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A survey on the geometry of production models in economics

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Abstract. In this article we survey selected recent results on the geometry of production models, focussing on the main production functions that are usually analyzed in economics, namely homogeneous, homothetic, quasi-sum and quasi-product production functions.

Keywords: Homogeneous production function; Homothetic production function; Quasi-sum production function; Quasi-product production function; Production hypersurface; Gauss–Kronecker curvature; Mean curvature; Flat space; Marginal rate of substitution; Constant elasticity of substitution; Return to scale

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1. Introduction

It is well known that the notion of production function is one of the key concepts of mainstream neoclassical theories, being used in the mathematical modeling of the relationship between the output of a production process and the inputs that have been used in obtaining it. According to Humphrey [28] and Mishra [32], it seems that the German mathematical economist, location theorist, and agronomist Johann Heinrich von Thünen was the first person to algebraically formulate the relationship between output and input as a mapping $f : \mathbb{R}_+^n \to \mathbb{R}_+$, $f = f(x_1, \ldots, x_n)$, where f is the quantity of output, n is the number of the inputs and $x_1, \ldots, x_n$ are the factor inputs, such as labor, capital, land, raw materials etc.

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Because these functions should also model the economic reality, they are required to have several properties [27,33,34]. As it was pointed in [29], the classical treatment of the production functions makes use of the projections of production functions on a plane, but this approach leads to limited conclusions. Fortunately, this problem can be solved by identifying a production function f with the graph of f, i.e. the nonparametric hypersurface of the $(n+1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$, defined by [36]

$$L(x_1,\dots,x_n) = (x_1,\dots,x_n, f(x_1,\dots,x_n)),\ (x_1,\dots,x_n)\in\mathbb{R}^n_+, \tag{1}$$

and applying a differential geometric treatment. Using this approach, several basic properties of production models can be interpreted in terms of the geometry of their graphs [22].

In 2011, the present authors obtained an unexpected link between some fundamental notions in the theory of production functions and the differential geometry of hypersurfaces in Euclidean spaces [36,37], giving an impulse to construct a differential geometrical theory of production models in economics. It is the aim of this paper to give a survey of the main results on the geometry of production functions obtained after the discovery of this link. The article is focussed on the main production models that are often analyzed both in microeconomics and macroeconomics, namely homogeneous, homothetic, quasi-sum and quasi-product production models.

2. Preliminaries on the production models in economics

The simplest production model used in economics is the famous Cobb–Douglas production function. It was introduced in 1928 by the mathematician C.W. Cobb and the economist P.H. Douglas [24] in order to describe the distribution of the national income of the United States in the following form

$$f = CK^\alpha L^\beta,$$

where f stands for total production, K for capital input, L for labor input and C is a positive constant which signifies the total factor productivity. We note that in the original definition it is required that $\alpha+\beta=1$, but this condition has been later relaxed; usually, C, α and β are estimated from empirical data. A generalized Cobb–Douglas production function depending on n-inputs is given by

$$f(x_1,\dots,x_n) = A\cdot\prod_{i=1}^n x_i^{\alpha_i}, \tag{2}$$

where $A,\alpha_1,\dots,\alpha_n>0$. The advantages of the Cobb–Douglas production function in its generalized form were highlighted by K.V. Bhanumurthy [10].

The Cobb–Douglas production model was generalized by K.J. Arrow, H.B. Chenery, B.S. Minhas and R.M. Solow in [3]. They introduced the so-called Constant Elasticity of Substitution (CES) production function. This model was extended to the n-inputs by H. Uzawa [35] and D. McFadden [31], who defined a new production function, usually called generalized CES production function, Armington aggregator or ACMS (Arrow-Chenery-Minhas-Solow) production function, by

$$f(x_1,\dots,x_n) = A\left(\sum_{i=1}^n c_i x_i^\rho\right)^{\frac{\gamma}{\rho}}, \tag{3}$$

where $A > 0$, $\rho < 1$, $\rho \neq 0$, $\gamma > 0$ and $c_i > 0$, for all $i \in \{1, \dots, n\}$. We note that CES production functions are of great interest in economics because of their invariant characteristic, namely that the elasticity of substitution between the parameters is constant on their domains [30].

We also remark that generalized CES production functions include as special cases many other famous production models, like multinomial production functions or Leontief production functions (see [32] for other examples and definitions). We recall that a multinomial production function is obtained by taking $\rho \to 1$ in (3). Moreover, if $\gamma = 1$, then a multinomial production model is said to be a linear production model, also called a perfect substitutes production model.

We recall next the main indicators of production. If f is a production function with n inputs $x_1, x_2, \dots, x_n$, $n \geq 2$, then the elasticity of production with respect to a certain factor of production x_i is defined as

$$E_i = \frac{x_i}{f} f_i$$

and the marginal rate of technical substitution of input x_j for input x_i is given by

$$\mathrm{MRS}_{ij} = \frac{f_j}{f_i},$$

where the subscripts denote partial derivatives of the function f with respect to the corresponding variables.

A production function is said to satisfy the proportional marginal rate of substitution property if and only if

$$\mathrm{MRS}_{ij} = \frac{x_i}{x_j},$$

for all $1 \leq i \neq j \leq n$.

On the other hand, the function H_{ij} defined by

$$H_{ij}(x_1, \dots, x_n) = \frac{\frac{1}{x_i f_i} + \frac{1}{x_j f_j}}{-\frac{f_{ii}}{f_i^2} + \frac{2 f_{ij}}{f_i f_j} - \frac{f_{jj}}{f_j^2}},$$

for all $(x_1, \dots, x_n) \in \mathbb{R}_+^n$, is called the Hicks elasticity of substitution of the ith production factor with respect to the jth production factor, where $i, j \in \{1, \dots, n\}$, $i \neq j$.

Moreover, the function A_{ij} defined as

$$A_{ij}(x_1, \dots, x_n) = -\frac{x_1 f_1 + \dots + x_n f_n}{x_i x_j} \frac{\Delta_{ij}}{\Delta},$$

for all $(x_1, \dots, x_n) \in \mathbb{R}_+^n$, where Δ is the determinant of the bordered matrix

$$A(f) = \begin{pmatrix} 0 & f_1 & \dots & f_n \\ f_1 & f_{11} & \dots & f_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ f_n & f_{1n} & \dots & f_{nn} \end{pmatrix}$$

and Δ_{ij} is the co-factor of the element f_{ij} in the determinant Δ ($\Delta \neq 0$ is assumed), is called the Allen elasticity of substitution of the ith production factor with respect to the jth production factor, where $i, j \in \{1, \ldots, n\}$, $i \neq j$. Moreover, $A(f)$ is said to be the Allen matrix of f and we call Δ the Allen determinant [4].

We note that in the case of two inputs, Hicks elasticity of substitution and Allen elasticity of substitution coincide. So $H_{ij} = A_{ij}$ and in this case the indicator is simply called the elasticity of substitution between the two factors of production.

3. Preliminaries on the geometry of hypersurfaces

For general references on the geometry of hypersurfaces, we refer to [12,13,15]. If M is a hypersurface of the Euclidean space $\mathbb{E}^{n+1}$, then it is known that the Gauss map $\nu : M \to S^n$ maps M to the unit hypersphere S^n of $\mathbb{E}^{n+1}$. With the help of the differential $d\nu$ of ν one can define a linear operator on the tangent space T_pM, denoted by S_p and known as the *shape operator*, by $g(S_p v, w) = g(d\nu(v), w)$, for $v, w \in T_pM$, where g is the metric tensor on M induced by the Euclidean metric on $\mathbb{E}^{n+1}$. The determinant of the shape operator S_p, denoted by $K(p)$, is called the Gauss–Kronecker curvature. When $n = 2$, the Gauss–Kronecker curvature is simply called the Gauss curvature, which is intrinsic due to Gauss's famous Theorema Egregium. The trace of the shape operator S_p is called the mean curvature of the hypersurfaces. In contrast to the Gauss–Kronecker curvature, the mean curvature is extrinsic, depending on the immersion of the hypersurface. A hypersurface is said to be minimal if its mean curvature vanishes identically. The Riemann curvature tensor R of M is given by

$$R(u, v)w = \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w,$$

where ∇ is the Levi-Civita connection of g. It is well known that R measures the non-commutativity of the covariant derivative, and as such is the integrability obstruction for the existence of an isometry with Euclidean space [21]. A Riemannian manifold is said to be flat if its Riemann curvature tensor vanishes identically. We recall now the following result, which is a basic tool in proving the results presented throughout this paper.

Lemma 3.1 (*[13]*). *For the production hypersurface defined by* (1) *and* $w = \sqrt{1 + \sum_{i=1}^{n} f_i^2}$, *we have:*

i. *The Gauss–Kronecker curvature K is given by*

$$K = \frac{\det(f_{x_i x_j})}{w^{n+2}}. \tag{4}$$

ii. *The mean curvature H is given by*

$$H = \frac{1}{n} \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{f_{x_i}}{w} \right). \tag{5}$$

iii. *The sectional curvature K_{ij} of the plane section spanned by* $\frac{\partial}{\partial x_i}$, $\frac{\partial}{\partial x_j}$ *is*

$$K_{ij} = \frac{f_{x_i x_i} f_{x_j x_j} - f_{x_i x_j}^2}{w^2 \left(1 + f_{x_i}^2 + f_{x_j}^2\right)}. \tag{6}$$

iv. *The Riemann curvature tensor R and the metric tensor g satisfy*

$$g\left(R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_\ell}\right) = \frac{f_{x_i x_\ell} f_{x_j x_k} - f_{x_i x_k} f_{x_j x_\ell}}{w^4}. \tag{7}$$

4. On homogeneous production models

There are some special classes of production functions that are often analyzed in both macroeconomics and microeconomics. The first one is given by the class of homogeneous production functions. A production function f defined on a set D for which $(\lambda x_1, \dots, \lambda x_n) \in D$ whenever $\lambda > 0$ and $(x_1, \dots, x_n) \in D$, is said to be homogeneous of degree p if there exists a real number p such that

$$f(\lambda x_1, \dots, \lambda x_n) = \lambda^p f(x_1, \dots, x_n),$$

for all $(x_1, \dots, x_n) \in D$ and all $\lambda > 0$. This means that if the inputs are multiplied by same factor, then the output is multiplied by some power of this factor. If $p = 1$ then the function is said to have a constant return to scale, if $p > 1$ then we have an increased return to scale and if $p < 1$ then we say that the function has a decreased return to scale. We remark that both generalized Cobb–Douglas and CES production functions are homogeneous.

In [36], the author established a link between some fundamental notions in the theory of production functions and the differential geometry of hypersurfaces, proving the following result.

Theorem 4.1 (*[36]*)**.** *If $(GCDPF)$ denotes the generalized Cobb–Douglas production function and (CDH) denotes the corresponding Cobb–Douglas hypersurface, then:*

i. *$(GCDPF)$ has constant return to scale if and only if (CDH) has vanishing Gauss–Kronecker curvature.*
ii. *$(GCDPF)$ has decreasing return to scale if and only if (CDH) has positive Gauss–Kronecker curvature.*
iii. *$(GCDPF)$ has increasing return to scale if and only if (CDH) has negative Gauss–Kronecker curvature.*

In [37], the above theorem was generalized to the case of the CES production functions with n-inputs.

Theorem 4.2 (*[37]*)**.** *If $(GCESPF)$ denotes the generalized CES production function and $(CESH)$ denotes the corresponding CES hypersurface, then:*

i. *$(GCESPF)$ has constant return to scale if and only if $(CESH)$ has vanishing Gauss–Kronecker curvature.*
ii. *$(GCESPF)$ has decreasing return to scale if and only if $(CESH)$ has positive Gauss–Kronecker curvature.*
iii. *$(GCESPF)$ has increasing return to scale if and only if $(CESH)$ has negative Gauss–Kronecker curvature.*

Because a similar result is also valid for other production functions, a natural question arises, namely whether a general result of this type can be stated for all homogeneous production functions. A first answer was given by B.-Y. Chen in [14]: the decreasing/increasing return to scale property cannot be determined by the Gauss–Kronecker curvature of production hypersurface; even for two-factor homogeneous production functions, as follows from [14, Example 4.1]. Moreover, an interesting result on the homogeneous production functions defining flat hypersurfaces follows from the next theorem, which generalizes a classical result in differential geometry concerning homogeneous functions of degree two due to F. Brickell [11].

Theorem 4.3 ([21]). *Let f be a twice differentiable, r-homogeneous, non-constant, real valued function of n variables $(x_1, \dots, x_n)$ on an open domain $D \subset \mathbb{R}^n$, $n \geq 2$. Then the hypersurface of $\mathbb{E}^{n+1}$ defined by*

$$z = f(x_1, \dots, x_n), \ \ (x_1, \dots, x_n) \in D,$$

is flat if and only if either f is linearly homogeneous, i.e. $r = 1$, or

$$f = (c_1 x_1 + c_2 x_2 + \cdots + c_n x_n)^r \, , \ \ r \neq 1, \tag{8}$$

for some real constants $c_1, \dots, c_n$.

However, we note that the above theorem is false if we replace the flatness of the hypersurface by the vanishing of the Gauss–Kronecker curvature, as was shown in [21, Remark 4.1].

Now, from Theorem 4.3 we obtain a complete classification of homogeneous production functions with an arbitrary number of inputs whose production hypersurfaces are flat, as follows.

Theorem 4.4 ([21]). *A homogeneous production function with an arbitrary number of inputs defines a flat hypersurface if and only if either it has constant return to scale or it is a multinomial production function.*

Recently, X. Wang [41] proved that if the sectional curvature for a homogeneous graph hypersurface is constant, then it must be null. Consequently, applying the above theorem, it follows that in fact a homogeneous production function defines a hypersurface with constant sectional curvature if and only if either it has constant return to scale or it is a multinomial production function.

On the other hand, concerning the minimality of the generalized Cobb–Douglas and CES production hypersurfaces, we have the following results proved by X. Wang and Y. Fu.

Theorem 4.5 ([42]).

i. *There does not exist a minimal generalized Cobb–Douglas production hypersurface in the Euclidean space $\mathbb{E}^{n+1}$.*
ii. *A generalized CES production hypersurface in $\mathbb{E}^{n+1}$ is minimal if and only if the generalized CES production function is a perfect substitute.*

We note that B.-Y. Chen proved in [14] a more general result in the case of 2 inputs: a homogeneous production function is a perfect substitute if and only if the production surface is a minimal surface.

In 2010, L. Losonczi [30] showed that twice differentiable two-inputs homogeneous production functions with CES property are Cobb–Douglas and ACMS production functions. This result was later generalized to an arbitrary number of inputs in [16], the author classifying all homogeneous production functions which satisfy the CES property. More precisely, he proves the following.

Theorem 4.6 (*[16]*). *Let f be a twice differentiable, homogeneous of degree γ, non-constant, real valued function of n variables $(x_1, \ldots, x_n)$ on an open domain $D \subset \mathbb{R}^n$. If f satisfies the constant elasticity of substitution property, then it is either the generalized Cobb–Douglas production function given by (2), with $\alpha_1 + \cdots + \alpha_n = \gamma$, or the generalized ACMS production function given by (3).*

On the other hand, homogeneous production functions with proportional marginal rate of substitution, and with constant elasticity with respect to any factor of production, were classified in [38].

Theorem 4.7 (*[38]*). *Let f be a twice differentiable, r-homogeneous, non-constant, real valued function of n variables $(x_1, x_2, \ldots, x_n)$ on an open domain $D \subset \mathbb{R}^n$. Then:*

i. *The elasticity of production is a constant k_i with respect to a certain factor of production x_i if and only if*

$$f(x_1, x_2, \ldots, x_n) = x_i^{k_i} x_j^{r-k_i} F(u_1, \ldots, u_{n-2}),$$

where j is any element selected from the set $\{1, \ldots, n\} \setminus \{i\}$ and F is a twice differentiable real valued function of $n - 2$ variables

$$\{u_1, \ldots, u_{n-2}\} = \left\{ \frac{x_k}{x_j} | k \in \{1, \ldots, n\} \setminus \{i, j\} \right\}.$$

ii. *The elasticity of production is a constant k_i with respect to any factor of production x_i, $i \in \{1, 2, \ldots, n\}$, if and only if*

$$k_1 + k_2 + \cdots + k_n = r$$

and f reduces to the Cobb–Douglas production function given by

$$f(x_1, x_2, \ldots, x_n) = C x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n},$$

where C is a positive constant.

iii. *The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the Cobb–Douglas production function given by*

$$f(x_1, x_2, \ldots, x_n) = C x_1^{\frac{r}{n}} x_2^{\frac{r}{n}} \ldots x_n^{\frac{r}{n}},$$

where C is a positive constant.

Further results concerning the geometry of homogeneous production functions can be found in [14,17,40].

5. On quasi-sum production models

A second class of production models of interest in economic analysis is given by the quasi-sum production models. A production function f is called *quasi-sum* [9,18] if there are strict monotone functions $G, h_1, \ldots, h_n$ with $G' > 0$ such that

$$f(x) = G(h_1(x_1) + \cdots + h_n(x_n)), \tag{9}$$

where $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+$. We note that these functions are of great interest because they appear as solutions of the general bisymmetry equation, being related to the problem of consistent aggregation [1]. A quasi-sum production function is said to be quasi-linear if at most one of the functions $G, h_1, \dots, h_n$ is nonlinear. The following theorem gives a very simple necessary and sufficient condition for a quasi-sum production function with more than two factors to be quasi-linear.

Theorem 5.1 ([18]). *A twice differentiable quasi-sum production function with more than two factors is quasi-linear if and only if its production hypersurface is a flat space.*

We recall now that a production function given by

$$f(x_1, \dots, x_n) = G\left(\prod_{i=1}^{n} x_i^{k_i}\right), \tag{10}$$

where G is a strictly increasing function and $k_1, \dots, k_n$ are nonzero real numbers, is called a homothetic generalized Cobb–Douglas production function. Similarly, a production function defined by

$$f(x_1, \dots, x_n) = G\left(\sum_{i=1}^{n} k_i x_i^{\rho}\right), \tag{11}$$

where G is a strictly increasing function and $k_1, \dots, k_n, \rho$ are nonzero real numbers, is said to be a homothetic generalized ACMS production function.

The quasi-sum production models whose production hypersurfaces have vanishing Gauss–Kronecker curvature were completely classified by B.-Y. Chen in [18]. Moreover, M.E. Aydin and A. Mihai proved in [9] that the production hypersurface of a quasi-sum production function f has vanishing Gauss–Kronecker curvature if and only if the Allen matrix of f is singular, provided that one of the $h_1, \dots, h_n$ is a linear function and $G'' \neq 0$. We also note that quasi-sum production functions whose Allen matrices are singular were classified as follows.

Theorem 5.2 ([9]). *Let f be a twice differentiable quasi-sum production function. Then the Allen matrix $A(f)$ is singular if and only if, up to translations, f is one of the following:*

i. $f = G(c_1x_1 + c_2x_2 + h_3(x_3) + \cdots + h_n(x_n))$, *where c_1, c_2 are nonzero constants and $G, h_3, \dots, h_n$ are strict monotone functions;*
ii. $f = G(\sum_{i=1}^{n} c_i \ln|d_i x_i + e_i|)$, *where c_i are nonzero constants and d_i, e_i are some constants, $i = 1, \dots, n$.*

On the other hand, the classification of quasi-sum production functions satisfying the constant elasticity of substitution property was obtained by B.-Y. Chen as follows.

Theorem 5.3 ([19]). *Let f be a twice differentiable quasi-sum production function given by (9). Then f satisfies the constant elasticity of substitution property if and only if, up to suitable translations, f is one of the following functions:*

i. *a homothetic generalized ACMS production function given by*

$$f(x_1, \dots, x_n) = G(c_1 x_1^{\frac{\sigma-1}{\sigma}} + \cdots + c_n x_n^{\frac{\sigma-1}{\sigma}}),$$

where $\sigma \neq 1$ and $c_1, \dots, c_n$ are nonzero constants;

ii. *a homothetic generalized Cobb–Douglas production function given by (10);*
iii. *a two-input production function of the form*

$$f(x_1, x_2) = G\left(\frac{x_2}{x_1}\right),$$

where G is a strictly increasing function.

Quasi-sum production functions with constant elasticity of production with respect to any factor of production, and with proportional marginal rate of substitution, were recently classified in [39], the authors proving the following result.

Theorem 5.4 (*[39]*). *Let f be a quasi-sum production function given by (9). Then:*

i. *The elasticity of production is a constant k_i with respect to a certain factor of production x_i if and only if f reduces to*

$$f(x_1, \dots, x_n) = A \cdot x_i^{k_i} \cdot \exp\left(D \sum_{j \neq i} h_j(x_j)\right), \tag{12}$$

where A and D are positive constants.

ii. *The elasticity of production is a constant k_i with respect to all factors of production x_i, $i = 1, \dots, n$, if and only if f reduces to the generalized Cobb–Douglas production function given by (2).*

iii. *The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the homothetic generalized Cobb–Douglas production function given by*

$$f(x_1, \dots, x_n) = G\left(\prod_{i=1}^{n} x_i^k\right), \tag{13}$$

where k is a nonzero real number.

iv. *If the production function satisfies the proportional marginal rate of substitution property, then:*

iv$_1$. *The production hypersurface has vanishing Gauss–Kronecker curvature if and only if, up to a suitable translation, f reduces to the following generalized Cobb–Douglas production function with constant return to scale:*

$$f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^{n} x_i^{\frac{1}{n}}. \tag{14}$$

iv$_2$. *The production hypersurface cannot be minimal.*

iv$_3$. *The production hypersurface has vanishing sectional curvature if and only if, up to a suitable translation, f reduces to the following generalized Cobb–Douglas production function:*

$$f(x_1, \dots, x_n) = A \cdot \prod_{i=1}^{n} \sqrt{x_i}. \tag{15}$$

From Theorem 5.4 we can easily deduce the following result for quasi-sum production models with two inputs.

Corollary 5.5. *Let f be a quasi-sum production function with two-inputs given by $f(K,L) = G(g(K)+h(L))$ where K is the capital and L is the labor. Then:*

i. *f has a constant elasticity of labor k if and only if f reduces to $f(K,L) = A\cdot K^k e^{D\cdot h(L)}$, where A and D are positive constants.*
ii. *f has a constant elasticity of capital ℓ if and only if f reduces to $f(K,L) = A\cdot L^\ell e^{D\cdot g(K)}$, where A and D are positive constants.*
iii. *f has constant elasticities with respect to both labor and capital if and only if f reduces to the Cobb–Douglas production function given by $f(K,L) = AK^kL^\ell$, where A is a positive constant.*
iv. *f satisfies the proportional rate of substitution property between capital and labor if and only if f is a homothetic Cobb–Douglas production function given by $f(K,L) = F(K^kL^k)$, where k is a positive constant.*
v. *If f satisfies the proportional marginal rate of substitution property, then the production surface cannot be minimal.*
vi. *If f satisfies the proportional marginal rate of substitution property, then the production surface is flat if and only if f reduces to the following Cobb–Douglas production function with constant return to scale: $f(K,L) = A\sqrt{KL}$.*

6. On homothetic and quasi-product production models

Along with the homogeneous and quasi-sum production models presented in the previous sections, there are two other classes of production models of great interest in microeconomics and macroeconomics, namely homothetic and quasi-product production models. A production function of the form

$$f(x_1,\dots,x_n) = G(h(x_1,\dots,x_n)), \tag{16}$$

where G is a strictly increasing function and h is a homogeneous function of any given degree p, is said to be a homothetic production function [22].

In [22], B.-Y. Chen classified homothetic functions satisfying the homogeneous Monge–Ampère equation. As a direct application to production models in economics, the following result is obtained.

Theorem 6.1 (*[22]*)**.** *Let f be a homothetic function given by (16) such that h is a p-homogeneous function with $p \neq 1$. Then the graph of h has null Gauss–Kronecker curvature if and only if either*

i. *h satisfies the homogeneous Monge–Ampère equation $\det(h_{ij}) = 0$, or*
ii. *up to constants, $f = G\circ h$ is a linearly homogeneous function.*

On the other hand, the classification of homothetic production functions satisfying the constant elasticity of substitution property was realized by B.-Y. Chen as follows.

Theorem 6.2 (*[20]*)**.** *Let f be a homothetic function given by (16). Then f satisfies the constant elasticity of substitution property if and only if the homogeneous function h is either a generalized Cobb–Douglas production function or a generalized ACMS production function.*

A production function is said to be quasi-product [26] if the function has the form

$$f(x_1, \dots, x_n) = F\left(\prod_{i=1}^{n} g_i(x_i)\right), \tag{17}$$

where $F, g_1, \dots, g_n$ are continuous positive functions with nowhere zero first derivatives on their domain of definition. We note that quasi-product production functions include the generalized CES production functions. We also remark that Y. Fu and W.G. Wang obtained in [26] the following classification of quasi-product productions, provided their corresponding graph hypersurfaces are flat spaces.

Theorem 6.3 (*[26]*). *Let f be a quasi-product production function given by (17). If the corresponding production hypersurface is flat, then, up to translations, f is given by one of the following functions:*

(a) $f(x_1, \dots, x_n) = F(\exp(\sum_{i=1}^{n} c_i x_i))$, *where* $c_i \in \mathbb{R} - \{0\}$, $i \in \{1, \dots, n\}$*;*
(b) $f(x_1, \dots, x_n) = C_1 \ln(g_1(x_1)) + \sum_{i=2}^{n} C_i x_i$, *where* f_1 *satisfies* $g_1 g_1'' \neq {g_1'}^2$ *and* $C_i \in \mathbb{R} - \{0\}$, $i \in \{1, \dots, n\}$*;*
(c) $f(x_1, \dots, x_n) = A\sqrt{x_1 \cdot \dots \cdot x_n}$, *where A is a positive constant.*

Recently, the following classification results were proved in [2] for quasi-product production models whose production hypersurfaces have null Gauss–Kronecker curvature, with constant elasticity of production with respect to any factor of production, with proportional marginal rate of substitution, and with constant elasticity of substitution property.

Theorem 6.4. *Let f be a quasi-product production function given by (17), where the functions $F, g_1, \dots, g_n$ are twice differentiable. Then:*

i. *The elasticity of production is a constant k_i with respect to a certain factor of production x_i if and only if f reduces to*

$$f(x_1, \dots, x_n) = A \cdot x_i^{k_i} \cdot \prod_{j \neq i} g_j^k(x_j), \tag{18}$$

where A is a positive constant and k is a nonzero real constant.

ii. *The elasticity of production is constant with respect to all factors of production if and only if f reduces to the generalized Cobb–Douglas production function given by (2).*
iii. *The production function satisfies the proportional marginal rate of substitution property if and only if it reduces to the homothetic generalized Cobb–Douglas production function given by*

$$f(x_1, \dots, x_n) = F\left(A \cdot \prod_{i=1}^{n} x_i^k\right), \tag{19}$$

where A is a positive constant and k is a nonzero real constant.

iv. *If the production function satisfies the proportional marginal rate of substitution property, then:*
  iv_1. *The production hypersurface cannot be minimal.*
  iv_2. *The production hypersurface has vanishing sectional curvature if and only if, up to a suitable translation, f reduces to the following generalized Cobb–Douglas*

production function:

$$f(x_1, \dots, x_n) = A \cdot \sqrt{x_1 \dots x_n}, \tag{20}$$

where A is a positive constant.

v. *The production hypersurface has vanishing Gauss–Kronecker curvature if and only if, up to a suitable translation, f reduces to one of the following forms:*
   (a) *a generalized Cobb–Douglas production function with constant return to scale;*
   (b) $f(x_1, \dots, x_n) = A \cdot \ln \left[\exp(A_1 x_1) \cdot \prod_{j=2}^{n} g_j(x_j) \right]$, *where A, A_1 are nonzero real constants;*
   (c) $f(x_1, \dots, x_n) = F \left(A \cdot \exp(A_1 x_1 + A_2 x_2) \cdot \prod_{j=3}^{n} g_j(x_j) \right)$, *where A is a positive constant and A_1, A_2 are nonzero real constants;*
   (d) *an Armington aggregator with constant return to scale, given by*

$$f(x_1, \dots, x_n) = \left(\sum_{i=1}^{n} C_i x_i^{\frac{A}{A-1}} \right)^{\frac{A-1}{A}},$$

   where A is a nonzero real constant, $A \neq 1$, and $C_1, \dots, C_n$ are nonzero real constants;
   (e) $f(x_1, \dots, x_n) = A \cdot \ln \left(\sum_{i=1}^{n} B_i \exp(A_i x_i) \right)$, *where A, A_i, B_i are nonzero real constants for $i = 1, \dots, n$.*

vi. *The production function satisfies the constant elasticity of substitution property if and only if, up to a suitable translation, f reduces to one of the following forms:*
   (a) *a homothetic generalized Cobb–Douglas production function;*
   (b) $f(x_1, \dots, x_n) = F \left(A \cdot \prod_{i=1}^{n} \exp \left(A_i x_i^{\frac{\sigma-1}{\sigma}} \right) \right)$, *where A is a positive constant and $A_1, \dots, A_n, \sigma$ are nonzero real constants, $\sigma \neq 1$;*
   (c) *a two-input production function given by*

$$f(x_1, x_2) = F \left(A \cdot \left(\frac{x_1^{\frac{\sigma-1}{\sigma}} + A_1}{x_2^{\frac{\sigma-1}{\sigma}} + A_2} \right)^{\frac{\sigma}{k}} \right),$$

   where A, A_1, A_2, k, σ are nonzero real constants, $\sigma \neq 1$;
   (d) *a two-input production function given by*

$$f(x_1, x_2) = F \left(A \cdot \left(\frac{\ln(A_1 x_1)}{\ln(A_2 x_2)} \right)^{\frac{1}{k}} \right),$$

   where A, k are nonzero real constants and A_1, A_2 are positive constants.

For further results on the geometry of the quasi-product and homothetic production functions see [5–7]. Many other results concerning production models in economics from the viewpoint of isotropic geometry can be found in [8,23,25].

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