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A remark on the existence of positive solutions for variable exponent elliptic systems

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Abstract. In this article, we consider the system of differential equations

$$\begin{cases} -\Delta_{p(x)} u = \lambda^{p(x)}[a(x)u^{\alpha(x)}v^{\gamma(x)} + h_1(x)] & \text{in } \Omega, \\ -\Delta_{q(x)} v = \lambda^{q(x)}[b(x)u^{\delta(x)}v^{\beta(x)} + h_2(x)] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary $\partial\Omega$, $1 < p(x)$, $q(x) \in C^1(\overline{\Omega})$ are functions. The operator $-\Delta_{p(x)}\, u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian. When α, β, δ, γ satisfy some suitable conditions, we prove the existence of positive solution via sub-supersolution arguments without assuming sign conditions on the functions h_1 and h_2.

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1. Introduction and preliminaries

The study of differential equations and variational problems with nonstandard $p(x)$-growth conditions is a new and interesting topic. It arises from nonlinear elasticity theory, electrorheological fluids, etc. (see [1,2,14,20]). Many existence results have been obtained on this kind of problems, see for example [4,9,10,12,15–18]. In [6–8], Fan

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et al. studied the regularity of solutions for differential equations with nonstandard $p(x)$-growth conditions.

In this paper, we mainly consider the existence of positive weak solutions for the system

$$\begin{cases} -\Delta_{p(x)}u = \lambda^{p(x)}[a(x)u^{\alpha(x)}v^{\gamma(x)} + h_1(x)] & \text{in } \Omega, \\ -\Delta_{q(x)}v = \lambda^{q(x)}[b(x)u^{\delta(x)}v^{\beta(x)} + h_2(x)] & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2-boundary $\partial\Omega$, $1 < p(x), q(x) \in C^1(\overline{\Omega})$ are two functions. The operator $-\Delta_{p(x)}u = -\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$ is called the $p(x)$-Laplacian and the corresponding equation is called a variable exponent equation. Especially, if $p(x) \equiv p$ (a constant), (1.1) is the well-known (p,q)-Laplacian system and the corresponding equation is called a constant exponent equation. We have known that the existence of solutions for p-Laplacian elliptic systems has been intensively studied in the last decades, we refer to [3,11,13]. In [11], Hai et al. considered the existence of positive weak solutions for the p-Laplacian problem

$$\begin{cases} -\Delta_p u = \lambda f(v) & \text{in } \Omega, \\ -\Delta_p v = \lambda g(u) & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

in which the first eigenfunction was used for constructing the subsolution of p-Laplacian problems. Under the condition

$$\lim_{u\to+\infty} \frac{f\Big(M(g(u))^{1/(p-1)}\Big)}{u^{p-1}} = 0 \text{ for all } M > 0,$$

the authors showed that the problem (1.2) has at least one positive solution provided that $\lambda > 0$ is large enough.

In [3], the author studied the existence and nonexistence of positive weak solution to the following quasilinear elliptic system

$$\begin{cases} -\Delta_p u = \lambda f(u,v) = \lambda u^{\alpha}v^{\gamma} & \text{in } \Omega, \\ -\Delta_q v = \lambda g(u,v) = \lambda u^{\delta}v^{\beta} & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

The first eigenfunction is used to construct the subsolution of problem (1.3), the main results are as follows:

(*i*) If $\alpha,\beta \geqslant 0, \gamma,\delta > 0$, $\theta = (p - 1 - \alpha)(q - 1 - \beta) - \gamma\delta > 0$, then problem (1.3) has a positive weak solution for each $\lambda > 0$;

(*ii*) If $\theta = 0$ and $p\gamma = q(p - 1 - \alpha)$, then there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, then problem (1.3) has no nontrivial nonnegative weak solution.

In recent papers [15–18], Zhang has developed problems (1.2) and (1.3) in the variable exponent Sobolev space. On the $p(x)$-Laplacian problems, maybe the first eigenvalue and the first eigenfunction of $p(x)$-Laplacian do not exist. Even if the first eigenfunction of $p(x)$-Laplacian exists, because of the nonhomogeneity of $p(x)$-Laplacian, the first eigenfunction cannot be used to construct the subsolution of $p(x)$-Laplacian problems. Motivated by the above papers, in this note, we are interested in the existence of positive solution for problem (1.1), where a, b are continuous functions in $\overline{\Omega}$ and λ is a positive parameter. Our main goal is to improve the result introduced in [18], in which $a = b \equiv 1$.

To be more precise, we assume that $\Omega \subset \mathbb{R}^N$ is an open bounded domain with C^2-boundary $\partial\Omega$ and the following conditions hold:

(H_1) $p(x), q(x) \in C^1(\overline{\Omega})$ and $1 < p^- \leqslant p^+$ and $1 < q^- \leqslant q^+$;
(H_2) $h_1, h_2, \alpha, \beta, \gamma, \delta \in C^1(\overline{\Omega})$ satisfy $\alpha,\beta \geqslant 0$ on $\overline{\Omega}$, and $\gamma,\delta > 0$ on $\overline{\Omega}$;
(H_3) $0 \leqslant \alpha^+ < p^- - 1, 0 \leqslant \beta^+ < q^- - 1$ and $\varpi := (p^- - 1 - \alpha^+)(q^- - 1 - \beta^+) - \delta^+\gamma^+ > 0$;
(H_4) $a, b : \overline{\Omega} \to (0, \infty)$ are continuous functions such that $a_1 = \min_{x\in\overline{\Omega}} a(x)$, $b_1 = \min_{x\in\overline{\Omega}} b(x), a_2 = \max_{x\in\overline{\Omega}} a(x)$ and $b_2 = \max_{x\in\overline{\Omega}} b(x)$.

To study $p(x)$-Laplacian problems, we need some theories on the spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and properties of $p(x)$-Laplacian which we will use later (see [5]). If $\Omega \subset \mathbb{R}^N$ is an open domain, write

$$C_+(\Omega) = \{h : h \in C(\Omega), h(x) > 1 \text{ for } x \in \Omega\},$$

$h^+ = \sup_{x\in\Omega} h(x), h^- = \inf_{x\in\Omega} h(x)$, for any $h \in C(\Omega)$, and

$$L^{p(x)}(\Omega) = \left\{ u | u \text{ is a measurable real-valued function such that } \int_\Omega |u|^{p(x)} dx < \infty \right\}.$$

Throughout the paper, we assume that $p, q \in C_+(\Omega)$ and $1 < \inf_{x\in\Omega} p(x) \leqslant \sup_{x\in\Omega} p(x) < N, 1 < \inf_{x\in\Omega} q(x) \leqslant \sup_{x\in\Omega} q(x) < N$. We introduce the norm $L^{p(x)}(\Omega)$ by

$$|u|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leqslant 1 \right\},$$

and $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ becomes a Banach space, we call it the generalized Lebesgue space. The space $(L^{p(x)}(\Omega), |\cdot|_{p(x)})$ is a separable, reflexive and uniform convex Banach space (see [5, Theorem 1.10, 1.14]).

The space $W^{1,p(x)}(\Omega)$ is defined by $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$, and it is equipped with the norm

$$\|u\| = |u|_{p(x)} + |\nabla u|_{p(x)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable, reflexive and uniform convex Banach space (see [5, Theorem 2.1]). We define

$$(L(u), v) = \int_{\mathbb{R}^N} |\nabla u|^{p(x)-2} \nabla u \nabla v dx, \quad \forall u, v \in W^{1,p(x)}(\Omega),$$

then $L: W^{1,p(x)}(\Omega) \to (W^{1,p(x)}(\Omega))^*$ is a continuous, bounded and is a strictly monotone operator, and it is a homeomorphism [9, Theorem 3.11].

Definition 1.1. If $(u, v) \in \left(W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega)\right)$, (u, v) is called a weak solution of (1.1) if it satisfies

$$\begin{cases} \int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi dx = \int_\Omega \lambda^{p(x)} [a(x) u^{\alpha(x)} v^{\gamma(x)} + h_1(x)] \varphi dx, & \forall \varphi \in W_0^{1,p(x)}(\Omega), \\ \int_\Omega |\nabla v|^{q(x)-2} \nabla v \cdot \nabla \psi dx = \int_\Omega \lambda^{q(x)} [b(x) u^{\delta(x)} v^{\beta(x)} + h_1(x)] \psi dx, & \forall \psi \in W_0^{1,q(x)}(\Omega). \end{cases}$$

Define $A : W^{1,p(x)}(\Omega) \to (W_0^{1,p(x)}(\Omega))^*$ as

$$\langle Au, \varphi \rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} \nabla u \nabla \varphi + h(x, u) \varphi \right) dx, \quad \forall u, \varphi \in W^{1,p(x)}(\Omega),$$

where $h(x,u)$ is continuous on $\overline{\Omega} \times \mathbb{R}$, and $h(x, \cdot)$ is increasing and satisfies

$$|h(x,t)| \leqslant C_1 + C_2 |t|^{p^*(x)-1},$$

where

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N \\ \infty, & p(x) \geqslant N. \end{cases}$$

It is easy to check that A is a continuous bounded mapping. Copying the proof of [19], we have the following lemma.

Lemma 1.2 (Comparison principle). *Let $u,v \in W^{1,p(x)}(\Omega)$ satisfy $Au - Av \geqslant 0$ in $(W_0^{1,p(x)}(\Omega))^*$, $\varphi(x) = \min\{u(x) - v(x), 0\}$. If $\varphi(x) \in W_0^{1,p(x)}(\Omega)$ (i.e. $u \geqslant v$ on $\partial\Omega$), then $u \geqslant v$ a.e. in Ω.*

Here and hereafter, we will use the notation $d(x, \partial\Omega)$ to denote the distance of $x \in \Omega$ to the boundary of Ω. Denote $d(x) = d(x, \partial\Omega)$ and $\partial\Omega_\epsilon = \{x \in \Omega | \, d(x, \partial\Omega) < \epsilon\}$. Since $\partial\Omega$ is C^2 regularly, then there exists a constant $l \in (0,1)$ such that $d(x) \in C^2(\overline{\partial\Omega_{3l}})$, and $|\nabla d(x)| \equiv 1$.

Denote

$$v_1(x) = \begin{cases} \zeta d(x), & d(x) < l, \\ \zeta l + \int_l^{d(x)} \zeta \left(\frac{2l-t}{l}\right)^{\frac{2}{p^- - 1}} (a_1 + 1)^{\frac{2}{p^- - 1}} dt, & l \leqslant d(x) < 2l, \\ \zeta l + \int_l^{2l} \zeta \left(\frac{2l-t}{l}\right)^{\frac{2}{p^- - 1}} (a_1 + 1)^{\frac{2}{p^- - 1}} dt, & 2l \leqslant d(x). \end{cases}$$

Obviously, $0 \leqslant v_1(x) \in C^1(\overline{\Omega})$. Consider the problem

$$\begin{cases} -\Delta_{p(x)} w(x) = \eta & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

where η is a parameter. The following result plays an important role in our argument whose proof can be found in [18] or [4].

Lemma 1.3 (see [18]). *If the positive parameter η is large enough and w is the unique solution of (1.4), then we have*

(i) For any $\theta \in (0, 1)$ there exists a positive constant C_1 such that

$$C_1 \eta^{\frac{1}{p^+-1+\theta}} \leqslant \max_{x \in \overline{\Omega}} w(x);$$

(ii) *There exists a positive constant C_2 such that*

$$\max_{x \in \overline{\Omega}} w(x) \leqslant C_2 \eta^{\frac{1}{p^--1}}.$$

Proof

(i) By computation,

$$-\Delta_{p(x)} v_1(x) = \begin{cases} -\zeta^{p(x)-1}[(\nabla p \nabla d) \ln \zeta + \Delta d(x)], \quad d(x) < \sigma, \\ \left\{\frac{1}{l} \frac{2(p(x)-1)}{p^- -1} - \left(\frac{2l-d}{l}\right)\left[\left(\ln \zeta \left(\frac{2l-d}{l}\right)^{\frac{2}{p^--1}}\right) \nabla p \nabla d + \Delta d\right]\right\} \\ \times \zeta^{p(x)-1} \left(\frac{2l-d}{2l-\sigma}\right)^{\frac{2(p(x)-1)}{p^--1}-1} (a_1 + 1), \quad l < d(x) < 2l, \\ 0, \quad 2l < d(x). \end{cases}$$

Then $\left| -\Delta_{p(x)} v_1(x)\right| \leqslant C_* \zeta^{p(x)-1+\theta}$ a.e. on Ω, for any $\theta \in (0,1)$, where $C_* = C_*(l,\theta,p,\Omega)$ is a positive constant depending on ζ.

When $C_* \zeta^{p^+-1+\theta} = \frac{1}{2}\eta$, we can see that $v_1(x)$ is a subsolution of (1.1). According to the comparison principle, it follows that $v_1(x) \leqslant \Omega(x)$ on $\overline{\Omega}$. Obviously, $\zeta l \leqslant \max_{x \in \overline{\Omega}} v_1(x) \leqslant 2\zeta l$, there exists a positive constant C_1 such that

$$\max_{x \in \overline{\Omega}} w(x) \geqslant \max_{x \in \overline{\Omega}} v_1(x) \geqslant C_1 \eta^{\frac{1}{p^+-1+\theta}}.$$

(ii) It is easy to see from Lemma 1.2 of [4]. This completes the proof. □

2. Existence of solutions

In the following, when there is no misunderstanding, we always use C_i to denote positive constants. Our main result of this paper is the following theorem.

Theorem 2.1. *On the conditions of $(H_1)-(H_4)$, then problem (1.1) has positive solution when λ is large enough.*

Proof. We shall establish Theorem 2.1 by constructing a positive subsolution (ϕ_1, ϕ_2) and supersolution (z_1, z_2) of (1.1), such that $\phi_1 \leqslant z_1$ and $\phi_2 \leqslant z_2$. That is (ϕ_1, ϕ_2) and (z_1, z_2) satisfies

$$\begin{cases} \int_\Omega |\nabla \phi_1|^{p(x)-2} \nabla \phi_1 \cdot \nabla \varphi dx \leqslant \int_\Omega \lambda^{p(x)} [a(x)\phi_1^{\alpha(x)} \phi_2^{\gamma(x)} + h_1(x)] \varphi dx \\ \int_\Omega |\nabla \phi_2|^{q(x)-2} \nabla \phi_2 \cdot \nabla \psi dx \leqslant \int_\Omega \lambda^{q(x)} [b(x)\phi_1^{\delta(x)} \phi_2^{\beta(x)} + h_2(x)] \psi dx, \end{cases}$$

and

$$\begin{cases} \int_\Omega |\nabla z_1|^{p(x)-2} \nabla z_1 \cdot \nabla \varphi dx \geqslant \int_\Omega \lambda^{p(x)} [a(x) z_1^{\alpha(x)} z_2^{\gamma(x)} + h_1(x)] \varphi dx \\ \int_\Omega |\nabla z_2|^{q(x)-2} \nabla z_2 \cdot \nabla \psi dx \geqslant \int_\Omega \lambda^{p(x)} [b(x) z_1^{\delta(x)} z_2^{\beta(x)} + h_2(x)] \psi dx, \end{cases}$$

for all $(\varphi, \psi) \in (W_0^{1,p(x)}(\Omega), W_0^{1,q(x)}(\Omega))$ with $\varphi, \psi \geqslant 0$. According to the sub-supersolution method for $p(x)$-Laplacian equations (see [4]), then (1.1) has a positive solution.

Step 1. We construct a subsolution of (1.1).
Let $\sigma \in (0,l)$

$$\phi_1(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma} \left(\frac{2l-t}{2l-\sigma}\right)^{\frac{2}{p^- - 1}} (a_1 + 1)^{\frac{2}{p^- - 1}} dt, \quad \sigma \leqslant d(x) < 2l, \\ e^{k\sigma} - 1 + \int_\sigma^{2l} k e^{k\sigma} \left(\frac{2l-t}{2l-\sigma}\right)^{\frac{2}{p^- - 1}} (a_1 + 1)^{\frac{2}{p^- - 1}} dt, \quad 2l \leqslant d(x). \end{cases}$$

$$\phi_2(x) = \begin{cases} e^{kd(x)} - 1, \quad d(x) < \sigma, \\ e^{k\sigma} - 1 + \int_\sigma^{d(x)} k e^{k\sigma} \left(\frac{2l-t}{2l-\sigma}\right)^{\frac{2}{p^- - 1}} (b_1 + 1)^{\frac{2}{p^- - 1}} dt, \quad \sigma \leqslant d(x) < 2l, \\ e^{k\sigma} - 1 + \int_\sigma^{2l} k e^{k\sigma} \left(\frac{2l-t}{2l-\sigma}\right)^{\frac{2}{p^- - 1}} (b_1 + 1)^{\frac{2}{p^- - 1}} dt, \quad 2l \leqslant d(x). \end{cases}$$

It is easy to see that $\phi_1, \phi_2 \in C^1(\overline{\Omega})$. Denote

$$\alpha = \min\left\{ \frac{\inf p(x) - 1}{4(\sup |\nabla p(x)| + 1)}, \frac{\inf q(x) - 1}{4(\sup |\nabla q(x)| + 1)}, 1 \right\},$$
$$b = \min\{a_1 + |h_1(0)|, b_1 + |h_2(0)|, -1\}$$

By computation

$$-\Delta_{p(x)}\phi_1 = \begin{cases} -k(ke^{kd(x)})^{p(x)-1}\left[(p(x)-1) + \left(d(x) + \frac{\ln k}{k}\right)\nabla p \nabla d + \frac{\Delta d}{k}\right], \quad d(x) < \sigma, \\ \left\{\frac{1}{2l-\sigma}\frac{2(p(x)-1)}{p^- - 1} - \left(\frac{2l-d}{2l-\sigma}\right)\left[\left(\ln k e^{k\sigma}\left(\frac{2l-d}{2l-\sigma}\right)^{\frac{2}{p^- - 1}}\right)\nabla p \nabla d + \Delta d\right]\right\} \\ \quad \times (ke^{k\sigma})^{p(x)-1}\left(\frac{2l-d}{2l-\sigma}\right)^{\frac{2(p(x)-1)}{p^- - 1} - 1}(a_1 + 1), \quad \sigma < d(x) < 2l, \\ 0, \quad 2l < d(x). \end{cases}$$

$$-\Delta_{p(x)}\phi_2 = \begin{cases} -k(ke^{kd(x)})^{p(x)-1}\left[(p(x)-1) + \left(d(x) + \frac{\ln k}{k}\right)\nabla p \nabla d + \frac{\Delta d}{k}\right], \quad d(x) < \sigma, \\ \left\{\frac{1}{2l-\sigma}\frac{2(p(x)-1)}{p^- - 1} - \left(\frac{2l-d}{2l-\sigma}\right)\left[\left(\ln k e^{k\sigma}\left(\frac{2l-d}{2l-\sigma}\right)^{\frac{2}{p^- - 1}}\right)\nabla p \nabla d + \Delta d\right]\right\} \\ \quad \times (ke^{k\sigma})^{p(x)-1}\left(\frac{2l-d}{2l-\sigma}\right)^{\frac{2(p(x)-1)}{p^- - 1} - 1}(b_1 + 1), \quad \sigma < d(x) < 2l, \\ 0, \quad 2l < d(x). \end{cases}$$

From (H_2) and (H_3), there exists a positive constant $M > 2$ such that

$$b(x)\phi_1^{\delta(x)}\phi_2^{\beta(x)} + h_2(x) \geqslant 1, a(x)\phi_1^{\alpha(x)}\phi_2^{\gamma(x)} + h_1(x) \geqslant 1, \quad \forall x \in \overline{\Omega} \quad \text{when} \quad \phi_1, \phi_2 \geqslant M - 1.$$

Let $\sigma = \frac{1}{k}\ln M$. Then

$$\sigma k = \ln M. \tag{2.1}$$

If k is sufficiently large, from (2.1), we have

$$-\Delta_{p(x)}\phi \leqslant -k^{p(x)}\alpha, \quad d(x) < \sigma. \tag{2.2}$$

Let $-\lambda b = k\alpha$, then

$$k^{p(x)}\alpha \geqslant \lambda^{p(x)}b,$$

from (2.2) and the definition of b, we have

$$-\Delta_{p(x)}\phi_1 \leqslant \lambda^{p(x)}(a_1 + 1) \leqslant \lambda^{p(x)}(a(x)\phi_1^{\alpha(x)}\phi_2^{\gamma(x)} + h_1(x)), \quad d(x) < \sigma. \tag{2.3}$$

Since $d(x) \in C^2(\overline{\partial\Omega_{3l}})$, then there exists a positive constant C_3 such that

$$\begin{aligned}
-\Delta_{p(x)}\phi_1 &\leqslant (ke^{k\sigma})^{p(x)-1}\left(\frac{2l-d}{2l-\sigma}\right)^{\frac{2(p(x)-1)}{p^- -1}-1}(a_1+1) \\
&\quad \times \left|\frac{2(p(x)-1)}{(2l-\sigma)(p^- -1)} - \left(\frac{2l-d}{2l-\sigma}\right)\left[\left(\ln ke^{k\sigma}\left(\frac{2l-d}{2l-\sigma}\right)^{\frac{2}{p^- -1}}\right)\nabla p\nabla d + \Delta d\right]\right| \\
&\leqslant C_3(ke^{k\sigma})^{p(x)-1}(a_1+1)\ln k, \quad \sigma < d(x) < 2l.
\end{aligned}$$

If k is sufficiently large, let $-\lambda\zeta = k\alpha$, we have

$$(a_1 + 1)C_3(ke^{k\sigma})^{p(x)-1}\ln k = (a_1 + 1)C_3(kM)^{p(x)-1}\ln k \leqslant \lambda^{p(x)}(a_1 + 1), \tag{2.4}$$

then

$$-\Delta_{p(x)}\phi_1 \leqslant \lambda^{p(x)}(a_1 + 1), \quad \sigma < d(x) < 2l. \tag{2.5}$$

Since $\phi_1(x)$,$\phi_2(x) \geqslant 0$ and combining (2.4) and (2.5) when λ is large enough, then we have

$$-\Delta_{p(x)}\phi_1 \leqslant \lambda^{p(x)}(a(x)\phi_1^{\alpha(x)}\phi_2^{\gamma(x)} + h_1(x)), \quad \sigma < d(x) < 2l. \tag{2.6}$$

Obviously,

$$-\Delta_{p(x)}\phi_1 = 0 \leqslant \lambda^{p(x)}(a_1 + 1) \leqslant \lambda^{p(x)}(a(x)\phi_1^{\alpha(x)}\phi_2^{\gamma(x)} + h_1(x)), \quad 2l < d(x). \tag{2.7}$$

Combining (2.5)–(2.7), we can conclude that

$$-\Delta_{p(x)}\phi_1 \leqslant \lambda^{p(x)}(\lambda_1\phi_1^{\alpha(x)}\phi_2^{\gamma(x)} + h_1(x)), \quad \text{a.e. in } \Omega. \tag{2.8}$$

Similarly,

$$-\Delta_{q(x)}\phi_2 \leqslant \lambda^{q(x)}(b(x)\phi_1^{\delta(x)}\phi_2^{\beta(x)} + h_2(x)), \quad \text{a.e. in } \Omega. \tag{2.9}$$

From (2.8) and (2.9), we can see that (ϕ_1, ϕ_2) is a subsolution of (1.1).

Step 2. We construct a supersolution of (1.1).
We consider

$$\begin{cases} -\Delta_{p(x)} z_1 = \lambda^{p^+}(a_2+1)\mu_1 & \text{in } \Omega, \\ -\Delta_{q(x)} z_2 = \lambda^{q^+}(b_2+1)\mu_2 & \text{in } \Omega, \\ z_1 = z_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

when μ_1, μ_2 satisfy some conditions.

If we could prove that

$$(a_2+1)\mu_1 \geqslant a(x)\left[\max_{x\in\overline{\Omega}} z_1\right]^{\alpha+}\left[\max_{x\in\overline{\Omega}} z_2\right]^{\gamma+} + \max_{x\in\overline{\Omega}}|h_1(x)|, \tag{2.10}$$

and

$$(b_2+1)\mu_2 \geqslant b(x)\left[\max_{x\in\overline{\Omega}} z_1\right]^{\delta+}\left[\max_{x\in\overline{\Omega}} z_2\right]^{\beta+} + \max_{x\in\overline{\Omega}}|h_2(x)|, \tag{2.11}$$

we would see that (z_1, z_2) is a supersolution for (1.1).
From Lemma 1.3, we have

$$\max_{x\in\overline{\Omega}} z_1(x) \leqslant C_2\left(\lambda^{p^+}(a_2+1)\mu_1\right)^{\frac{1}{p^--1}} \quad \text{and} \quad \max_{x\in\overline{\Omega}} z_2(x) \leqslant C_2\left(\lambda^{p^+}(a_2+1)\mu_1\right)^{\frac{1}{p^--1}}.$$

Let

$$\mu_1 = 2\left[C_2(\lambda^{p^+}(a_2+1)\mu_1)^{\frac{1}{p^--1}}\right]^{\alpha+}\left[C_2(\lambda^{p^+}(b_2+1)\mu_2)^{\frac{1}{q^--1}}\right]^{\gamma+}.$$

We only need

$$\mu_2 \geqslant 2\left[C_2\left(\lambda^{p^+}(a_2+1)\mu_1\right)^{\frac{1}{p^--1}}\right]^{\delta+}\left[C_2(\lambda^{p^+}(b_2+1)\mu_2)^{\frac{1}{q^--1}}\right]^{\beta+}, \tag{2.12}$$

when μ_1, μ_2 are large enough.
Indeed, since $0 \leqslant \alpha^+ < p^- - 1$ and $0 \leqslant \beta^+ < q^- - 1$, from (2.11), we can see that μ_2 is large enough when μ_1 is large enough. From (H_2) and (H_3), relation (2.12) is satisfied.

According to (2.10) and (2.11), we can conclude that (z_1, z_2) is a supersolution for (1.1). It only remains to prove that $\phi_1 \leqslant z_1$ and $\phi_2 \leqslant z_2$.

In the definition of $v_1(x)$, let

$$\zeta = \frac{2}{l}\left(\max_{x\in\overline{\Omega}} \phi_1(x) + \max_{x\in\overline{\Omega}}|\nabla\phi_1(x)|\right).$$

We will claim that

$$\phi_1(x) \leqslant v_1(x), \quad \forall x \in \Omega. \tag{2.13}$$

From the definition of v_1, it is easy to see that

$$\phi_1(x) \leqslant 2\max_{x\in\overline{\Omega}}\phi_1(x) \leqslant v_1(x), \quad \text{when } d(x) = l,$$

and

$$\phi_1(x) \leqslant 2\max_{x\in\overline{\Omega}}\phi_1(x) \leqslant v_1(x), \quad \text{when } d(x) \geqslant l.$$

It only remains to prove that

$$\phi_1(x) \leqslant v_1(x), \quad \text{when } d(x) < l.$$

Since $v_1 - \phi_1 \in C^1(\overline{\partial\Omega_l})$, then there exists a point $x_0 \in \overline{\partial\Omega_l}$ such that

$$v_1(x_0) - \phi_1(x_0) = \min_{x_0\in\overline{\partial\Omega_l}}[v_1(x) - \phi_1(x)].$$

If $v_1(x_0) - \phi_1(x_0) < 0$, it is easy to see that $0 < d(x) < l$, and then

$$\nabla v_1(x_0) - \nabla\phi_1(x_0) = 0.$$

From the definition of v_1, we have

$$|\nabla v_1(x_0)| = \zeta = \frac{2}{l}\left(\max_{x\in\overline{\Omega}}\phi_1(x) + \max_{x\in\overline{\Omega}}|\nabla\phi_1(x)|\right) > |\nabla\phi_1(x_0)|.$$

It is a contradiction to $\nabla v_1(x_0) - \nabla\phi_1(x_0) = 0$. Thus (2.13) is valid.

Obviously, there exists a positive constant C_3 such that

$$\zeta \leqslant C_3\lambda.$$

Since $d(x) \in C^2(\overline{\partial\Omega_{3l}})$, according to the proof of Lemma 1.3, then there exists a positive constant C_4 such that

$$-\Delta_{p(x)}v_1(x) \leqslant C_*\zeta^{p(x)-1+\theta} \leqslant C_4\lambda^{p(x)-1+\theta}, \quad \text{a.e. in } \Omega, \text{where } \theta \in (0,1).$$

When $\eta \geqslant \lambda^{p^+}$ is large enough, we have

$$-\Delta_{p(x)}v_1(x) \leqslant \eta.$$

According to the comparison principle, we have

$$v_1(x) \leqslant w(x), \quad \forall x \in \Omega. \tag{2.14}$$

From (2.13) and (2.14), when $\eta \geqslant \lambda^{p^+}$ and the parameter $\lambda \geqslant 1$ is sufficiently large, we have

$$\phi_1(x) \leqslant v_1(x) \leqslant w(x), \quad \forall x \in \Omega. \tag{2.15}$$

According to the comparison principle, when μ is large enough, we have

$$v_1(x) \leqslant w(x) \leqslant z_1(x), \quad \forall x \in \Omega.$$

Combining the definition of $v_1(x)$ and (2.15), it is easy to see that

$$\phi_1(x) \leqslant v_1(x) \leqslant w(x) \leqslant z_1(x), \quad \forall x \in \Omega.$$

When $\mu \geqslant 1$ and the parameter λ is large enough, from Lemma 1.3, we can see that $\beta\Big(\lambda^{p^+}(\lambda_1+\mu_1)\mu\Big)$ is large enough, then $\lambda^{p^+}(\lambda_2+\mu_2)h\Big(\beta(\lambda^{p^+}(\lambda_1+\mu_1)\mu)\Big)$ is large enough. Similarly, we have $\phi_2 \leqslant z_2$. This completes the proof. $\square$

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