) = 1$ for all $n \geq 1$;
(2) nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \geq 1$ and $\lim_{n\to\infty}\eta(T^n) = 1$;
(3) nearly uniformly k-Lipschitzian if $\eta(T^n) \leq k$ for all $n \geq 1$.

Note that every asymptotically nonexpansive mapping is nearly asymptotically nonexpansive.

Definition 3. Let K be a nonempty subset of a metric space (X, d). A mapping $T : K \to K$ is said to satisfy condition (I) if there exists a nondecreasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ and $g(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq g(d(x, F(T)))$ for all $x \in K$.

Let (X, d) be a metric space. A geodesic path joining $x(\in X)$ to $y(\in X)$(or more briefly, a geodesic from x to y) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x$, $c(l) = y$ and $d(c(t), c(s)) = |t - s|$ for all $t, s \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique, this geodesic is denoted by $[x, y]$. The space (X, d) is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset Y of X is said to be convex if Y includes every geodesic segment joining any two of its points.

Let D be a positive number. A metric space (X, d) is called a D-geodesic space if any two points of X with distance less than D are joined by a geodesic. If this holds in a convex set Y, then Y is said to be D-convex. Let M_k denotes the 2-dimensional, complete and simply connected spaces of curvature k, where k is a constant.

We define the diameter D_k of M_k $(k \geq 0)$ by $D_K = \frac{\pi}{\sqrt{k}}$ for $k > 0$ and $D_k = \infty$ for $k = 0$. It is well known that any ball in X with radius less than $D_k/2$ is convex (see [4]). A geodesic triangle $\Delta(x, y, z)$ in the metric space (X, d) consists of three points x, y, z in X (the vertices of Δ) and three geodesic segments between each pair of vertices. For $\Delta(x, y, z)$ in a geodesic space X satisfying

$$d(x, y) + d(y, z) + d(z, x) < 2D_k,$$

there exist points $\bar{x}, \bar{y}, \bar{z} \in M_k$ such that $d(x, y) = d_k(\bar{x}, \bar{y})$, $d(y, z) = d_k(\bar{y}, \bar{z})$ and $d(z, x) = d_k(\bar{z}, \bar{x})$ where d_k is the metric of M_k. The triangle having vertices $\bar{x}, \bar{y}, \bar{z} \in M_k$ is called a comparison triangle of $\Delta(x, y, z)$. A geodesic triangle $\Delta(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_k$ is said to satisfy the CAT(k) inequality if, for any $p, q \in \Delta(x, y, z)$ and for their comparison points $\bar{p}, \bar{q} \in \bar{\Delta}(\bar{x}, \bar{y}, \bar{z})$, we have $d(p, q) \leq d_k(\bar{p}, \bar{q})$.

Definition 4. A metric space (X, d) is called a CAT(k) space if and only if

(1) (for $k \leq 0$) X is a geodesic space such that all of its geodesic triangles satisfy the CAT(k) inequality;
(2) (for $k > 0$) X is D_k-geodesic and any geodesic triangle $\Delta(x, y, z)$ in X with $d(x, y) + d(y, z) + d(z, x) < 2D_k$ satisfies the CAT(k) inequality.

Notice that in a CAT(0) space (X, d) if $x, y, z \in X$, then the CAT(0) inequality implies

$$d^2\left(x, \frac{y \oplus z}{2}\right) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d^2(y, z), \tag{CN}$$

which is called the (CN) inequality given by Bruhat and Tits [5]. Dhompongsa and Panyanak [6] extended the (CN) inequality as follows:

$$d^2(z, \alpha x \oplus (1-\alpha)y) \le \alpha d^2(z, x) + (1-\alpha)d^2(z, y) - \alpha(1-\alpha)d^2(x, y) \qquad (\mathrm{CN}^*)$$

for all $\alpha \in [0, 1]$ and $x, y, z \in X$. In fact, if X is a geodesic space, then the following statements are equivalent:

(1) X is a CAT(0) space;
(2) X satisfies the (CN) inequality;
(3) X satisfies the (CN*) inequality.

Let $R \in (0, 2]$. A geodesic space (X, d) is said to be R-convex for R [20] if for $x, y, z \in X$, we have

$$d^2(z, \alpha x \oplus (1-\alpha)y) \le \alpha d^2(z, x) + (1-\alpha)d^2(z, y) - \frac{R}{2}\alpha(1-\alpha)d^2(x, y). \qquad (2.1)$$

It follows from (CN*) that a geodesic space (X, d) is a CAT(0) space if and only if (X, d) is R-convex for $R = 2$.

Lemma 1 (*[4]*). *Let (X, d) be a complete CAT(k) space with diam$(X) = \frac{\pi/2-\epsilon}{\sqrt{k}}$, $k > 0$ for some $\epsilon \in (0, \pi/2)$. Then*

$$d((1-\alpha)x \oplus \alpha y, z) \le (1-\alpha)d(x, z) + \alpha d(y, z)$$

for all $x, y, z \in X$ and $\alpha \in [0, 1]$.

Now, we recall some elementary facts about CAT(k) spaces.

Let (X, d) be a CAT(k) space and $\{x_n\}$ a bounded sequence in X. For $x \in X$, we set

$$r(x, \{x_n\}) = \lim_{n\to\infty} \sup d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set

$$A(\{x_n\}) = \{x \in X : r(\{x_n\}) = r(x, \{x_n\})\}.$$

It is clear that a CAT(k) space with diam$(X) = \frac{\pi}{2\sqrt{k}}$, $A(\{x_n\})$ consists of exactly one point (see [7]).

Definition 5 (*[15]*). A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$.

We write Δ-$\lim_{n\to\infty} x_n = x$ where x is called the Δ-limit of $\{x_n\}$.

We state the results in a CAT(k) space with $k > 0$.

Lemma 2 (*[7]*). *Let (X, d) be a complete CAT(k) space with diam$(X) = \frac{\pi/2-\epsilon}{\sqrt{k}}$, $k > 0$ for some $\epsilon \in (0, \pi/2)$. Then the following statements hold:*

(1) *Every sequence in X has a* Δ*-convergent subsequence.*
(2) *If* $\{x_n\} \subseteq X$ *and* $\Delta\text{-}\lim_{n\to\infty} x_n = x$, *then* $x \in \bigcap_{k=1}^{\infty} \overline{conv}\{x_k, x_{k+1}, \ldots\}$, *where* $\overline{conv}(A) = \bigcap\{B : B \supseteq A$ *and B is closed and convex*$\}$.

Lemma 3 (*[6]*). *Let* (X, d) *be a complete CAT*(k) *space with diam*$(X) = \frac{\pi/2-\epsilon}{\sqrt{k}}$, $k > 0$ *for some* $\epsilon \in (0, \pi/2)$. *If* $\{x_n\}$ *is a bounded sequence in X with* $A(\{x_n\}) = \{x\}$ *and* $\{v_n\}$ *is a subsequence of* $\{x_n\}$ *with* $A(\{v_n\}) = \{v\}$ *and the sequence* $\{d(x_n, v)\}$ *converges, then* $x = v$.

Lemma 4 (*[31]*). *Let* $\{p_n\}_{n=1}^{\infty}$, $\{q_n\}_{n=1}^{\infty}$ *and* $\{r_n\}_{n=1}^{\infty}$ *be sequences of nonnegative numbers satisfying the inequality*

$$p_{n+1} \le (1 + q_n)p_n + r_n, \quad \forall\, n \ge 1.$$

If $\sum_{n=1}^{\infty} q_n < \infty$ *and* $\sum_{n=1}^{\infty} r_n < \infty$, *then* $\lim_{n\to\infty} p_n$ *exists.*

Proposition 1 (*[18]*). *Let* $\{x_n\}$ *be a bounded sequence in a CAT*(0) *space X, and K a closed convex subset of X which contains* $\{x_n\}$. *Then*

(1) $\Delta\text{-}\lim_{n\to\infty} x_n = x$, *implies that* $\{x_n\} \rightharpoonup x$,
(2) *the converse is true if* $\{x_n\}$ *is regular.*

For approximating fixed point, Mann [17] and Ishikawa [10] introduced iteration schemes for a mapping $T : K \to K$, which are respectively described in the following lines: $x_1 \in K$,

$$x_{n+1} = a_n T x_n \oplus (1 - a_n)x_n, \quad n \ge 1, \tag{2.2}$$

$$\begin{cases} y_n = b_n T x_n \oplus (1 - b_n)x_n, \\ x_{n+1} = a_n T y_n \oplus (1 - a_n)x_n, \quad n \ge 1, \end{cases} \tag{2.3}$$

where $\{a_n\}$ and $\{b_n\}$ are appropriate sequences in (0, 1). He et al. [9] and Jun [11] proved that the sequence $\{x_n\}$ generated by (2.2) and (2.3) converges and Δ-converges respectively to a fixed point of T in CAT(k) spaces.

Thianwan [33] defined the two step iteration as follows: $x_1 \in K$,

$$\begin{cases} y_n = b_n T x_n \oplus (1 - b_n)x_n, \\ x_{n+1} = a_n T y_n \oplus (1 - a_n)y_n, \quad n \ge 1, \end{cases} \tag{2.4}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are appropriate sequences in (0, 1).

Further in [19], Noor defined the three step Noor iteration as follows: $x_1 \in K$,

$$\begin{cases} z_n = c_n T x_n \oplus (1 - c_n)x_n, \\ y_n = b_n T z_n \oplus (1 - b_n)x_n, \\ x_{n+1} = a_n T y_n \oplus (1 - a_n)x_n, \quad n \ge 1, \end{cases} \tag{2.5}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are appropriate sequences in (0, 1).

Recently, Phuengrattana and Suantai [21] defined the SP-iteration as follows: $x_1 \in K$,

$$\begin{cases} z_n = c_n T x_n \oplus (1 - c_n)x_n, \\ y_n = b_n T z_n \oplus (1 - b_n)z_n, \\ x_{n+1} = a_n T y_n \oplus (1 - a_n)y_n, \quad n \ge 1, \end{cases} \tag{2.6}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are appropriate sequences in (0, 1).

It has been shown that a three-step iterative scheme gives better numerical results than the two-step and one-step approximate iterations. Thus we conclude that a three-step iterative scheme plays an important and significant role in solving various problems which arise in pure and applied sciences. These facts motivated us to study a class of three-step modified SP-iterative scheme in the setting of CAT(k) spaces with $k > 0$. For more details, one can see [7,9,12,27].

In the sequel, we need the following lemmas:

Lemma 5 (*[25]*). *Let (X, d) be a complete CAT(k) space with diam(X) $= \frac{\pi/2-\epsilon}{\sqrt{k}}$, $k > 0$ for some $\epsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ a uniformly continuous nearly asymptotically nonexpansive mapping. Then T has a fixed point.*

Lemma 6 (*[25]*). *Let (X, d) be a complete CAT(k) space with diam(X) $= \frac{\pi/2-\epsilon}{\sqrt{k}}$, $k > 0$ for some $\epsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ a uniformly continuous nearly asymptotically nonexpansive mapping. If $\{x_n\}$ is an AFPS for T such that Δ-$\lim_{n\to\infty} x_n = z$, then $z \in K$ and $z = Tz$.*

3. Main results

In this section, we approximate the fixed points of nearly asymptotically nonexpansive mapping of modified SP-iterative scheme in complete CAT(k) spaces and establish some strong and Δ-convergence theorems.

Theorem 1. *Let (X, d) be a complete CAT(k) space with diam(X) $= \frac{\pi/2-\epsilon}{\sqrt{k}}$, $k > 0$ for some $\epsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(s_n, \eta(T^n))\}$. For arbitrary $x_1 \in K$, the sequence $\{x_n\}$ be the modified SP-iteration defined as follows:*

$$\begin{cases} z_n = c_n T^n x_n \oplus (1-c_n)x_n, \\ y_n = b_n T^n z_n \oplus (1-b_n)z_n, \\ x_{n+1} = a_n T^n y_n \oplus (1-a_n)y_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are appropriate sequences in $(0, 1)$ satisfying the following:

(1) $\lim_{n\to\infty} \inf a_n(1-a_n) > 0$, $\lim_{n\to\infty} \inf b_n(1-b_n) > 0$ *and* $\lim_{n\to\infty} \inf c_n(1-c_n) > 0$;
(2) $\sum_{n=1}^{\infty} s_n < \infty$ *and* $\sum_{n=1}^{\infty} (\eta(T^n) - 1) < \infty$.

Then $\{x_n\}$ Δ-converges to a fixed point of T.

Proof. From Lemma 5, it follows that $F(T) \neq \emptyset$. Let $p \in F(T)$. Since T is nearly asymptotically nonexpansive, from (3.1) and Lemma 1, we have

$$\begin{aligned} d(z_n, p) &= d(c_n T^n x_n \oplus (1-c_n)x_n, p) \\ &\leq c_n d(T^n x_n, p) + (1-c_n)d(x_n, p) \\ &\leq c_n[\eta(T^n)(d(x_n, p) + s_n)] + (1-c_n)d(x_n, p) \\ &\leq \eta(T^n)[c_n(d(x_n, p) + (1-c_n)d(x_n, p))] + c_n\eta(T^n)s_n \\ &\leq \eta(T^n)d(x_n, p) + \eta(T^n)s_n. \end{aligned} \tag{3.2}$$

Again by using (3.1), (3.2) and Lemma 1, we have

$$\begin{aligned} d(y_n, p) &= d(b_n T^n z_n \oplus (1-b_n) z_n, p) \\ &\leq b_n d(T^n z_n, p) + (1-b_n) d(z_n, p) \\ &\leq b_n[\eta(T^n)(d(z_n, p) + s_n)] + (1-b_n) d(z_n, p) \\ &\leq \eta(T^n)[b_n(d(z_n, p) + (1-b_n) d(z_n, p))] + b_n \eta(T^n) s_n \\ &\leq \eta(T^n) d(z_n, p) + \eta(T^n) s_n \\ &\leq \eta(T^n)[\eta(T^n) d(x_n, p) + \eta(T^n) s_n] + \eta(T^n) s_n \\ &= \eta(T^n)^2 d(x_n, p) + (\eta(T^n) + \eta(T^n)^2) s_n. \end{aligned} \tag{3.3}$$

Further by using (3.1), (3.3) and Lemma 1, we have

$$\begin{aligned} d(x_{n+1}, p) &= d(a_n T^n y_n \oplus (1-a_n) y_n, p) \\ &\leq a_n d(T^n y_n, p) + (1-a_n) d(y_n, p) \\ &\leq a_n[\eta(T^n)(d(y_n, p) + s_n)] + (1-a_n) d(y_n, p) \\ &\leq \eta(T^n)[a_n(d(y_n, p) + (1-a_n) d(y_n, p))] + a_n \eta(T^n) s_n \\ &\leq \eta(T^n) d(y_n, p) + \eta(T^n) s_n \\ &\leq \eta(T^n)[\eta(T^n)^2 d(x_n, p) + (\eta(T^n) + \eta(T^n)^2) s_n] + \eta(T^n) s_n \\ &= \eta(T^n)^3 d(x_n, p) + (\eta(T^n)^3 + \eta(T^n)^2 + \eta(T^n)) s_n \\ &= (1+\alpha_n) d(x_n, p) + \beta_n, \end{aligned} \tag{3.4}$$

where $\alpha_n = \eta(T^n)^3 - 1 = (\eta(T^n)^2 + \eta(T^n) + 1)(\eta(T^n) - 1)$ and $\beta_n = (\eta(T^n)^3 + \eta(T^n)^2 + \eta(T^n))s_n$. Since $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$ and $\sum_{n=1}^{\infty} s_n < \infty$, we have that $\sum_{n=1}^{\infty}\alpha_n < \infty$ and $\sum_{n=1}^{\infty}\beta_n < \infty$. Hence by Lemma 4, $\lim_{n\to\infty} d(x_n, p)$ exists.

Claim: We claim that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Since $\{x_n\}$ is bounded, there exists $R > 0$ such that $\{x_n\}, \{y_n\}, \{z_n\} \subset B_R(p)$ for all $n \geq 1$ with $R < D_k/2$. In view of (2.1) and (3.1), we have

$$d^2(z_n, p) = d^2(c_n T^n x_n \oplus (1 - c_n)x_n, p)$$

$$\leq c_n d^2(T^n x_n, p) + (1 - c_n)d^2(x_n, p) - \frac{R}{2}c_n(1 - c_n)d^2(T^n x_n, x_n)$$

$$\leq c_n[\eta(T^n)(d(x_n, p) + s_n)]^2 + (1 - c_n)d^2(x_n, p)$$

$$- \frac{R}{2}c_n(1 - c_n)d^2(T^n x_n, x_n)$$

$$\leq \eta(T^n)^2 d^2(x_n, p) + As_n - \frac{R}{2}c_n(1 - c_n)d^2(T^n x_n, x_n), \tag{3.5}$$

for some $A > 0$, where $A = \eta(T^n)^2[s_n + 2d(x_n, p)]$, which implies that

$$d^2(z_n, p) \leq \eta(T^n)^2 d^2(x_n, p) + As_n. \tag{3.6}$$

Again by using (2.1), (3.1) and (3.6), we have

$$d^2(y_n, p) = d^2(b_n T^n z_n \oplus (1 - b_n)z_n, p)$$

$$\leq b_n d^2(T^n z_n, p) + (1 - b_n)d^2(z_n, p) - \frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n)$$

$$\leq b_n[\eta(T^n)(d(z_n, p) + s_n)]^2 + (1 - b_n)d^2(z_n, p)$$

$$- \frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n)$$

$$\leq \eta(T^n)^2 d^2(z_n, p) + Bs_n - \frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n)$$

$$\leq \eta(T^n)^2[\eta(T^n)^2 d^2(x_n, p) + As_n] + Bs_n - \frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n)$$

$$\leq \eta(T^n)^4 d^2(x_n, p) + (A\eta(T^n)^2 + B)s_n - \frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n)$$

$$= \eta(T^n)^4 d^2(x_n, p) + (C + B)s_n - \frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n), \tag{3.7}$$

for some $B, C > 0$, where $B = \eta(T^n)^2[s_n + 2d(z_n, p)]$ and $C = \eta(T^n)^2 A$, which implies that

$$d^2(y_n, p) \leq \eta(T^n)^4 d^2(x_n, p) + (B + C)s_n. \tag{3.8}$$

Finally, by using (2.1), (3.1) and (3.8), we have

$$
\begin{aligned}
d^2(x_{n+1}, p) &= d^2(a_n T^n y_n \oplus (1-a_n)y_n, p) \\
&\le a_n d^2(T^n y_n, p) + (1-a_n)d^2(y_n, p) - \frac{R}{2}a_n(1-a_n)d^2(T^n y_n, y_n) \\
&\le a_n[\eta(T^n)(d(y_n, p) + s_n)]^2 + (1-a_n)d^2(y_n, p) \\
&\quad - \frac{R}{2}a_n(1-a_n)d^2(T^n y_n, y_n) \\
&\le \eta(T^n)^2 d^2(y_n, p) + Ds_n - \frac{R}{2}a_n(1-a_n)d^2(T^n y_n, y_n) \\
&\le \eta(T^n)^2[\eta(T^n)^4 d^2(x_n, p) + (B+C)s_n] + Ds_n \\
&\quad - \frac{R}{2}a_n(1-a_n)d^2(T^n y_n, y_n) \\
&= \eta(T^n)^6 d^2(x_n, p) + (D+E)s_n - \frac{R}{2}a_n(1-a_n)d^2(T^n y_n, y_n) \\
&= [1 + (\eta(T^n)^6 - 1)]d^2(x_n, p) + (D+E)s_n \\
&\quad - \frac{R}{2}a_n(1-a_n)d^2(T^n y_n, y_n) \\
&= [1 + (\eta(T^n) - 1)\delta]d^2(x_n, p) + (D+E)s_n \\
&\quad - \frac{R}{2}a_n(1-a_n)d^2(T^n y_n, y_n) \qquad (3.9)
\end{aligned}
$$

for some $D, E, \delta > 0$, where $D = \eta(T^n)^2[s_n + 2d(y_n, p)]$, $E = \eta(T^n)^2(B+C)$, and

$$\delta = \eta(T^n)^5 + \eta(T^n)^4 + \eta(T^n)^3 + \eta(T^n)^2 + \eta(T^n) + 1,$$

which implies that

$$
\begin{aligned}
\frac{R}{2}a_n(1-a_n)d^2(T^n y_n, y_n) \le d^2(x_n, p) - d^2(x_{n+1}, p) + (\eta(T^n) - 1)\delta d^2(x_n, p) \\
+ (D+E)s_n.
\end{aligned}
$$

Since $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$ and also $d(x_n, p) < R$, we have

$$\frac{R}{2}a_n(1-a_n)d^2(T^n y_n, y_n) < \infty.$$

Since $\lim_{n\to\infty} \inf a_n(1-a_n) > 0$, we have

$$\lim_{n\to\infty} d(T^n y_n, y_n) = 0. \qquad (3.10)$$

Now, consider (3.7) we have

$$\begin{aligned} d^2(y_n, p) &\le [1 + (\eta(T^n)^4 - 1)]d^2(x_n, p) + (B + C)s_n - \frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n) \\ &\le [1 + (\eta(T^n) - 1)\lambda]d^2(x_n, p) + (B + C)s_n - \frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n), \end{aligned}$$

for some $\lambda > 0$, where $\lambda = (\eta(T^n) + 1)(\eta(T^n)^2 + 1)$, which yields that

$$\frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n) \le d^2(x_n, p) - d^2(y_n, p) + (\eta(T^n) - 1)\lambda d^2(x_n, p) + (B + C)s_n.$$

Since $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$ and also $d(x_n, p) < R$, $d(y_n, p) < R$, we have

$$\frac{R}{2}b_n(1 - b_n)d^2(T^n z_n, z_n) < \infty.$$

Since $\lim_{n\to\infty} \inf b_n(1 - b_n) > 0$, we have

$$\lim_{n\to\infty} d(T^n z_n, z_n) = 0. \tag{3.11}$$

Further, consider (3.5), we have

$$\begin{aligned} d^2(z_n, p) &\le [1 + (\eta(T^n)^2 - 1)]d^2(x_n, p) + As_n - \frac{R}{2}c_n(1 - c_n)d^2(T^n x_n, x_n) \\ &\le [1 + (\eta(T^n) - 1)\mu]d^2(x_n, p) + As_n - \frac{R}{2}c_n(1 - c_n)d^2(T^n x_n, x_n), \end{aligned}$$

for some $\mu > 0$, where $\mu = \eta(T^n) + 1$, which yields that

$$\frac{R}{2}c_n(1 - c_n)d^2(T^n x_n, x_n) \le d^2(x_n, p) - d^2(z_n, p) + (\eta(T^n) - 1)\mu d^2(x_n, p) + As_n.$$

Since $\sum_{n=1}^{\infty} s_n < \infty$, $\sum_{n=1}^{\infty}(\eta(T^n) - 1) < \infty$ and also $d(x_n, p) < R$, $d(z_n, p) < R$, we have

$$\frac{R}{2}c_n(1 - c_n)d^2(T^n x_n, x_n) < \infty.$$

As $\lim_{n\to\infty} \inf c_n(1 - c_n) > 0$, we have

$$\lim_{n\to\infty} d(T^n x_n, x_n) = 0. \tag{3.12}$$

Now, from (3.10)–(3.12), we have

$$\begin{aligned} d(x_{n+1}, y_n) &= d(a_n T^n y_n \oplus (1 - a_n)y_n, y_n) \\ &\le a_n d(T^n y_n, y_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} d(y_n, z_n) &= d(b_n T^n z_n \oplus (1 - b_n)z_n, z_n) \\ &\le b_n d(T^n z_n, z_n) \to 0 \text{ as } n \to \infty, \end{aligned}$$

and

$$\begin{aligned} d(z_n, x_n) &= d(c_n T^n x_n \oplus (1 - c_n)x_n, x_n) \\ &\le c_n d(T^n x_n, x_n) \to 0 \text{ as } n \to \infty. \end{aligned}$$

It follows that

$$d(x_{n+1}, x_n) \leq d(x_{n+1}, y_n) + d(y_n, z_n) + d(z_n, x_n) \to 0 \text{ as } n \to \infty.$$

Since T is uniformly continuous, we have

$$\begin{aligned} d(x_n, Tx_n) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_{n+1}, T^{n+1}x_n) + d(T^{n+1}x_n, Tx_n) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) + \eta(T^{n+1})d(x_{n+1}, x_n) \\ &\quad + s_{n+1} + d(T^{n+1}x_n, Tx_n) \\ &= (1 + \eta(T^{n+1}))d(x_n, x_{n+1}) + d(x_{n+1}, T^{n+1}x_{n+1}) \\ &\quad + d(T^{n+1}x_n, Tx_n) + s_{n+1} \\ &\to 0 \text{ as } n \to \infty, \end{aligned} \tag{3.13}$$

which implies that $\lim_{n\to\infty} d(Tx_n, x_n) = 0$. Let $w_w(x_n) := \cup A(\{v_n\})$, where the union is taken over all subsequences $\{v_n\}$ of $\{x_n\}$. Now, we show that $w_w(x_n) \subseteq F(T)$ and $w_w(x_n)$ consists of exactly one point. Let $v \in w_w(x_n)$. Then there exists a subsequence $\{v_n\}$ of $\{x_n\}$ such that $A(\{v_n\}) = \{v\}$. By Lemma 2, there exists a subsequence $\{t_n\}$ of $\{v_n\}$ such that $\Delta\text{-}\lim_{n\to\infty} t_n = t \in K$. Hence by (3.13) and Lemma 6, we have $t \in F(T)$. Since $\lim_{n\to\infty} d(x_n, t)$ exists, so by Lemma 3, $t = v$, that is, $w_w(x_n) \subseteq F(T)$. Now, we show that $\{x_n\}$ Δ-converges to a fixed point of T. For this, it suffices to show that $w_w(x_n)$ consists of exactly one point. Let $\{w_n\}$ be a subsequence of $\{x_n\}$ with $A(\{w_n\}) = \{w\}$. Let $A(\{x_n\}) = \{x\}$. Since from above $w \in w_w(x_n) \subseteq F(T)$, therefore $\lim_{n\to\infty} d(x_n, w)$ exists. Further from above $x = w \in F(T)$. Thus $w_w(x_n) = \{x\}$. Therefore the sequence $\{x_n\}$ Δ-converges to a fixed point of T. This completes the proof of the theorem. □

From Theorem 1, we deduce the following Corollary in CAT(0) space:

Corollary 1. *Let (X, d) be a complete CAT(0) space and K a nonempty closed and convex subset of X. Let $T : K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(s_n, \eta(T^n))\}$. Let $\{x_n\}$ be a sequence in K defined by (3.1) and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences in $(0, 1)$ satisfying the conditions (1) and (2) (of Theorem 1). Then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.*

Theorem 2. *Let (X, d) be a complete CAT(k) space with diam$(X) = \frac{\pi/2-\epsilon}{\sqrt{k}}$, $k > 0$ for some $\epsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(s_n, \eta(T^n))\}$. Let $\{x_n\}$ be a sequence in K defined by (3.1) and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences in $(0, 1)$ satisfying the conditions (1) and (2) (of Theorem 1). Suppose that T^q is semi compact for some $q \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Theorem 1, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Since T is uniformly continuous, we have

$$d(x_n, T^q x_n) \leq d(x_n, Tx_n) + d(Tx_n, T^2 x_n) + \cdots \\ + d(T^{q-1} x_n, T^q x_n) \to 0, \quad \text{as } n \to \infty,$$

that is, $\{x_n\}$ is an AFPS for T^q. As T^q is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to\infty} x_{n_k} = p$, where $p \in K$. Again, by the uniform continuity of T, we have

$$d(p, Tp) \leq d(Tp, Tx_{n_k}) + d(Tx_{n_k}, x_{n_k}) + d(x_{n_k}, p) \to 0, \text{ as } k \to \infty,$$

that is, $p \in F(T)$. Again, by Theorem 1, $\lim_{n\to\infty} d(x_n, p)$ exists. Thus the sequence $\{x_n\}$ has the strong limit p. Therefore $\{x_n\}$ converges strongly to a fixed point of T. This completes the proof of the theorem. □

Remark 1. Since T is completely continuous, then for some $q \in \mathbb{N}$, the image of T^q is semi-compact. Since $\{x_n\}$ is a bounded sequence and from Theorem 2, $d(x_n, T^q x_n) \to 0$ as $n \to \infty$. Therefore for some $q \in \mathbb{N}$, T^q is semi-compact, that is, the continuous image of a semi-compact space is semi-compact.

Example 1. Let $X = K = [0, 1]$ with the usual metric. Define $T : K \to K$ by

$$T(x) = \begin{cases} \dfrac{x}{4}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

Then T is semi-compact. However, T is not continuous. In fact, if $\{x_n\}$ is a bounded sequence in K satisfying $|x_n - Tx_n| \to 0$ as $n \to \infty$, then by Bolzano–Weierstrass theorem, $\{x_n\}$ has a convergent subsequence.

Also, there is an example which shows that a semi-compact mapping is not necessarily compact.

Example 2 (*[2]*)**.**

Let $X = l^2$ and $K = \{e_1, e_2, \ldots, e_n, \ldots\}$ be the usual orthonormal basis for l^2. Define $T : K \to K$ by $T(e_j) = e_{j+1},\ j \in \mathbb{N}$. Then T is continuous and also an isometry but not compact. However, T is semi-compact. Indeed, if $\{e_j\}_{j\in\mathbb{N}}$ is a bounded sequence in K such that $e_j - Te_j$ converges, $\{e_j\}_{j\in\mathbb{N}}$ must be finite.

From Theorem 2, we have the following result as a Corollary:

Corollary 2. *Let (X, d) be a complete CAT(0) space and K a nonempty closed and convex subset of X. Let $T : K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(s_n, \eta(T^n))\}$. Let $\{x_n\}$ be a sequence in K defined by (3.1) and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences in $(0, 1)$ satisfying the conditions (1) and (2) (of Theorem 1). Suppose that T^q is semi-compact for some $q \in \mathbb{N}$. Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Now, we have strong convergence theorems.

Theorem 3. *Let (X, d) be a complete CAT(k) space with diam$(X) = \frac{\pi/2-\epsilon}{\sqrt{k}}$, $k > 0$ for some $\epsilon \in (0, \pi/2)$. Let K be a nonempty closed convex subset of X, and $T : K \to K$ a uniformly* *continuous nearly asymptotically nonexpansive mapping with sequence* $\{(s_n, \eta(T^n))\}$. *Let* $\{x_n\}$ *be a sequence in* K *defined by* (3.1) *and* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *the sequences in* $(0, 1)$ *satisfying the conditions (1) and (2) (of Theorem 1). Suppose that* T *satisfies condition* (I). *Then the sequence* $\{x_n\}$ *converges strongly to a fixed point of* T *if and only if* $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$.

Proof. It is easy to see that if $\{x_n\}$ converges to a point $x \in F(T)$, then $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$.

For converse part, suppose that $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$. Since from Theorem 1, we have

$$d(x_{n+1}, p) \le d(x_n, p) \quad \text{for } p \in F(T)$$

so that

$$d(x_{n+1}, F(T)) \le d(x_n, F(T)).$$

Hence, $\lim_{n\to\infty} d(x_{n+1}, F(T))$ exists. By hypothesis, $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$, so $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Now, we show that $\{x_n\}$ is a Cauchy sequence in K. Let $\epsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, there exists n_0 such that for all $n \ge n_0$, we have

$$d(x_n, F(T)) < \frac{\epsilon}{4}.$$

In particular, $\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\epsilon}{4}$, so there must exist a $p^* \in F(T)$ such that

$$d(x_{n_0}, p^*) < \frac{\epsilon}{2}.$$

Now, for $m, n \ge n_0$, we have

$$d(x_{n+m}, x_n) \le d(x_{n+m}, p^*) + d(x_n, p^*) < 2d(x_{n_0}, p^*) < \epsilon.$$

Therefore $\{x_n\}$ is a Cauchy sequence in a closed subset K of a complete CAT(k) space X, and hence $\{x_n\}$ must converge in K. Let $\lim_{n\to\infty} x_n = q$. Now, since T is uniformly continuous and from Theorem 1, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have

$$d(q, Tq) = d(q, x_n) + d(x_n, Tx_n) + d(Tx_n, Tq) \to 0 \text{ as } n \to \infty,$$

which implies that $q \in F(T)$. This completes the proof of the theorem. □

Now, we prove strong convergence theorem using condition (I).

Theorem 4. *Let* (X, d) *be a complete CAT*(k) *space with diam*$(X) = \frac{\pi/2-\epsilon}{\sqrt{k}}$, $k > 0$ *for some* $\epsilon \in (0, \pi/2)$. *Let* K *be a nonempty closed convex subset of* X, *and* $T : K \to K$ *a uniformly continuous nearly asymptotically nonexpansive mapping with sequence* $\{(s_n, \eta(T^n))\}$. *Let* $\{x_n\}$ *be a sequence in* K *defined by* (3.1) *and* $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *the sequences in* $(0, 1)$ *satisfying the conditions (1) and (2) (of Theorem 1). Suppose that* T *satisfies condition* (I). *Then* $\{x_n\}$ *converges strongly to a fixed point of* T.

Proof. From Theorem 1, $\lim_{n\to\infty} d(x_n, F(T))$ exists and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Therefore by using condition (I), we have

$$\lim_{n\to\infty} g(d(x_n, F(T))) \le \lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

that is, $\lim_{n\to\infty} g(d(x_n, F(T))) = 0$. Since g is a nondecreasing function satisfying $g(0) = 0$ and $g(r) > 0$ for all $r \in (0, \infty)$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, the result follows from Theorem 3. □

The following example shows that a nearly asymptotically nonexpansive mapping need not be continuous and Lipschitzian.

Example 3. Let $X = \mathbb{R}$ and $K = [0, 1]$. Define a mapping $T : K \to K$ by

$$T(x) = \begin{cases} \frac{1}{3}, & x \in \left[0, \frac{1}{3}\right], \\ 0, & x \in \left(\frac{1}{3}, 1\right]. \end{cases}$$

Hence $F(T) = \frac{1}{3}$. Then obviously T is a discontinuous and non-Lipschitzian mapping. However, T is nearly nonexpansive mapping and hence a nearly asymptotically nonexpansive mapping with sequence $\{(s_n, \eta(T^n))\} = \left\{\frac{1}{3^n}, 1\right\}$. In fact, for a sequence $\{s_n\}$ with $s_1 = \frac{1}{3}$ and $s_n \to 0$ as $n \to \infty$, we have

$$d(Tx, Ty) \le d(x, y) + s_1 \text{ for all } x, y \in K$$

and

$$d(T^n x, T^n y) \le d(x, y) + s_n \text{ for all } x, y \in K \text{ and } n \ge 2,$$

since $T^n x = \frac{1}{3}$ for all $x \in [0, 1]$ and $n \ge 2$.

Further, it can be easily shown that strong convergence $\Rightarrow$ Δ-convergence $\Rightarrow$ weak convergence. For details, (see [25]). But the converse is not true in general. The following example shows that if the sequence $\{x_n\}$ is weakly convergent, then it is not Δ-convergent.

Example 4 (*[18]*)**.** Let $X = \mathbb{R}$ with the usual metric d and $K = [-1, 1]$. Let $\{x_n\}$, $\{u_n\}$ and $\{v_n\}$ be the sequences in K defined by $\{x_n\} = \{1, -1, 1, -1, \ldots\}$, $\{u_n\} = \{-1, -1, -1, \ldots\}$ and $\{v_n\} = \{1, 1, 1, \ldots\}$. Then $A(\{x_n\}) = A_K(\{x_n\}) = \{0\}$, $A(\{u_n\}) = \{-1\}$ and $A(\{v_n\}) = \{1\}$. Thus the sequence $\{x_n\}$ converges weakly to 0 but it does not have a Δ-limit.

From Theorems 3 and 4, we deduce the following Corollaries in CAT(0) space:

Corollary 3. *Let (X, d) be a complete CAT(0) space and K a nonempty closed and convex subset of X. Let $T : K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(s_n, \eta(T^n))\}$. Let $\{x_n\}$ be a sequence in K defined by (3.1) and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences in $(0, 1)$ satisfying the conditions (1) and (2) (of Theorem 1). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T if and only if* $\lim_{n\to\infty} \inf d(x_n, F(T)) = 0$.

Corollary 4. *Let (X, d) be a complete CAT(0) space and K a nonempty closed and convex subset of X. Let $T : K \to K$ be a uniformly continuous nearly asymptotically nonexpansive mapping with sequence $\{(s_n, \eta(T^n))\}$. Let $\{x_n\}$ be a sequence in K defined by (3.1) and $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ be sequences in $(0, 1)$ satisfying the conditions (1) and (2) (of Theorem 1). Suppose that T satisfies condition (I). Then the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

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Conflict of interest

The author declares that there are no conflicts of interest.

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