ARAB JOURNAL OF
MATHEMATICAL SCIENCES

Arab J Math Sci **19**(2) (2013), 223–241

A stochastic maximum principle in mean-field optimal control problems for jump diffusions

FARID CHIGHOUB, BRAHIM MEZERDI *

Laboratory of Applied Mathematics, University Mohamed Khider, P.O. Box 145, Biskra (07000), Algeria

Received 1 April 2012; revised 22 January 2013; accepted 2 February 2013
Available online 13 February 2013

Abstract. This paper is concerned with the study of a stochastic control problem, where the controlled system is described by a stochastic differential equation (SDE) driven by a Poisson random measure and an independent Brownian motion. The cost functional involves the mean of certain nonlinear functions of the state variable. The inclusion of this mean terms in the running and the final cost functions introduces a major difficulty when applying the dynamic programming principle. A key idea of solving the problem is to use the stochastic maximum principle method (SMP). In the first part of the paper, we focus on necessary optimality conditions while the control set is assumed to be convex. Then we prove that these conditions are in fact sufficient provided some convexity conditions are fulfilled. In the second part, the results are applied to solve the mean-variance portfolio selection problem in a jump setting.

MSC 2000 subject classifications: 93E20; 60H10; 60H30

Keywords: Stochastic systems with jumps; Mean-field control problem; Stochastic maximum principle; Optimal control

1. INTRODUCTION

In this paper, we discuss stochastic control models which are driven by a stochastic differential equation with jumps, taking the following form

$$\begin{cases} dx_t = b(t, x_t, u_t)dt + \sigma(t, x_t, u_t)dB_t + \int_Z \gamma(t, x_{t-}, u_t, z)\widetilde{N}(dt, dz), \\ x_0 = x. \end{cases} \tag{1.1}$$

* Corresponding author. Tel./fax: +213 33 74 77 88.
E-mail addresses: chighoub_farid@yahoo.fr (F. Chighoub), bmezerdi@yahoo.fr (B. Mezerdi).

Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.ajmsc.2013.02.001

The objective is to minimize the following expected cost functional,

$$J(u)=\mathbb{E}\left[\int_0^T f(t,x_t,\mathbb{E}[\varphi(x_t)],u_t)dt+g(x_T,\mathbb{E}[\eta(x_T)])\right],\quad \text{for } u\in\mathcal{U}, \tag{1.2}$$

over the set of the admissible controls. The models where the coefficients f and g depend on the marginal probability law of the solution are not simple extensions from the mathematical point of view, but also provide interesting models in applications [1,16]. The typical example is the continuous-time Markowitz's mean-variance portfolio selection model, where one should minimize an objective function involving a quadratic function of the expectation, due to the variance term, see [3,15,2,22]. The main difficulty when facing a general mean-field controlled diffusion is that, the setting is non-Markovian, and hence, the dynamic programming and HJB techniques based on the law of iterated expectations on J do not hold in general. The stochastic maximum principle provides a powerful tool for handling this problem, see [2,17].

It is well known that the maximum principle for a stochastic optimal control problem provides necessary conditions of optimality obtained by duality theory, involving the so-called adjoint process, which solves a linear backward stochastic differential equation (BSDE in short). Some results that cover the controlled jump diffusion processes are discussed in [20,3,14,21,5,18]. Necessary and sufficient conditions of optimality for partial information control problems have been obtained in [3]. In [14] the sufficient maximum principle and the link with the dynamic programming principle are given. The second order stochastic maximum principle for optimal controls of nonlinear dynamics with jumps and convex state constraints was developed via spike variation method by Tang and Li [21]. These conditions are described in terms of two adjoint processes, which are linear backward SDEs [19]. Such equations have important applications in hedging problems, see [13]. Existence and uniqueness of solutions of BSDEs with jumps and nonlinear coefficients have been treated by Tang and Li [21], Barles et al. [4]. The link with integral-partial differential equations is studied in [4]. See Bouchard and Elie [7] for discrete time approximation of decoupled FBSDE with jumps. The notion of mean-field BSDE appears in [8,9]. Equations of this type are essentially generalizations of BSDEs, which allow the generator term to be an expectation of certain nonlinear function. In [8], a general notion of mean-field BSDE associated with a mean-field SDE is obtained in a natural way as a limit of some high dimensional system of FBSDEs governed by a d-dimensional Brownian motion, and influenced by positions of a large number of other particles. The study of mean-field BSDEs in a Markovian framework, associated with a mean-field SDE is given in [9]. By combining classical BSDE theory with specific arguments from mean-field BSDEs, it was shown that this mean-field BSDE describes the viscosity solution of a nonlocal PDE.

Control problems for jump diffusions have been treated in [10], to derive near-optimality conditions rather than exact optimality conditions. The control domain was supposed to be non convex, and the coefficients in the cost functional (1.2) do not depend explicitly on the mean-field terms. By using the spike variation method and Ekeland's variational principle, these conditions are obtained in terms of two adjoint processes.

Our main concern in the present paper is to derive the stochastic maximum principle for a stochastic control problem in which the mean-field terms appear in the cost functional but not in the dynamics of the state process. The inclusion of these terms introduces a major difference with respect to the case without mean-field terms, see [10]. Furthermore, since the set of controls is convex, the key fact to establish necessary optimality conditions for this kind of problems, is to use the classical way of the convex perturbation method instead of the spike variation technique, see [6,12]. Note that, the adjoint process that appears in our SMP is defined as the solution to a certain mean-field type backward stochastic differential equation.

Such problems have been studied by many authors, see e.g. [17], where the SMP is constructed via results from Malliavin calculus. Another relevant work is that of [2], where, the SMP is proved for mean-field stochastic control problems, where both the state dynamics and the cost functional are of mean-field type. This SMP is obtained in the non-jump case as an extension of the Bensoussan approach [6].

The plan of the paper is as follows. In the second section, we formulate the problem and give the necessary notations and preliminaries. Section 3 is devoted to the presentation to the necessary as well as sufficient conditions for optimality, which are our main results. In the last section, we apply the results of Section 3 to solve the mean-variance portfolio selection problem.

2. Assumptions and problem formulation

In all what follows, we will work on the classical probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\leqslant T}, \mathbb{P})$, such that $\mathcal{F}_0$ contains the $\mathbb{P}$-null sets, $\mathcal{F}_T = \mathcal{F}$ for an arbitrarily fixed time horizon T, and $(\mathcal{F}_t)_{t\leqslant T}$ satisfies the usual conditions. We assume that $(\mathcal{F}_t)_{t\leqslant T}$ is generated by a d-dimensional standard Brownian motion B and an independent Poisson measure N on $[0, T] \times Z$ where $Z \subset \mathbb{R}^m \setminus \{0\}$ for some $m \geqslant 1$. We denote by $\left(\mathcal{F}_t^B\right)_{t\leqslant T}$ (resp. $\left(\mathcal{F}_t^N\right)_{t\leqslant T}$) the $\mathbb{P}$-augmentation of the natural filtration of B (resp. N). Obviously, we have

$$\mathcal{F}_t = \sigma\left[\int\int_{A\times(0,s]} N(dz, dr); s \leqslant t, A \in \mathcal{B}(Z)\right] \vee \sigma[B_s; s \leqslant t] \vee \mathcal{N}, \tag{2.1}$$

where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets, and $\sigma_1 \vee \sigma_2$ denotes the σ-field generated by $\sigma_1 \cup \sigma_2$. We assume that the compensator of N has the form $\mu(dt,dz) = v(dz)dt$ for some positive and σ-finite Lévy measure v on Z, endowed with its Borel σ-field $\mathcal{B}(Z)$. We suppose that $\int_Z 1 \wedge |z|^2 v(dz) < \infty$ and write $\widetilde{N} = N - vdt$ for the compensated jump martingale random measure of N.

Notation. Let $n \in \mathbb{N}^*$, any element $x \in \mathbb{R}^n$ will be identified to a column vector with i-th component, and the norm $|x| = |x^1| + \ldots + |x^n|$. The scalar product of any two vectors x and y on $\mathbb{R}^n$ is denoted by xy. An $n \times n$ matrix will be considered as an element $M \in \mathbb{R}^{n\times n}$, we note xM for $M^{\mathrm{T}} x \in \mathbb{R}^n$, where M^{T} is the transpose of the matrix M. For a function h, we denote by h_x (resp. h_{xx}) the gradient or Jacobian (resp. the Hessian) of h with respect to the variable x.

We need to define some additional notations. Given $s < t$, let us introduce the following spaces

- $\mathcal{S}^2([s, t], \mathbb{R}^n)$ the set of $\mathbb{R}^n$-valued adapted cadlag processes P such that

$$\|P\|_{\mathcal{S}^2([s,t],\mathbb{R}^n)} := \mathbb{E}\left[\sup_{r\in[s,t]}|P_r|^2\right]^{\frac{1}{2}} < \infty.$$

- $\mathcal{M}^2([s,t],\mathbb{R}^n)$ is the set of progressively measurable $\mathbb{R}^n$-valued processes Q such that

$$\|Q\|_{\mathcal{M}^2([s,t],\mathbb{R}^n)} := \mathbb{E}\left[\int_s^t |Q_r|^2 dr\right]^{\frac{1}{2}} < \infty.$$

- $\mathcal{L}^2([s,t],\mathbb{R}^n)$ is the set of $\mathcal{B}([0,T]\times\Omega)\otimes\mathcal{B}(Z)$ mesurable maps $R : [0,T]\times\Omega\times Z \to \mathbb{R}^n$ such that

$$\|R\|_{\mathcal{L}^2([s,t],\mathbb{R}^n)} := \mathbb{E}\left[\int_s^t \int_Z |R_r(z)|^2 \nu(de)dr\right]^{\frac{1}{2}} < \infty.$$

We shall omit $([s,t],\mathbb{R}^n)$ in these notations when $(s,t) = (0,T)$.

Definition 2.1. Let A be a nonempty convex closed subset of $\mathbb{R}^m$. An admissible control is a A-valued measurable $\mathcal{F}_t$-adapted process u, such that $\|u\|_{\mathcal{S}^2} < \infty$. We denote by $\mathcal{U}$ the set of all admissible controls.

In order to state the considered stochastic control problem, we first assume that, for $u \in \mathcal{U}$, the state of the controlled jump diffusion in $\mathbb{R}^n$ is described by the following stochastic differential equation:

$$\begin{cases} dx_t = b(t,x_t,u_t)dt + \sigma(t,x_t,u_t)dB_t + \int_Z \gamma(t,x_{t-},u_t,z)\widetilde{N}(dt,dz), & \text{for } t\in[0,T], \\ x_0 = x. \end{cases} \tag{2.2}$$

where $x \in \mathbb{R}^n$ represents the initial state of the system. Here $b : [0,T]\times\mathbb{R}^n\times\mathcal{U} \to \mathbb{R}^n, \sigma : [0,T]\times\mathbb{R}^n\times\mathcal{U} \to \mathbb{R}^{n\times d}$, and $\gamma : [0,T]\times\mathbb{R}^n\times\mathcal{U}\times Z \to \mathbb{R}^n$, are given. Suppose that the cost functional has the form

$$J(u) = \mathbb{E}\left[\int_0^T f(t,x_t,\mathbb{E}[\varphi(x_t)],u_t)dt + g(x_T,\mathbb{E}[\eta(x_T)])\right], \quad \text{for } u \in \mathcal{U}, \tag{2.3}$$

where $\mathbb{E}$ denotes the expectation with respect to $\mathbb{P}$, and $\varphi : \mathbb{R}^n \to \mathbb{R}, \eta : \mathbb{R}^n \to \mathbb{R}, f : [0,T]\times\mathbb{R}^{n+1}\times\mathcal{U} \to \mathbb{R}, g : \mathbb{R}^{n+1} \to \mathbb{R}$, are measurable functions. We deal with the following stochastic control problem. Find $u^\star$ such that

$$J(u^\star) = \inf_{u\in\mathcal{U}} J(u). \tag{2.4}$$

Any admissible control $u^\star$ that achieves the minimum is called an optimal control.

The basic assumptions made on the coefficients b, σ, γ, φ, f, g, and η are the following

(H1) The maps b, σ, γ, and f are continuously differentiable with respect to (x,u), and they are bounded by $K(1+|x|)$. The derivatives b_x, b_u, σ_x, σ_u, γ_x and γ_u are continuous in (x,u) and uniformly bounded.

(H2) f and g are continuously differentiable with respect to (x, y, u), and they are bounded by $K(1 + |x|)$. φ and η are continuously differentiable with respect to x. The derivatives $f_x, f_u, g_x, g_u, \varphi_x$, and η_x are continuous and uniformly bounded.

(H3) For all $(t, u, z) \in [0, T] \times A \times Z$, the map

$$(x, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n \to a(t, u, x, \zeta; z) := \zeta^{\mathrm{T}}(\gamma_x(t, x, u, z) + I)\zeta,$$

satisfies uniformly in $(x, \zeta) \in \mathbb{R}^n \times \mathbb{R}^n$,

$$a(t, u, x, \zeta; z) \geqslant |\zeta|^2 K_2^{-1}, \quad \text{or } a(t, u, x, \zeta; z) \leqslant -|\zeta|^2 K_2^{-1}, \quad \text{for some } K_2 > 0.$$

Under the above hypothesis, the SDE (2.2) has a unique solution, and the functional J is well defined from $\mathcal{U}$ into $\mathbb{R}$.

3. The stochastic maximum principle

The purpose of the stochastic maximum principle is to find necessary optimality conditions satisfied by an optimal control, assuming that the solution exists. In [6] the SMP is constructed for control problems, where the state is described by a stochastic differential equation, driven by a Brownian motion only. The author constructs his SMP by considering a convex perturbation of the optimal control defined by $u_t^{\varepsilon} = u_t^{\star} + \varepsilon\left(v_t - u_t^{\star}\right)$, for all $v \in \mathcal{U}$. This perturbation plays a key role in deriving the first order adjoint process and the variational inequality which reduces to the computation of a Gâteaux derivative.

The sufficient condition of optimality is of significant importance in the stochastic maximum principle for computing optimal controls. It says that if an admissible control satisfies the maximum condition on the Hamiltonian, then the control is indeed optimal for the stochastic control problem. This allows one to solve examples of control problems where one can find, a smooth solution to the associated adjoint equation.

3.1. Necessary conditions of optimality

Our approach extends Bensoussan's ideas to the framework of mean-field stochastic control problems in a jump-diffusion setting. Since the control domain is convex, the classical way to derive necessary conditions of optimality is to use convex perturbations method. More precisely, we consider a family of perturbed controls u^{ε} of the optimal control $u^{\star}$, which are defined by $u_t^{\varepsilon} = u_t^{\star} + \varepsilon v$, for $v \in \mathcal{U}, \varepsilon \in (0, 1)$. Then we compute the Gateaux derivative of the cost functional in terms of the Taylor expansion of the state process. We will deduce a maximum principle for the control problem in terms of some adjoint process. This adjoint process is solution of some linear BSDE in which the generator term and the terminal value depend on the marginal probability law of the optimal state process.

Define the usual Hamiltonian associated to the control problem, for $(t, x, u, p, q, r) \in [s, T] \times \mathbb{R}^n \times \mathcal{U} \times \mathbb{R}^n \times \mathbb{R}^{n \times n} \times \mathbb{R}^n$

$$H(t,x,\mathbb{E}[\varphi(x)],u,p,q,r) = f(t,x,\mathbb{E}[\varphi(x)],u) + pb(t,x,u) + \sum_{j=1}^{n} q^j\sigma^j(t,x,u) + \int_Z r(z)\gamma(t,x,u,z)v(dz). \tag{3.1}$$

where q^j and σ^j for $j = 1,\ldots,n$, denote the *jth* column of the matrix q and σ, respectively. The stochastic maximum principle involves an admissible pair $(u^\star, x^\star)$ and a triple $(p,q,r(.))$ of square integrable adapted processes associated to $(u^\star, x^\star)$, with values in $\mathbb{R}^n \times \mathbb{R}^{n\times n} \times \mathbb{R}^n$, such that

$$\begin{cases} dp_t = -\Big\{p_t b_x\big(t,x_t^\star,u_t^\star\big) + \sum_{j=1}^{n} q_t^j\sigma_x^j\big(t,x_t^\star,u_t^\star\big) + \int_Z r_t(z)\gamma_x\big(t,x_t^\star,u_t^\star,z\big)v(dz) \\ \qquad + f_x\big(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],u_t^\star\big) + \mathbb{E}\big[f_y\big(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],u_t^\star\big)\big]\varphi_x\big(x_t^\star\big)\Big\}dt + q_t dB_t \\ \qquad + \int_Z r_t(z)\widetilde{N}(dt,dz), \\ p_T = g_x\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big) + \mathbb{E}\big[g_y\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big)\big]\eta_x\big(x_T^\star\big), \end{cases} \tag{3.2}$$

where f_x and f_y denote the derivatives of f with respect to its second and third argument. Similarly, g_x and g_y denote the derivatives of g with respect to the first and the second variables.

Obviously the above BSDE admits a unique solution $(p,q,r(.)) \in \mathcal{S}^2 \times \mathcal{M}^2 \times \mathcal{L}^2$ under assumptions (**H1**) and (**H2**). Note that the existence and uniqueness for solutions to BSDE with jumps have been treated by Tang and Li [21].

Remark 3.1. BSDE (3.2) reduces to the standard one, when the coefficients do not depend explicitly on the marginal law of the underlying diffusion, see [21].

We can state the stochastic maximum principle, which is the first main result in this paper.

Theorem 3.2 (*Necessary conditions of optimality*). *Let $u^\star$ be an optimal control of the problem* (2.3) *subject to the controlled system* (2.2), *then there exists a unique adapted process $(p,q,r(.)) \in \mathcal{S}^2 \times \mathcal{M}^2 \times \mathcal{L}^2$ which is the unique solution to the BSDE* (3.2), *such that*

$$H_u(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],u_t^\star,p_t,q_t,r_t(z))\big(a-u_t^\star\big) \geqslant 0, \quad dt-a.e.,\mathbb{P}-a.s. \quad \text{for all } a\in A, \tag{3.3}$$

where H is the Hamiltonian (3.1) *orresponding to the optimal trajectory $x^\star$.*

To prove the main result, we need some preliminary results given in the following.

Let $u^\star$ be an optimal control and $x^\star$ denote the corresponding optimal trajectory. Let $v \in \mathcal{U}$ be such that $u^\star + v \in \mathcal{U}$. The convexity condition on the control domain ensures that, for $\varepsilon \in (0,1)$ the control $u^\star + \varepsilon v$ is also in $\mathcal{U}$.

We define the process $y_t = y_t^{u^\star,v}$ by $y_t = \frac{d}{d\varepsilon} x_t^{u^\star+\varepsilon v}\big|_{\varepsilon=0}$. From the differentiability result on solutions of stochastic differential equations which depend on a parameter, we deduce that

$$\begin{cases} dy_t = \{b_x(t,x_t^\star,u_t^\star)y_t + b_u(t,x_t^\star,u_t^\star)v_t\}dt + \sum_{j=1}^n \{\sigma_x^j(t,x_t^\star,u_t^\star)y_t + \sigma_u^j(t,x_t^\star,u_t^\star)v_t\}dB_t^j \\ \quad + \int_Z \{\gamma_x(t,x_{t-}^\star,u_t^\star,z)y_{t-} + \gamma_u(t,x_{t-}^\star,u_t^\star,z)v_t\}\widetilde{N}(dt,dz), \\ y_0 = 0. \end{cases} \tag{3.4}$$

It is clear that the variational Eq. (3.4) has a unique solution.

Let Φ be the fundamental solution of the linear matrix equation, for $0 \leqslant s < t \leqslant T$

$$\begin{cases} d\Phi_{s,t} = b_x(t,x_t^\star,u_t^\star)\Phi_{s,t}dt + \sum_{j=1}^n \sigma_x^j(t,x_t^\star,u_t^\star)\Phi_{s,t}dB_t^j + \int_Z \gamma_x(t,x_{t-}^\star,u_t^\star,z)\Phi_{s,t-}\widetilde{N}(dt,dz), \\ \Phi_{s,s} = I. \end{cases} \tag{3.5}$$

where I is the $n \times n$ identity matrix. This equation is linear with bounded coefficients, then it admits a unique strong solution. Moreover, the condition **(H3)** ensures that the process Φ is invertible, with an inverse satisfying the following equation

$$\begin{cases} d\Psi_{s,t} = -\Psi_{s,t}\left\{ b_x(t,x_t^\star,u_t^\star) - \sum_{j=1}^n \sigma_x^j(t,x_t^\star,u_t^\star)^2 - \int_Z \gamma_x(t,x_t^\star,u_t^\star,z)\nu(dz) \right\}dt \\ \qquad - \sum_{j=1}^n \Psi_{s,t}\sigma_x^j(t,x_t^\star,u_t^\star)dB_t^j \\ \qquad - \Psi_{s,t-}\int_Z \left(\gamma_x(t,x_{t-}^\star,u_t^\star,z) + I\right)^{-1}\gamma_x(t,x_{t-}^\star,u_t^\star,z)N(dt,dz), \\ \Psi_{s,s} = I, \end{cases} \tag{3.6}$$

so $\Psi = \Phi^{-1}$, if $s = 0$ we simply write $\Phi_{0,t} = \Phi t$, and $\Psi_{0,t} = \Psi_t$. By the integration by parts formula, we can see that the solution of (3.4) is given by $y_t = \Phi_t\beta_t$, where β_t is the solution of the stochastic differential equation

$$\begin{cases} d\beta_t = \Psi_t\left\{ b_u(t,x_t^\star,u_t^\star)v_t - \sum_{j=1}^n \sigma_x^j(t,x_t^\star,u_t^\star)\sigma_u^j(t,x_t^\star,u_t^\star)v_t - \int_Z \gamma_u(t,x_t^\star,u_t^\star,z)v_t\nu(dz) \right\}dt \\ \quad + \sum_{j=1}^n \Psi_t\sigma_u^j(t,x_t^\star,u_t^\star)v_t dB_t^j + \Psi_t\int_Z \left(\gamma_x(t,x_t^\star,u_t^\star,e) + I\right)^{-1}\gamma_u(t,x_t^\star,u_t^\star,e)v_t N(dt,dz), \\ \beta_0 = 0. \end{cases} \tag{3.7}$$

From standard estimates in the theory of SDEs, the assumptions **(H1)**-**(H3)** imply the following

$$\|\Phi\|_{\mathcal{S}^2} + \|\Psi\|_{\mathcal{S}^2} \leqslant C, \quad \text{for somme } C > 0. \tag{3.8}$$

Let us introduce the following convex perturbation of the optimal control

$$u^{\star,\varepsilon} = u^{\star} + \varepsilon v, \quad \text{for any } v \in \mathcal{U}, \text{ and } \varepsilon \in (0,1). \tag{3.9}$$

The control $u^{\star} \in \mathcal{U}$ is an optimal control, then

$$J(u^{\star}) \leqslant J(u^{\star} + \varepsilon v) = J(u^{\star}) + \varepsilon \frac{d}{d\varepsilon} J(u^{\star} + \varepsilon v)\Big|_{\varepsilon=0} + \circ(\varepsilon),$$

if the derivative exists. Thus a necessary condition for optimality is that

$$\frac{d}{d\varepsilon} J(u^{\star} + \varepsilon v)\Big|_{\varepsilon=0} \geqslant 0. \tag{3.10}$$

The rest of this subsection is devoted to the computation of the differential of the cost functional at $\varepsilon = 0$. We shall see that this relation leads to a precise characterization of $u^{\star}$, in terms of the adjoint process.

Proof. Differentiating the cost functional (2.3) with respect to ε, we obtain

$$\begin{aligned}
\frac{d}{d\varepsilon} J(u^{\star} + \varepsilon v)\Big|_{\varepsilon=0} &= \mathbb{E}\Big[\int_0^T \Big\{ \big(f_x(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star}) + \mathbb{E}[f_y(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star})]\varphi_x(x_t^{\star})\big) \frac{d}{d\varepsilon} x_t^{u^{\star}+\varepsilon v}\Big|_{\varepsilon=0} \\
&\quad + f_u(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star}) v_t \} dt + g_x(x_T^{\star}, \mathbb{E}[\eta(x_T^{\star})]) \frac{d}{d\varepsilon} x_T^{u^{\star}+\varepsilon v}\Big|_{\varepsilon=0} \\
&\quad + \mathbb{E}[g_y(x_T^{\star}, \mathbb{E}[\eta(x_T^{\star})])]\eta_x(x_T^{\star}) \frac{d}{d\varepsilon} x_T^{u^{\star}+\varepsilon v}\Big|_{\varepsilon=0}\Big], \\
&= \mathbb{E}\Big[\int_0^T \{ f_x(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star}) y_t + \mathbb{E}[f_y(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star})]\varphi_x(x_t^{\star}) y_t \\
&\quad + f_u(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star}) v_t \} dt + g_x(x_T^{\star}, \mathbb{E}[\eta(x_T^{\star})]) y_T + \mathbb{E}[g_y(x_T^{\star}, \mathbb{E}[\eta(x_T^{\star})])]\eta_x(x_T^{\star}) y_T\Big],
\end{aligned}$$

for all $v \in \mathcal{U}$. Substituting by y_t by $\Phi_t \beta_t$, leads to

$$\begin{aligned}
\frac{d}{d\varepsilon} J(u^{\star} + \varepsilon v)|_{\varepsilon=0} &= \mathbb{E}\Big[\int_0^T \{ f_x(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star}) \Phi_t \beta_t + \mathbb{E}[f_y(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star})]\varphi_x(x_t^{\star}) \Phi_t \beta_t \\
&\quad + f_u(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star}) v_t \} dt + g_x(x_T^{\star}, \mathbb{E}[\eta(x_T^{\star})]) \Phi_T \beta_T \\
&\quad + \mathbb{E}[g_y(x_T^{\star}, \mathbb{E}[\eta(x_T^{\star})])]\eta_x(x_T^{\star}) \Phi_T \beta_T\Big].
\end{aligned} \tag{3.11}$$

Consider the right continuous version of the square integrable martingale

$$\begin{aligned}
M_t := \mathbb{E}\Big[\int_0^T \{ f_x(s, x_s^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_s^{\star}) + \mathbb{E}[f_y(t, x_t^{\star}, \mathbb{E}[\varphi(x_t^{\star})], u_t^{\star})]\varphi_x(x_t^{\star}) \} \Phi_s ds \\
+ \{ g_x(x_T^{\star}, \mathbb{E}[\eta(x_T^{\star})]) + \mathbb{E}[g_y(x_T^{\star}, \mathbb{E}[\eta(x_T^{\star})])]\eta_x(x_T^{\star}) \} \Phi_T | \mathcal{F}_t\Big].
\end{aligned}$$

By the martingale representation theorem, there exist two processes $Q = (Q^1, \ldots, Q^n)$ where $Q^j \in \mathcal{M}^2$, for $j = 1, \ldots, n$, and $U \in \mathcal{L}^2$, satisfing

$$M_t = \mathbb{E}\Big[\int_0^T \{f_x(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star) + \mathbb{E}[f_y(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star)]\varphi_x(x_t^\star)\}\Phi_t dt$$
$$+ \{g_x(x_T^\star, \mathbb{E}[\eta(x_T^\star)]) + \mathbb{E}[g_y(x_T^\star, \mathbb{E}[\eta(x_T^\star)])]\eta_x(x_T^\star)\}\Phi_T\Big]$$
$$+ \sum_{j=1}^n \int_0^t Q_s^j dB_s^j + \int_0^t \int_Z U(s,e)\widetilde{N}(ds,de).$$

Let us denote

$$\xi_t^\star = M_t - \int_0^t \{f_x(s, x_s^\star, \mathbb{E}[\varphi(x_s^\star)], u_s^\star) + \mathbb{E}[f_y(s, x_s^\star, \mathbb{E}[\varphi(x_s^\star)], u_s^\star)]\varphi_x(x_s^\star)\}\Phi_s ds. \tag{3.12}$$

The adjoint variable is the process defined by

$$\begin{cases} p_t = \xi_t^\star \Psi_t, \\ q_t^j = Q_t^j\Psi_t - p_t\sigma_x^j(t, x_t^\star, u_t^\star), \quad \text{for } j = 1, \ldots, n. \\ r_t(z) = U(t,z)\Psi_t(\gamma_x(t, x_t^\star, u_t^\star, z) + I)^{-1} + p_t\Big((\gamma_x(s, x_t^\star, u_t^\star, z) + I)^{-1} - I\Big). \end{cases} \tag{3.13}$$

From the integration by parts formula applied to $\xi_t^\star\beta_t$, and the above definition of p_t, q_t^j for $j = 1,\ldots,n$, and $r_t(z)$, we have

$$\mathbb{E}[\xi_T^\star\beta_T] = \mathbb{E}\Bigg[\int_0^T \Bigg\{p_t b_u(t, x_t^\star, u_t^\star) + \sum_{j=1}^n q_s^j\sigma_u^j(t, x_t^\star, u_t^\star)$$
$$+ \int_Z r_t(z)\gamma_u(t, x_t^\star, u_t^\star, z)\nu(dz)\Bigg\}v_t dt - \int_0^T \{f_x(s, x_s^\star, \mathbb{E}[\varphi(x_t^\star)], u_s^\star)$$
$$+ \mathbb{E}[f_y(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star)]\varphi_x(x_t^\star)\}\Phi_t\beta_t dt\Bigg]. \tag{3.14}$$

Also, we easily verify that the following representation holds

$$\frac{d}{d\varepsilon}J(u^\star + \varepsilon v)\Big|_{\varepsilon=0} = \mathbb{E}\Big[\int_0^T \{f_x(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star) + \mathbb{E}[f_y(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star)]\varphi_x(x_t^\star)\}\Phi_t\beta_t dt$$
$$+ \int_0^T f_u(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star)v_t dt + \xi_T^\star\beta_T\Big]. \tag{3.15}$$

By using (3.14) and (3.15), we obtain

$$\mathbb{E}\Bigg[\int_0^T \Bigg\{f_u(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star) + p_t b_u(t, x_t^\star, u_t^\star) + \sum_{j=1}^n q_t^j\sigma_u^j(t, x_t^\star, u_t^\star) + \int_Z r_t(z)\gamma_u(t, x_t^\star, u_t^\star, z)\nu(dz)\Bigg\}v_t dt\Bigg] \geqslant 0.$$

The previous inequality combined with (3.1) implies

$$\frac{d}{d\varepsilon}J(u^\star + \varepsilon v)\Big|_{\varepsilon=0} = \mathbb{E}\left[\int_0^T H_u\big(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star, p_t, q_t, r_t(z)\big) v_t dt\right] \geqslant 0. \quad (3.16)$$

Since A is a convex set, we may choose

$$u_t^\varepsilon = u_t^\star + \varepsilon\big(v_t - u_t^\star\big) \in \mathcal{U}, \text{ for } \varepsilon \in [0,1].$$

Thus, since $u_t^\star$ is optimal, we have

$$\frac{d}{d\varepsilon}J(u^\star + \varepsilon(v - u^\star))\Big|_{\varepsilon=0} = \mathbb{E}\left[\int_0^T H_u\big(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star, p_t, q_t, r_t(z)\big)\big(v_t - u_t^\star\big) dt\right] \geqslant 0.$$

Now let

$$v_t = \begin{cases} a & \text{on } B \times (t_0, t_0 + h), \\ u_t^\star & \text{otherwise,} \end{cases}$$

where B is a $\mathcal{F}_{t_0}$-measurable set. Then

$$\mathbb{E}\left[\int_0^T H_u\big(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star, p_t, q_t, r_t(z)\big)\big(v_t - u_t^\star\big) dt\right] = \mathbb{E}\left[\int_{t_0}^{t_0+h} H_u\big(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star, p_t, q_t, r_t(z)\big)\big(v_t - u_t^\star\big) 1_B dt\right].$$

Dividing by h and sending h to 0, we get

$$\lim_{h\to 0}\mathbb{E}\left[\int_{t_0}^{t_0+h} H_u\big(t, x_t^\star, \mathbb{E}[\varphi(x_t^\star)], u_t^\star, p_t, q_t, r_t(z)\big)\big(v_t - u_t^\star\big) 1_B dt\right] = \mathbb{E}\left[H_u\left(t_0, x_{t_0}^\star, \mathbb{E}\left[\varphi\left(x_{t_0}^\star\right)\right], u_{t_0}^\star, p_{t_0}, q_{t_0}, r_{t_0}(z)\right)\left(a - u_{t_0}^\star\right) 1_B\right],$$

for all t_0 outside a Lebesgue negligible set.

The last equality is valid for all $B \in \mathcal{F}_{t_0}$, which implies that

$$\mathbb{E}\left[H_u(t_0, x_{t_0}^\star, \mathbb{E}\left[\varphi\left(x_{t_0}^\star\right)\right], u_{t_0}^\star, p_{t_0}, q_{t_0}, r_{t_0}(z))\left(a - u_{t_0}^\star\right) / \mathcal{F}_{t_0}\right] \geqslant 0.$$

Now since the quantity inside the conditional expectation is $\mathcal{F}_{t_0}$-measurable, then inequality (3.3) holds $dt - a.e$, $\mathbb{P} - a.s.$, for all $a \in A$. □

3.2. Sufficient conditions of optimality

In this section, we focus on proving that provided some convexity assumptions are satisfied, the necessary condition (3.3) turns out to be sufficient.

Theorem 3.3. [Sufficient conditions of optimality]*Given an admissible control* $u^\star$, *we denote by* $x^\star$ *the associated controlled state process, and* $(p,q,r(.))$ *be the solution to the corresponding BSDE* (3.2). *We assume that the following conditions hold.*

(i) $(x,y,u) \to H(t,.,.,.,p_t,q_t,r_t(z))$ *is convex.*
(ii) $(x,y) \to g(.,.), x \to \varphi(.)$ *and* $x \to \eta(.)$ *are convex functions.*
(iii) *The derivatives* f_y *and* g_y *are non-negative.*
iv) $dt - a.e., \mathbb{P} - a.s.$

$$H\big(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],u_t^\star,p_t,q_t,r_t(z)\big) = \inf_{a\in A} H\big(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],a,p_t,q_t,r_t(z)\big), \tag{3.17}$$

Then $u^\star$ *is an optimal control.*

Remark 3.4. Under assumption (i) in Theorem 3.3., conditions (3.3) and (3.17) are equivalent (see [11] Proposition 2.2.1 pp. 36–37 or [2] Remark 4.1).

Proof of Theorem 3.3. Let u be an arbitrary admissible control and consider the difference

$$\begin{aligned} J(u^\star) - J(u) = \mathbb{E}\bigg[\int_0^T \big\{f\big(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],u_t^\star\big) - f(t,x_t,\mathbb{E}[\varphi(x_t)],u_t)\big\}dt\bigg] \\ + \mathbb{E}\big[g\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big) - g(x_T,\mathbb{E}[\eta(x_T)])\big]. \end{aligned} \tag{3.18}$$

We first note that, by convexity of g and η, it yields

$$\begin{aligned} &\mathbb{E}\big[g\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big) - g(x_T,\mathbb{E}[\eta(x_T)])\big] \\ &\quad \leqslant \mathbb{E}\big[\big(x_T^\star - x_T\big)g_x\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big) + g_y\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big)\mathbb{E}\big[\eta\big(x_T^\star\big) - \eta(x_T)\big]\big], \\ &\quad \leqslant \mathbb{E}\big[\big(x_T^\star - x_T\big)\big\{g_x\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big) + \mathbb{E}\big[g_y\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big)\big]\eta_x\big(x_T^\star\big)\big\}\big], \end{aligned} \tag{3.19}$$

Recall that the adjoint process $p : [0,T] \times \Omega \to \mathbb{R}^n$ solution to the BSDE (3.2) has by construction the terminal value

$$p_T = g_x\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big) + \mathbb{E}\big[g_y\big(x_T^\star,\mathbb{E}\big[\eta\big(x_T^\star\big)\big]\big)\big]\eta_x\big(x_T^\star\big). \tag{3.20}$$

By convexity of the Hamiltonian and the map φ, we have

$$\begin{aligned} &\mathbb{E}\big[H\big(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],u_t^\star,p_t,q_t,r_t(z)\big) - H(t,x_t,\mathbb{E}[\varphi(x_t)],u_t,p_t,q_t,r_t(z))\big] \\ &\quad \leqslant \mathbb{E}\big[\big(x_t^\star - x_t\big)H_x\big(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],u_t^\star,p_t,q_t,r_t(z)\big)\big] \\ &\qquad + \mathbb{E}\big[\big(x_t^\star - x_t\big)\mathbb{E}\big[f_y\big(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],u_t^\star\big)\big]\varphi_x\big(x_t^\star\big)\big] \\ &\qquad + \mathbb{E}\big[\big(u_t^\star - u_t\big)H_u\big(t,x_t^\star,\mathbb{E}\big[\varphi\big(x_t^\star\big)\big],u_t^\star,p_t,q_t,r_t(z)\big)\big]. \end{aligned} \tag{3.21}$$

From Ito's lemma, applied to $\left(x_T^\star - x_T\right)p_T$, it follows that

$$\begin{aligned}
\mathbb{E}\left[\left(x_T^\star - x_T\right)p_T\right] = &-\mathbb{E}\left[\int_0^T \left(x_t^\star - x_t\right)H_x\left(t,x_t^\star,\mathbb{E}\left[\varphi\left(x_t^\star\right)\right],u_t^\star,p_t,q_t,r_t(z)\right)dt\right] \\
&-\mathbb{E}\left[\int_0^T \left(x_t^\star - x_t\right)\mathbb{E}\left[f_y\left(t,x_t^\star,\mathbb{E}\left[\varphi\left(x_t^\star\right)\right],u_t^\star\right)\right]\varphi_x\left(x_t^\star\right)dt\right] \\
&+\mathbb{E}\left[\int_0^T \left\{p_t\left(b\left(t,x_t^\star,u_t^\star\right) - b(t,x_t,u_t)\right) + \sum\nolimits_{j=1}^n \left(\sigma^j\left(t,x_t^\star,u_t^\star\right) - \sigma^j(t,x_t,u_t)\right)q_t^j\right\}dt\right] \\
&+\mathbb{E}\left[\int_0^T \int_Z \left(\gamma\left(t,x_{t-}^\star,u_t^\star,z\right) - \gamma(t,x_{t-},u_t,z)\right)r_{t-}(z)N(dt,dz)\right] \\
&+\mathbb{E}\left[\int_0^T \left\{\left(x_t^\star - x_t\right)q_t + p_t\left(\sigma\left(t,x_t^\star,u_t^\star\right) - \sigma(t,x_t,u_t)\right)\right\}dB_t\right] \\
&+\mathbb{E}\left[\int_0^T \int_Z \left\{\left(x_{t-}^\star - x_{t-}\right)r_{t-}(z) + p_{t-}\left(\gamma\left(t,x_{t-}^\star,u_t^\star,z\right) - \gamma(t,x_{t-},u_t,z)\right)\right\}\widetilde{N}(dt,dz)\right].
\end{aligned} \tag{3.22}$$

According to Definition 2.1., and by the fact that $(p,q^j,r(.)) \in \mathcal{S}^2 \times \mathcal{M}^2 \times \mathcal{L}^2$ for $j = 1,..,n$, we deduce that, the stochastic integrals with respect to the Brownian motion and the Poisson random measure have zero expectation. By combining (3.19), (3.20), (3.21) and (3.22) the following holds

$$\begin{aligned}
&\mathbb{E}\left[g\left(x_T^\star,\mathbb{E}\left[\eta\left(x_T^\star\right)\right]\right) - g(x_T,\mathbb{E}[\eta(x_T)])\right] \\
&\quad\leqslant -\mathbb{E}\left[\int_0^T \left\{H\left(t,x_t^\star,\mathbb{E}\left[\varphi\left(x_t^\star\right)\right],u_t^\star,p_t,q_t,r_t(z)\right) - H(t,x_t,\mathbb{E}[\varphi(x_t)],u_t,p_t,q_t,r_t(z))\right\}dt\right] \\
&\qquad + \mathbb{E}\left[\int_0^T H_u\left(t,x_t^\star,\mathbb{E}\left[\varphi\left(x_t^\star\right)\right],u_t^\star,p_t,q_t,r_t(z)\right)\left(u_t^\star - u_t\right)dt\right] \\
&\qquad + \mathbb{E}\left[\int_0^T p_t\left(b\left(t,x_t^\star,u_t^\star\right) - b(t,x_t,u_t)\right)dt\right] + \mathbb{E}\left[\int_0^T \sum_{j=1}^n \left(\sigma^j\left(t,x_t^\star,u_t^\star\right) - \sigma^j(t,x_t,u_t)\right)q_t^j dt\right] \\
&\qquad + \mathbb{E}\left[\int_0^T \int_Z \left(\gamma\left(t,x_t^\star,u_t^\star,z\right) - \gamma(t,x_t,u_t,z)\right)r_t(z)\nu(dz)dt\right].
\end{aligned}$$

On the other hand, by the definition of the Hamiltonian, on has

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left\{f\left(t,x_t^\star,\mathbb{E}\left[\varphi\left(x_t^\star\right)\right],u_t^\star\right) - f(t,x_t,\mathbb{E}[\varphi(x_t)],u_t)\right\}dt\right] \\
&\quad = \mathbb{E}\left[\int_0^T \left\{H\left(t,x_t^\star,\mathbb{E}\left[\varphi\left(x_t^\star\right)\right],u_t^\star,p_t,q_t,r_t(z)\right) - H(t,x_t,\mathbb{E}[\varphi(x_t)],u_t,p_t,q_t,r_t(z))\right\}dt\right] \\
&\qquad - \mathbb{E}\left[\int_0^T p_t\left(b\left(t,x_t^\star,u_t^\star\right) - b(t,x_t,u_t)\right)dt\right] \\
&\qquad - \mathbb{E}\left[\int_0^T \sum_{j=1}^n \left(\sigma^j\left(t,x_t^\star,u_t^\star\right) - \sigma^j(t,x_t,u_t)\right)q_t^j dt\right] \\
&\qquad - \mathbb{E}\left[\int_0^T \int_Z \left(\gamma\left(t,x_t^\star,u_t^\star,z\right) - \gamma(t,x_t,u_t,z)\right)r_t(z)\nu(dz)dt\right].
\end{aligned}$$

Adding the above inequalities up, we get

$$J(u^\star) - J(u) \leqslant \mathbb{E}\left[\int_0^T H_u\big(t, x_t^\star, \mathbb{E}\big[\varphi\big(x_t^\star\big)\big], u_t^\star, p_t, q_t, r_t(z)\big)\big(u_t^\star - u_t\big)dt\right].$$

Due to the condition (3.17) which is equivalent to condition (3.3) by the convexity of the Hamiltonian, it yields

$$H_u\big(t, x_t^\star, \mathbb{E}\big[\varphi\big(x_t^\star\big)\big], u_t^\star, p_t, q_t, r_t(z)\big)\big(u_t^\star - u_t\big) \leqslant 0,\ dt - a.e., \mathbb{P} - a.s.$$

This leads to $J(u^\star) - J(u) \leqslant 0$, which means that $u^\star$ is an optimal control for the problem (2.4). □

4. An application to mean-variance portfolio selection problem

As an illustration, we apply a verification result for the stochastic maximum principle to solve the mean-variance portfolio problem, without using Lagrange multipliers technique. The key point in the explicit resolution of the problem is that the adjoint process may be separated into two functions of t and x. Then, one needs only to solve two linear ODEs in order to completely determine the mean-variance efficient portfolio and efficient frontier. Note that this problem has been first solved in the continuous Brownian motion case in [2].

We consider a financial market, in which two securities are traded continuously. One of them is a bond, with price S_t^0 at time $t \in [0,T]$ governed by

$$dS_t^0 = S_t^0 \rho_t dt, \quad S_0^0 = s_0 > 0. \tag{4.1}$$

There is also a stock with unit price S_t^1 at time $t \in [0,T]$ governed by

$$dS_t^1 = S_{t-}^1\left(b_t dt + \sigma_t dB_t + \int_{\mathbb{R}} \gamma_t(z)\widetilde{N}(dt, dz)\right), \quad S_0^1 = s^1 > 0. \tag{4.2}$$

The coefficients $\rho_t > 0, b_t, \sigma_t$, and $\gamma_t(z) \geqslant -1$ are assumed to be deterministic and bounded, we also assume a uniform ellipticity condition as follows $\sigma_t^2 + \int_Z \gamma_t^2(z)\nu(dz) \geqslant \epsilon, a.s.$, for some $\epsilon > 0$. For an investor, a portfolio π is a process representing the amount of money invested in the stock. The wealth process $x^{x_0,\pi}$ corresponding to initial capital $x_0 > 0$, and portfolio π, satisfies then the equation

$$\begin{cases} dx_t = (\rho_t x_t + \pi_t(b_t - \rho_t))dt + \pi_t\sigma_t dB_t + \pi_t \int_{\mathbb{R}} \gamma_t(z)\widetilde{N}(dt, dz), & \text{for } t \in [0, T], \\ x_0 = x. \end{cases} \tag{4.3}$$

The objective is to maximize the mean terminal wealth $\mathbb{E}\big[x_T^\pi\big]$, and at the same time to minimize the variance of the terminal wealth $\mathrm{Var}\big[x_T^\pi\big]$, over controls π valued in $\mathbb{R}$. Then, the mean-variance portfolio optimization problem is denoted as: minimizing the cost J, given by

$$J(\pi) = -\mathbb{E}[x_T] + \mu \mathrm{Var}[x_T], \tag{4.4}$$

subject to (4.3), where $\mu > 0$. The admissible portfolio is assumed to be progressively measurable square integrable process, and such that the corresponding $x_t^\pi \geqslant 0$, for all $t \in [0, T]$. We denote by Π the class of such strategies.

Remark 4.1. In the notation of the previous sections, let us take $u = \pi, \mathcal{U} = \Pi$, $A = \mathbb{R}, b(t, x, u) = \rho_t x + \pi(b_t - \rho_t), \sigma(t, x, u) = \pi\sigma_t, \gamma(t, x, u, z) = \pi\gamma_t(z),\ f(t, x, \mathbb{E}[\varphi(x)], u) = 0$, and $g(x, \mathbb{E}[\eta(x)]) = -x + \mu(x - \mathbb{E}[x])^2$, and thus, in this situation the final cost can be rewritten as $g(x,y) = -x + \mu(x + y)^2$, and $\eta(x) = -x$. Then all conditions in Theorem 3.3. on the coefficients are fulfilled.

The dynamic mean-variance portfolio selection problem has been formulated mostly for Itô processes and Brownian filtration. The basic idea presented in [15,22], is to embed the problem into a stochastic LQ control problem. Such an approach establishes a natural connection of the portfolio selection problems and standard stochastic control models. In [14] the continuous time mean-variance model where the wealth process is modulated by a controlled jump-diffusion is considered, with terminal constraint representing the expected payoff of the investor. Using Lagrange multiplier techniques, the problem is converted into an unconstrained problem parametrized by the Lagrange multiplier.

We show how to solve this optimization problem by applying the Theorem 3.3. In this case the Hamiltonian takes the form

$$H(t, x, \pi, p, q, r) = (\rho_t x + (b_t - \rho_t)\pi)p + \pi(\sigma_t q + \int_{\mathbb{R}} \gamma_t(z) r_t(z) \nu(de)).$$

Hence the adjoint Eq. (3.2) is written, given $\pi \in \Pi$

$$\begin{cases} dp_t = -\rho_t p_t dt + q_t dB_t + \int_{\mathbb{R}} r_{t-}(z) \widetilde{N}(dt, dz), & \text{for } t < T, \\ p_T = 2\mu(x_T - \mathbb{E}[x_T]) - 1. \end{cases} \tag{4.5}$$

As in the classical mean-variance control problem (without jump terms, see [2]), we attempt to look for a linear feedback optimal control. For this and due to the terminal condition in (4.5), we try a solution for the adjoint Eq. (4.5) of the form $p_t = \Phi_t(x_t - \mathbb{E}[x_t]) - \Psi_t$, where Φ_t and Ψ_t are deterministic C^1 functions with $\Phi_T = 2\mu$, and $\Psi_T = 1$. By differentiating, we get

$$dp_t = \Phi_t'(x_t - \mathbb{E}[x_t])dt + \Phi_t d(x_t - \mathbb{E}[x_t]) - \Psi_t' dt, \quad \text{for } t < T, \tag{4.6}$$

We remark that

$$d\mathbb{E}[x_t] = (\rho_t \mathbb{E}[x_t] + (b_t - \rho_t)\mathbb{E}[\pi_t])dt. \tag{4.7}$$

Then

$$d(x_t - \mathbb{E}[x_t]) = \{\rho_t(x_t - \mathbb{E}[x_t]) + (b_t - \rho_t)(\pi_t - \mathbb{E}[\pi_t])\}dt + \sigma_t \pi_t dB_t + \pi_t \int_{\mathbb{R}} \gamma_t(z) \widetilde{N}(dt, dz). \tag{4.8}$$

By invoking (4.6) and (4.8), and by comparing with (4.5), we easily check that

$$\begin{aligned}&\Phi_t'(x_t-\mathbb{E}[x_t])+\Phi_t\rho_t(x_t-\mathbb{E}[x_t])+\Phi_t(b_t-\rho_t)(\pi_t-\mathbb{E}[\pi_t])-\Psi_t'\\&\quad=-\rho_t\Phi_t(x_t-\mathbb{E}[x_t])+\rho_t\Psi_t.\end{aligned}\tag{4.9}$$

We have also $q_t = \Phi_t\sigma_t\pi_t$, and $r_t(z) = \Phi_t\gamma_t(z)\pi_t$.
The Theorem 3.3. then leads to the following theorem.

Theorem 4.2. *Let $\pi^\star \in \Pi$ be a candidate for an optimal control, and let $x^\star$ be the corresponding state process and*

$$\left(p_t^\star,q_t^\star,r_t^\star(z)\right)=\left(\Phi_t\left(x_t^\star-\mathbb{E}\left[x_t^\star\right]\right)-\Psi_t,\Phi_t\sigma_t\pi_t^\star,\Phi_t\gamma_t(z)\pi_t^\star\right),\tag{4.10}$$

the associated adjoint process which solves Eq. (4.5) for all $t \in [0,T]$. If

$$\begin{aligned}&\inf_\pi\left\{\{\rho_tx_t^\star+(b_t-\rho_t)\pi\}p_t^\star+\left\{\sigma_tq_t^\star+\int_{\mathbb{R}}\gamma_t(z)r_t^\star(z)\nu(de)\right\}\pi\right\}\\&\quad=\{\rho_tx_t^\star+(b_t-\rho_t)\pi_t^\star\}p_t^\star+\left\{\sigma_tq_t^\star+\int_{\mathbb{R}}\gamma_t(z)r_t^\star(z)\nu(de)\right\}\pi_t^\star.\end{aligned}\tag{4.11}$$

Then $\pi_t^\star$ is an optimal control for the problem (4.4) subject to the SDE (4.3).

In this case, the expression (4.9) under the optimal solution $(\pi^\star,x^\star)$, must be written

$$\begin{aligned}&\Phi_t'\left(x_t^\star-\mathbb{E}\left[x_t^\star\right]\right)+\Phi_t\rho_t\left(x_t^\star-\mathbb{E}[x_t^\star]\right)+\Phi_t(b_t-\rho_t)\left(\pi_t^\star-\mathbb{E}\left[\pi_t^\star\right]\right)-\Psi_t'\\&\quad=-\rho_t\Phi_t\left(x_t^\star-\mathbb{E}\left[x_t^\star\right]\right)+\rho_t\Psi_t.\end{aligned}\tag{4.12}$$

To find the control that satisfies condition (4.11), we differentiate that expression, we obtain from (4.10)

$$\pi_t^\star=\frac{(b_t-\rho_t)}{\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)\nu(dz)}\left(\frac{\Psi_t}{\Phi_t}+\mathbb{E}\left[x_t^\star\right]-x_t^\star\right).\tag{4.13}$$

Taking expectations, we obtain

$$\mathbb{E}\left[\pi_t^\star\right]=\frac{(b_t-\rho_t)}{\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)\nu(dz)}\left(\frac{\Psi_t}{\Phi_t}\right).\tag{4.14}$$

Putting the expressions of $\pi_t^\star$ and $\mathbb{E}\left[\pi_t^\star\right]$ obtained by (4.13) and (4.14), respectively, in (4.12), we obtain the following representation

$$\left\{\Phi_t'+(2\rho_t-\Lambda_t)\Phi_t\right\}\left(x_t^\star-\mathbb{E}\left[x_t^\star\right]\right)-\left\{\Psi_t'+\rho_t\Psi_t\right\}=0,\tag{4.15}$$

where

$$\Lambda_t=\frac{(b_t-\rho_t)^2}{\sigma_t^2+\int_{\mathbb{R}}\gamma_t^2(z)\nu(dz)},$$

By comparing the coefficients in (4.15) we get

$$\begin{cases}\Phi_t'+(2\rho_t-\Lambda_t)\Phi_t=0,\\\Phi_T=2\mu,\end{cases}$$

and

$$\begin{cases} \Psi_t' + \rho_t \Psi_t = 0, \\ \Psi_T = 1. \end{cases}$$

The solutions are respectively $\Phi_t = 2\mu e^{\int_t^T (2\rho_s - \Lambda_s)ds}$, and $\Psi_t = e^{\int_t^T \rho_s ds}$. Moreover, the optimal control is given by

$$\pi_t^\star = \frac{(b_t - \rho_t)}{\sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z)\nu(dz)} \left(\frac{1}{2\mu} e^{\int_t^T (\Lambda_s - \rho_s)ds} + \mathbb{E}[x_t^\star] - x_t^\star \right). \tag{4.16}$$

This completes the characterization of the optimal control from the Theorem 3.3.

In the next, we will find the efficient frontier of the mean-variance problem. Then by standard arguments based on Itô's formula, we will prove the following result.

Theorem 4.3. *The stochastic mean-variance control problem (4.4) subject to the SDE (4.3), has an optimal feedback solution*

$$\pi_t^\star = \frac{(b_t - \rho_t)}{\sigma_t^2 + \int_{\mathbb{R}} \gamma_t^2(z)\nu(dz)} \left(x_0 e^{\int_0^t \rho_s ds} + \frac{1}{2\mu} e^{\int_t^T -\rho_s ds + \int_0^T \Lambda_s ds} - x_t^\star \right). \tag{4.17}$$

Moreover, the optimal expected terminal wealth is

$$\mathbb{E}[x_T^\star] = x_0 e^{\int_0^T (\rho_t - \Lambda_t)dt} + \left(1 - e^{-\int_0^T \Lambda_t dt}\right)\left(\frac{1}{2\mu} e^{\int_0^T \Lambda_t dt} + x_0 e^{\int_0^T \rho_t dt} \right), \tag{4.18}$$

and the corresponding variance of the terminal wealth is

$$\mathrm{Var}[x_T^\star] = \left(\frac{e^{-\int_0^T \Lambda_t dt}}{1 - e^{-\int_0^T \Lambda_t dt}} \right) \left(\frac{1}{2\mu} e^{\int_0^T \Lambda_t dt} - \frac{1}{2\mu} \right)^2. \tag{4.19}$$

Proof. From (4.7), and by taking expectations on both sides of (4.16), we represent $\mathbb{E}[x_t^\star]$ as follows

$$d\mathbb{E}[x_t^\star] = \left\{ \rho_t \mathbb{E}[x_t^\star] + \frac{1}{2\mu} e^{\int_t^T (\Lambda_s - \rho_s)ds} \Lambda_t \right\} dt, \tag{4.20}$$

then, the unique solution to the linear ordinary differential equation (4.20) is explicitly given from the variation of constant formula by

$$\mathbb{E}[x_t^\star] = x_0 e^{\int_0^t \rho_s ds} - \frac{1}{2\mu} e^{\int_t^T (\Lambda_s - \rho_s)ds} + \frac{1}{2\mu} e^{\int_t^T -\rho_s ds + \int_0^T \Lambda_s ds}. \tag{4.21}$$

First, by subtituting (4.21) into (4.16), it follows immediately that the optimal feedback control is given by (4.17). This provided that the corresponding Eq. (4.3) under (4.17), has the following representation

$$\begin{cases} dx_t^\star = \left\{(\rho_t - \Lambda_t)x_t^\star + \Lambda_t e^{-\int_t^T \rho_s ds}\left(\frac{1}{2\mu}e^{\int_0^T \Lambda_t dt} + x_0 e^{\int_0^T \rho_t dt}\right)\right\}dt \\ \qquad + \frac{\Lambda_t \sigma_t}{(b_t-\rho_t)}\left\{e^{-\int_t^T \rho_s ds}\left(\frac{1}{2\mu}e^{\int_0^T \Lambda_t dt} + x_0 e^{\int_0^T \rho_t dt}\right) - x_t^\star\right\}dB_t \\ \qquad + \frac{\Lambda_t}{(b_t-\rho_t)}\left\{e^{-\int_t^T \rho_s ds}(\frac{1}{2\mu}e^{\int_0^T \Lambda_t dt} + x_0 e^{\int_0^T \rho_t dt}) - x_t^\star\right\}\int_{\mathbb{R}}\gamma_t(z)\widetilde{N}(dt,dz), \\ x_0^\star = x_0, \end{cases}$$

Applying Itô's formula to $x_t^{\star 2}$. Taking the expectation, we conclude that $\mathbb{E}\left[x_t^{\star 2}\right]$ satisfies the following linear ordinary differential equation

$$\begin{cases} d\mathbb{E}\left[x_t^{\star 2}\right] = \left\{(2\rho_t - \Lambda_t)\mathbb{E}\left[x_t^{\star 2}\right] + \Lambda_t e^{-\int_t^T 2\rho_s ds}\left(\frac{1}{2\mu}e^{\int_0^T \Lambda_t dt} + x_0 e^{\int_0^T \rho_t dt}\right)^2\right\}dt, \\ \mathbb{E}\left[x_0^{\star 2}\right] = x_0^2. \end{cases} \tag{4.22}$$

This equation can be solved explicitly, and we get from the variation of constant formula

$$\mathbb{E}\left[x_t^{\star 2}\right] = x_0^2 e^{\int_0^t (2\rho_s - \Lambda_s)ds} + \left(e^{-\int_t^T 2\rho_s ds} - e^{-\int_0^t \Lambda_s ds - \int_t^T 2\rho_s ds}\right)\left(\frac{1}{2\mu}e^{\int_0^T \Lambda_t dt} + x_0 e^{\int_0^T \rho_t dt}\right)^2.$$

At the terminal time T, we have

$$\mathbb{E}\left[x_T^{\star 2}\right] = x_0^2 e^{\int_0^T (2\rho_t - \Lambda_t)dt} + \left(1 - e^{-\int_0^T \Lambda_t dt}\right)\left(\frac{1}{2\mu}e^{\int_0^T \Lambda_t dt} + x_0 e^{\int_0^T \rho_t dt}\right)^2. \tag{4.23}$$

Also, from (4.21) it follows

$$\begin{aligned} \mathbb{E}\left[x_T^\star\right] &= x_0 e^{\int_0^T \rho_t dt} + \frac{1}{2\mu}\left(e^{\int_0^T \Lambda_t dt} - 1\right), \\ &= x_0 e^{\int_0^T (\rho_t - \Lambda_t)dt} + \left(1 - e^{-\int_0^T \Lambda_t dt}\right)\left(\frac{1}{2\mu}e^{\int_0^T \Lambda_t dt} + x_0 e^{\int_0^T \rho_t dt}\right). \end{aligned} \tag{4.24}$$

We shall use the following notations: $I_1 = e^{\int_0^T (\rho_t - \Lambda_t)dt}$, $I_2 = 1 - e^{-\int_0^T \Lambda_t dt}$, $I_3 = \frac{1}{2\mu}e^{\int_0^T \Lambda_t dt} + x_0 e^{\int_0^T \rho_t dt}$, and $I_4 = e^{\int_0^T (2\rho_t - \Lambda_t)dt}$. This suggest that (4.24) and (4.23) must be interpreted respectively by $\mathbb{E}\left[x_T^\star\right] = x_0 I_1 + I_2 I_3$, and $\mathbb{E}\left[x_T^{\star 2}\right] = x_0^2 I_4 + I_2 I_3^2$. From the same arguments as in the proof of Theorem 6.1. in [22], we can easily show that

$$\begin{aligned} \operatorname{Var}\left[x_T^\star\right] &= \mathbb{E}[x_T^{\star 2}] - (\mathbb{E}\left[x_T^\star\right])^2 \\ &= \frac{1 - I_2}{I_2}\left[(I_2 I_3)^2 - \frac{2 I_1 I_2^2 I_3}{1 - I_2}x_0 + \frac{I_2(I_4 - I_1^2)}{1 - I_2}x_0^2\right]. \end{aligned} \tag{4.25}$$

A simple computation shows that

$$\frac{I_1 I_2^2 I_3}{1-I_2} = \frac{I_1 I_2 I_3}{1-I_2} - I_1 I_2 I_3, \tag{4.26}$$

$$\frac{I_2(I_4 - I_1^2)}{1-I_2} = I_1^2 + \frac{I_2 I_4 - I_1^2}{1-I_2}. \tag{4.27}$$

By completing the square, we obtain by computing (4.26),(4.27) and (4.25)

$$\operatorname{Var}\left[x_T^\star\right] = \frac{1-I_2}{I_2}\left[(I_2 I_3 + I_1 x_0)^2 - \frac{2I_1 I_2 I_3}{1-I_2} x_0 + \frac{(I_2 I_4 - I_1^2)}{1-I_2} x_0^2\right].$$

Note that $\mathbb{E}\left[x_T^\star\right] = I_2 I_3 + I_1 x_0$, then we get

$$\operatorname{Var}\left[x_T^\star\right] = \frac{1-I_2}{I_2}\left[\left(\mathbb{E}\left[x_T^\star\right]\right)^2 - \frac{2I_1}{1-I_2} x_0 \mathbb{E}\left[x_T^\star\right] + \frac{I_2 I_4 + I_1^2}{1-I_2} x_0^2\right].$$

Finally, by replacing I_1,I_2,I_3 and I_4 by their values, we deduce (4.19). This completes the explicit representation of the efficient frontier of the mean-variance problem. □

5. Conclusion

In this paper, we have proved necessary optimality conditions in the form of a maximum principle for systems driven by stochastic differential equations with jumps. The cost functional is of mean field type, that is it contains not only the state process but also its marginal distribution. In the case where the data are convex, these necessary conditions are in fact sufficient. This gives a characterization of optimal controls and gives an effective way to compute them explicitly in some situations. This has been illustrated in our paper, by solving the mean variance portfolio selection problem in the presence of jumps.

Acknowledgments

The authors would like to thank the associate editor and the referees for valuable suggestions that led to a substantial improvement of the paper.

References

[1] N.U. Ahmed, X. Ding, Controlled McKean-Vlasov equations, Commun. Appl. Anal. 5 (2) (2001) 183–206.

[2] D. Andersson, B. Djehiche, A maximum principle for SDEs of mean-field type, Appl. Math. Optim. 63 (3) (2010) 341–356.

[3] F. Baghery, B. ∅ksendal, A maximum principle for stochastic control with partial information, Stoch. Anal. Appl. 25 (2007) 705–717.

[4] G. Barles, R. Buckdahn, E. Pardoux, BSDE's and integral–partial differential equations, Stochastics Stochastic Rep. 60 (1997) 57–83.

[5] D. Becherer, Bounded solutions to Backward SDE's with jumps for utility optimization and indifference hedging, Ann. Appl. Probab. 16 (4) (2006) 2027–2054.
[6] A. Bensoussan, Lectures on stochastic control, in: S.K. Mitter, A. Moro (Eds.), Lecture Notes in Mathematics: Nonlinear Filtering and Stochastic Control, vol. 972, Springer-Verlag, 1982, pp. 1–62.
[7] B. Bouchard, R. Elie, Discrete time approximation of decoupled Forward-Backward SDE with jumps, Stochastic Process. Appl. 118 (1) (2008) 53–75.
[8] R. Buckdahn, B. Djehiche, J. Li, S. Peng, Mean-field backward stochastic differential equations: a limit approach, The Ann. Probab. 37 (4) (2009) 1524–1565.
[9] R. Buckdahn, J. Li, S. Peng, Mean-field backward stochastic differential equations and related partial diferential equations, Stochastic Process. Appl. 119 (2009) 3133–3154.
[10] F. Chighoub, B. Mezerdi, Near Optimality Conditions in Stochastic Control of Jump Diffusion Processes, Syst. Control Lett. 60 (2011) 907–916.
[11] I. Ekeland, R. Temam, Convex Analysis and Variational Problems, Elsevier, New York, North-Holland, 1976.
[12] R.J. Elliott, The optimal control of diffusions, Appl. Math. Optim. 22 (1990) 229–240.
[13] A. Eyraud-Loisel, Backward Stochastic Differential Equations with Enlarged Filtration: Option hedging of an insider trader in a financial market with jumps, Stochastic Process. Appl. 115 (2005) 1745–1763.
[14] N.C. Framstad, B. Øksendal, A. Sulem, Sufficient stochastic maximum principle for the optimal control of jump diffusions and applications to finance, J. Optim. Theory Appl. 121 (2004) 77–98 [Errata 124 (2005) 511–512].
[15] M. Kohlmann, X.Y. Zhou, Relationship between backward stochastic differential equations and stochastic controls: a linear-quadratic approach, SIAM J. Control Optim. 38 (2000) 1392–1407 (electronic).
[16] J.M. Lasry, P.L. Lions, Mean-field games, Jpn. J. Math. 2 (2007) 229–260.
[17] T. Meyer-Brandis, B. Øksendal, X.Y. Zhou, A mean-field stochastic maximum principle via Malliavin calculus, Pure Mathematics, E-print 10, Dept. of Math./CMA University of Oslo, 2008. Revised May 7, 2010. ISSN 0806-2439 (to appear in Stochastics).
[18] B. Øksendal, A. Sulem, Applied Stochastic Control of Jump Diffusions, second ed., Springer, 2007.
[19] S. Peng, A general stochastic maximum principle for optimal control problems, SIAM J. Control Optim. 28 (1990) 966–979.
[20] M. Saksonov, Necessary optimality conditions for linear control problems of jump processes, in: Analysis in Probability for Control Problems of Economical Processes, CEMI, Academy of Science USSR, Moscow, 1985, pp. 117–130, in Russian.
[21] S. Tang, X. Li, Necessary conditions for optimal control for stochastic systems with random jumps, SIAM J. Control Optim. 32 (1994) 1447–1475.
[22] X.Y. Zhou, D. Li, Continuous-time mean-variance portfolio selection: a stochastic LQ framework, Appl. Math. Optim. 42 (2000) 19–33.