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A parameter uniform numerical method for singularly perturbed delay problems with discontinuous convection coefficient

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Abstract. In this paper a standard numerical method with piecewise linear interpolation on Shishkin mesh is suggested to solve singularly perturbed boundary value problem for second order ordinary delay differential equations with discontinuous convection coefficient and source term. An error estimate is derived by using the supremum norm and it is of almost first order convergence. Numerical results are provided to illustrate the theoretical results.

Keywords: Singularly perturbed problem; Convection–diffusion problem; Discontinuous convection coefficient; Shishkin mesh; Delay

2010 Mathematics Subject Classification: 65L10; 65L11; 65L12

1. INTRODUCTION

Singularly perturbed ordinary differential equations with a delay are ordinary differential equations in which the highest derivative is multiplied by a small parameter and involving at least one delay term. Such type of equations arises frequently from the mathematical modelling of various practical phenomena, for example, in the modelling of the human pupil-light reflex [14], the study of bistable devices [4] and variational problems in control theory [10], etc. It is important to develop suitable numerical methods to solve singularly perturbed differential equations with a delay, whose accuracy does not depend on the parameter ε, that is the methods are uniformly convergent with respect to the parameter.

In the past, only very few people had worked in the area Numerical Methods to Singularly Perturbed Delay Differential Equation(SPDDE). But in the recent years, there has been

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growing interest in this area. The authors of [12,6,16,1,2] suggested some numerical methods for singularly perturbed delay differential equations with continuous data. Recently few authors in [20,21,17] suggested some numerical method for singularly perturbed delay differential equations with discontinuous data.

In the present paper, as mentioned in the above abstract, motivated by the works of [7,3,13], we consider the following singularly perturbed boundary value problem (2.1) for second order ordinary delay differential equations with discontinuous convection coefficient and suggest a parameter uniform numerical method. It is proved that this method is uniformly convergent of order $O(N^{-1}\ln^2 N)$.

The present paper is organized as follows. In Section 2, the problem of study with discontinuous data is stated. Existence of the solution to the problem is established in Section 3. A maximum principle of the DDE is established in Section 4. Further a stability result is derived. Analytical results of the problem are derived in Section 5. The present numerical method is described in Section 6 and an error estimate is derived in Section 7. Section 8 presents numerical results.

2. Statement of the problem

Through out the paper, C, C_1 denote generic positive constants independent of the singular perturbation parameter ε and the discretization parameter N of the discrete problem. Further, I_N denotes $\{0, 1, \ldots, N\}$. The supremum norm is used for studying the convergence of the numerical solution to the exact solution to a singular perturbation problem: $\|u\|_\Omega = \sup_{x\in\Omega} |u(x)|$.

Motivated by the works of [8,3,13], we consider the following BVP for SPDDE.

Find $u \in Y = C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$ such that

$$\begin{cases} -\varepsilon u''(x) + a(x)u'(x) + b(x)u(x-1) = f(x), & x \in \Omega^*, \\ u(x) = \phi(x), \quad x \in [-1, 0], & u(2) = l, \end{cases} \tag{2.1}$$

$$a(x) = \begin{cases} a_1(x), & x \in [0,1], \\ a_2(x), & x \in (1,2], \end{cases} \qquad f(x) = \begin{cases} f_1(x), & x \in [0,1], \\ f_2(x), & x \in (1,2], \end{cases}$$

$$a_1(1-) \neq a_2(1+), \qquad f_1(1-) \neq f_2(1+),$$

$$a_1(x) \geq \alpha_1 > \alpha > 0, \qquad a_2(x) \leq -\alpha_2 < -\alpha < 0,$$

$$\alpha < \min\{\alpha_1, \alpha_2\}, \qquad \beta_0 \leq b(x) \leq \beta_1 < 0, \qquad \alpha + 2\beta_0 \geq \eta_0 > 0$$

where $0 < \varepsilon \ll 1$, a, f are sufficiently smooth and bounded in Ω^*. The function b is a sufficiently smooth function on $\overline{\Omega}$, $\Omega = (0,2)$, $\overline{\Omega} = [0,2]$, $\Omega^* = \Omega^- \cup \Omega^+$, $\Omega^- = (0,1)$, $\Omega^+ = (1,2)$ and ϕ is smooth on $[-1,0]$.

The above problem (2.1) is equivalent to

$$Pu(x) := \begin{cases} -\varepsilon u''(x) + a_1(x)u'(x) = f_1(x) - b(x)\phi(x-1), & x \in \Omega^-, \\ -\varepsilon u''(x) + a_2(x)u'(x) + b(x)u(x-1) = f_2(x), & x \in \Omega^+, \end{cases} \tag{2.2}$$

$$u(0) = \phi(0), \qquad u(1-) = u(1+), \qquad u'(1-) = u'(1+), \qquad u(2) = l,$$

where $u(1-)$ and $u(1+)$ denote the left and right limits of u at $x = 1$, respectively.

3. Existence result

For the reader's convenience some known results are briefly reported on this section and in Section 4. They can be used here with some modifications.

Theorem 3.1. *The problem (2.1) has a solution* $u \in C^0(\bar{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^*)$.

Proof. The proof is by construction. Let y_1 and y_2 be particular solutions of the DDEs,

$$-\varepsilon y_1''(x) + a_1(x)y_1'(x) + b(x)y_1(x-1) = f(x), \quad x \in \Omega^- \quad \text{and}$$
$$-\varepsilon y_2''(x) + a_2(x)y_2'(x) + b(x)y_2(x-1) = f(x), \quad x \in \Omega^+,$$

where $y_1 = \phi(x),\ x \in [-1, 0]$, $a_1,\ a_2 \in C^2(\bar{\Omega})$ with the above properties.

Consider the function

$$y(x) = \begin{cases} y_1(x) + A\phi_1(x), & x \in \Omega^-, \\ y_2(x) + \phi_2(x)[u(2) - y_2(2)] + B\phi_3(x), & x \in \Omega^+, \end{cases}$$

where ϕ_1, ϕ_2 and ϕ_3 are the solutions of the following problems, respectively:

$$\begin{cases} -\varepsilon\phi_1''(x) + a_1(x)\phi_1'(x) + b(x)\phi_1(x-1) = 0, & x \in \Omega, \\ \phi_1(x) = 0, \quad x \in [-1, 0], \quad \phi_1(2) = 1, \end{cases}$$
$$\begin{cases} -\varepsilon\phi_2''(x) + a_2(x)\phi_2'(x) + b(x)\phi_2(x-1) = 0, & x \in \Omega, \\ \phi_2(x) = 0, \quad x \in [-1, 0], \quad \phi_2(2) = 1, \end{cases}$$

and

$$\begin{cases} -\varepsilon\phi_3''(x) + a_2(x)\phi_3'(x) + b(x)\phi_3(x-1) = 0, & x \in \Omega, \\ \phi_3(x) = 1, \quad x \in [-1, 0], \quad \phi_3(2) = 0. \end{cases}$$

It is easy to see that the above function y satisfies the differential equation (2.1) and $u(0) = y(0)$ and $u(2) = y(2)$. Using the similar arguments given in [7, Theorem 1], and [19, Theorems 2,3] one can prove the existence of the solution. $\square$

Note: For the existence of $\phi_i,\ i = 1, 2, 3$ one may refer to [18,5].

4. Stability result

Theorem 4.1 (*Maximum Principle*). *Let* $w \in C^0(\overline{\Omega}) \cap C^2(\Omega^*)$ *be any function satisfying* $w(0) \geq 0$, $w(2) \geq 0$, $Pw(x) \geq 0$, $\forall x \in \Omega^*$ *and* $w'(1+) - w'(1-) = [w'](1) \leq 0$. *Then* $w(x) \geq 0$, $\forall x \in \overline{\Omega}$.

In the following we use the function

$$s(x) = \begin{cases} \dfrac{3}{2} + \dfrac{x}{2}, & x \in [0, 1], \\ 3 - x, & x \in [1, 2]. \end{cases} \tag{4.1}$$

Proof. Using the above function s and the procedure adopted in [20, Theorem 3.1], one can prove this theorem. $\square$

Corollary 4.2 (*Stability Result*). *For any* $u \in Y$ *we have*

$$|u(x)| \leq C \max\{|u(0)|, |u(2)|, \sup_{\xi \in \Omega^*} |Pu(\xi)|\}, \quad \forall\, x \in \overline{\Omega}. \tag{4.2}$$

Proof. Using the barrier function $\psi^{\pm}(x) = CC_1\ s(x) \pm u(x),\ x \in \overline{\Omega}$, where $C_1 = \max\{|u(0)|, |u(2)|,\ \sup_{\xi \in \Omega^*} |Pu(\xi)|\}$ and the procedure adopted in [20, Theorem 3.2], we can prove this corollary. $\square$

Note: An immediate consequence of the Corollary 4.2 is that, the solution of the BVP (2.1) is unique.

5. Analytical results

Theorem 5.1. *Let* u *be the solution of the problem* (2.1)*, then we have the following bounds*

$$\|u^{(k)}\|_{\Omega} \leq C\, \varepsilon^{-k}, \quad k = 0, 1,$$
$$\|u^{(k)}\|_{\Omega^*} \leq C\, \varepsilon^{-k}, \quad k = 2, 3.$$

Proof. Let $x \in \Omega^-$. Then we have,

$$\int_0^x a(s)u'(s)ds = [a(x)u(x) - a(0)u(0)] - \int_0^x a'(t)u(t)dt.$$

Integrating (2.2) from 0 to x we get,

$$\begin{aligned} -\varepsilon(u'(x) - u'(0)) &= -\int_0^x a(t)u'(t)dt + \int_0^x (f(t) - b(t)\phi(t-1))\, dt \\ &= -[a(x)u(x) - a(0)u(0)] + \int_0^x [a'(t)u(t) + (f(t) - b(t)\phi(t-1))]\, dt. \end{aligned}$$

Therefore,

$$\begin{aligned} \varepsilon u'(0) = \varepsilon u'(x) &- [a(x)u(x) - a(0)u(0)] \\ &+ \int_0^x [a'(t)u(t) + (f(t) - b(t)\phi(t-1))]\, dt. \end{aligned}$$

By the mean value theorem there exists a $z \in (0, \varepsilon)$ such that $|\varepsilon u'(z)| \leq 2\|u\|_{\overline{\Omega}}$. Therefore $\varepsilon|u'(0)| \leq C\left(\|u\|_{\overline{\Omega}} + \|f\|_{\Omega} + \|\phi\|_{[-1,0]}\right)$. Hence,

$$\varepsilon|u'(x)| \leq C \max\left\{\|u\|_{\overline{\Omega}}, \|f\|_{\Omega}, \|\phi\|_{[-1,0]}\right\}.$$

Similarly one can show that, $\varepsilon|u'(x)| \leq C,\ x \in \Omega^+$.

From (2.2) it is easy to show that $\|u^{(k)}\|_{\Omega^*} \leq C\varepsilon^{-k},\ k = 2, 3$. Hence the proof. $\square$

To derive uniform error estimates, we need sharper bounds on the derivatives of the solution u. We derive these using the following decomposition of the solution into smooth and singular components $u(x) = v(x) + w(x)$ where v can be written in the form $v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2$ and $v_0,\ v_1$ and v_2 are defined respectively to be the solutions of the following problems:

Find $v_0 \in C^0(\Omega^*) \cap C^1(\Omega^*)$ such that

$$a(x)v_0'(x) + b(x)v_0(x-1) = f(x), \quad x \in \Omega^*, \tag{5.1}$$

$$v_0(x) = \phi(x), \quad x \in [-1, 0], \qquad v_0(2) = l, \tag{5.2}$$

$v_1 \in C^0(\Omega^*) \cap C^1(\Omega^* \cup \{2\})$ such that

$$a(x)v_1'(x) + b(x)v_1(x-1) = v_0''(x), \quad x \in \Omega^*, \tag{5.3}$$

$$v_1(x) = 0, \quad x \in [-1, 0], \qquad v_1(2) = 0, \tag{5.4}$$

and $v_2 \in Y^*$ such that

$$Pv_2 = v_1''(x), \quad x \in \Omega^*, \tag{5.5}$$

$$v_2(x) = 0, \quad x \in [-1, 0], \qquad v_2(2) = 0. \tag{5.6}$$

We assume that, $\|v_0''\|_{\Omega^*} \le C$ and $\|v_1'''\|_{\Omega^*} \le C$.

Thus the smooth component v satisfies the following:
find $v \in C^0(\Omega^* \cup \{0,\ 2\}) \cap C^2(\Omega^*)$ such that

$$\begin{cases} Pv(x) = f(x), \quad x \in \Omega^*, \\ v(x) = v_0(x), \quad x \in [-1, 0], \qquad v(2) = v_0(2), \\ v(1) = v_0(1) + \varepsilon v_1(1) + \varepsilon^2 v_2(1). \end{cases} \tag{5.7}$$

Further w satisfies the problem, that is, find $w \in C^0(\Omega^* \cup \{0, 2\}) \cap C^2(\Omega^*)$ such that

$$\begin{cases} Pw(x) = 0, \quad x \in \Omega^*, \\ w(x) = 0, \quad x \in [-1, 0], \qquad [w](1) = -[v](1), \\ \quad [w'](1) = -[v'](1), \qquad w(2) = 0. \end{cases} \tag{5.8}$$

Note that $v + w = u \in Y^*$.

Theorem 5.2. *Let v and w be the solutions of the regular and singular components of the solution u. Then*

$$\|v^{(k)}\|_{\Omega^*} \le C(1 + \varepsilon^{2-k}), \quad k = 0, 1, 2, 3,$$

$$|w^{(k)}(x)| \le C \begin{cases} \varepsilon^{-k} \exp\left(-\alpha \dfrac{(1-x)}{\varepsilon}\right), \quad x \in \Omega^-, \\ \varepsilon^{-k} \exp\left(-\alpha \dfrac{(x-1)}{\varepsilon}\right) + \varepsilon^{-k+1} \exp\left(-\alpha \dfrac{(2-x)}{\varepsilon}\right), \\ \quad x \in \Omega^+,\ k = 0, 1, 2, 3. \end{cases}$$

Proof. Integrating the differential equation (5.1)–(5.4) separately on Ω^- and Ω^+, we get $\|v_i\| \le C,\ i = 0, 1$ and by the stability result we have $\|v_2\| \le C$. Therefore $\|v\|_{\Omega^*} \le C$. Similarly one can prove that $\|v^{(k)}\|_{\Omega^*} \le\ C(1 + \varepsilon^{2-k}),\ k = 0, 1, 2, 3.$

Note that $|w(x)| \leq |u(x)| + |v(x)|$. From the stability result we have $|u(1)| \leq C$. Further, $|v(1)| \leq C$. Therefore $|w(1)| \leq \eta$ (say). Now consider the barrier function

$$\varphi_1^{\pm}(x) = \eta \exp\left(\frac{-\alpha(1-x)}{\varepsilon}\right) \pm w(x), \quad x \in [0,1].$$

It is easy to check that $\varphi_1^{\pm}(0) = \eta \exp(\frac{-\alpha}{\varepsilon}) \pm w(0) \geq 0$, $\varphi_1^{\pm}(1) = \eta \pm w(1) \geq 0$.

Applying the result given in [9, Theorem 2.1] on $[0,1]$, we get $\varphi_1^{\pm}(x) \geq 0$.

Consider the barrier function

$$\varphi_2^{\pm}(x) = C_1 \left(\varepsilon + \exp\left(-\alpha\frac{(x-1)}{\varepsilon}\right) - \varepsilon \exp\left(-\alpha\frac{(2-x)}{\varepsilon}\right)\right) \pm w(x),$$
$$x \in [1,2].$$

It is easy to see that, $\varphi_2^{\pm}(1) = C_1\ (\varepsilon + 1 - \varepsilon \exp(\frac{-\alpha}{\varepsilon})) \pm w(1) \geq 0$, $\varphi_2^{\pm}(2) = C_1\ (\varepsilon + \exp(\frac{-\alpha}{\varepsilon}) - \varepsilon) \pm w(2) \geq 0$. Again applying the result given in [9, Theorem 2.1] on $[1,2]$, then we get $\varphi_2^{\pm}(x) \geq 0$. Using the procedure adopted in [7, Lemma 4], one can prove the rest of this theorem. □

Note: From the above theorem it is easy to show that

$$|u(x) - v(x)| \leq C \begin{cases} \exp(-\alpha(1-x)/\varepsilon), & x \in \Omega^-, \\ \exp(-\alpha(x-1)/\varepsilon) + \varepsilon \exp(-\alpha(2-x)/\varepsilon), & x \in \Omega^+. \end{cases} \tag{5.9}$$

6. DISCRETE PROBLEM

In this section, mesh selection strategy, namely piecewise uniform mesh (Shishkin mesh), is explained. Also upwind finite difference scheme with piecewise linear interpolation on Shishkin mesh for the problem (2.1) is described.

6.1. *Mesh selection strategy*

Since the BVP (2.1) exhibits strong interior layers at $x = 1$ and a weak boundary layer at $x = 2$, we choose a piecewise uniform Shishkin mesh on $[0,2]$. For this we divide the interval $[0,2]$ into five subintervals, namely $\Omega_1 = [0, 1-\tau_1]$, $\Omega_2 = [1-\tau_1, 1]$, $\Omega_3 = [1, 1+\tau_2]$, $\Omega_4 = [1+\tau_2, 2-\tau_2]$, $\Omega_5 = [2-\tau_2, 2]$, where $\tau_1 = \min\{0.5, \frac{2\varepsilon \ln N}{\alpha}\}$, $\tau_2 = \min\{0.25, \frac{2\varepsilon \ln N}{\alpha}\}$. Let $h_1 = 4N^{-1}(1-\tau_1)$, $h_2 = 4N^{-1}\tau_1$, $h_3 = 8N^{-1}\tau_2$, $h_4 = 4N^{-1}(1-2\tau_2)$. The mesh $\overline{\Omega}^N = \{x_0, x_1, \ldots, x_N\}$ is defined by

$$x_0 = 0.0, \qquad x_i = x_0 + ih_1, \quad i = 1(1)\frac{N}{4}, \qquad x_{i+\frac{N}{4}} = x_{\frac{N}{4}} + ih_2, \quad i = 1(1)\frac{N}{4},$$
$$x_{i+\frac{N}{2}} = x_{\frac{N}{2}} + ih_3, \quad i = 1(1)\frac{N}{8}, \qquad x_{i+\frac{5N}{8}} = x_{\frac{5N}{8}} + ih_4, \quad i = 1(1)\frac{N}{4},$$
$$x_{i+\frac{7N}{8}} = x_{\frac{7N}{8}} + ih_3, \quad i = 1(1)\frac{N}{8}.$$

6.2. A finite difference scheme for (2.2)

On $\overline{\Omega}^N$, we define the following scheme for the BVP (2.2):

$$P^N U(x_i) = -\varepsilon\delta^2 U(x_i) + a(x_i) DU(x_i) + b(x_i) U^I(x_i) = f^*(x_i),$$
$$x_i \in \Omega^* \cap \overline{\Omega}^N, \tag{6.1}$$
$$D^- U(x_{N/2}) = D^+ U(x_{N/2}), \tag{6.2}$$
$$U(x_0) = u(0), \qquad U(x_N) = u(2), \tag{6.3}$$

where

$$\delta^2 U(x_i) = \frac{2[D^+ U(x_i) - D^- U(x_i)]}{x_{i+1} - x_{i-1}}, \qquad D^- U(x_i) = \frac{U(x_i) - U(x_{i-1})}{x_i - x_{i-1}},$$

$$D^+ U(x_i) = \frac{U(x_{i+1}) - U(x_i)}{x_{i+1} - x_i}, \qquad DU(x_i) = \begin{cases} D^- U(x_i), & x_i \in \Omega^- \cap \overline{\Omega}^N, \\ D^+ U(x_i), & x_i \in \Omega^+ \cap \overline{\Omega}^N, \end{cases}$$

$$U^I(x_i) = \begin{cases} 0, \quad x_i \in \Omega^- \cap \overline{\Omega}^N, \\ U(x_j)\dfrac{x_{j+1} - (x_i - 1)}{x_{j+1} - x_j} + U(x_{j+1})\dfrac{(x_i - 1) - x_j}{x_{j+1} - x_j}, \\ \qquad x_i \in \Omega^+ \cap \overline{\Omega}^N, x_j \le x_i - 1 \le x_{j+1}, \end{cases}$$

$$f^*(x_i) = \begin{cases} f(x_i) - b(x_i)\phi(x_i - 1), & x_i \in \Omega^- \cap \overline{\Omega}^N, \\ f(x_i), & x_i \in \Omega^+ \cap \overline{\Omega}^N. \end{cases}$$

6.3. Discrete stability result

Lemma 6.1 (*Discrete Maximum Principle*)**.** *Let* $Z(x_i)$ *be a mesh function satisfying* $Z(x_0) \ge 0$, $Z(x_N) \ge 0$, $P^N Z(x_i) \ge 0$, $i \in I_N \setminus \{0, N/2, N\}$ *and* $(D^+ - D^-)Z(x_{N/2}) = [DZ](x_{N/2}) \le 0$. *Then* $Z(x_i) \ge 0$, $\forall x_i \in \overline{\Omega}^N$.

Proof. Define $s(x_i) = \begin{cases} \frac{3}{2} + \frac{x_i}{2}, & x_i \in [0,1] \cap \overline{\Omega}^N, \\ 3 - x_i, & x_i \in [1,2] \cap \overline{\Omega}^N. \end{cases}$

Note that $P^N s(x_i) > 0, \forall x_i \in \Omega^* \cap \overline{\Omega}^N$, $[Ds](x_{N/2}) < 0$, $s(x_i) > 0, \forall x_i \in \overline{\Omega}^N$.

Let $\mu^* = \max\{\frac{-Z(x_i)}{s(x_i)} : x_i \in \overline{\Omega}^N\}$. Then there exists $x_i^* \in \overline{\Omega}^N$ such that $Z(x_i^*) + \mu^* s(x_i^*) = 0$ and $Z(x_i) + \mu^* s(x_i) \ge 0$, $\forall x_i \in \overline{\Omega}^N$. Therefore the mesh function $(Z + \mu^* s)$ attains its minimum at $x_i = x_i^*$. Suppose the theorem does not hold true, then $\mu^* > 0$.

Case (i): $(x_i^* \in \Omega^- \cap \overline{\Omega}^N)$

$$0 < P^N(Z + \mu s)(x_i^*) = -\varepsilon\delta^2(Z + \mu^* s)(x_i^*) + a_1(x_i^*) D^-(Z + \mu^* s)(x_i^*) \le 0.$$

It is a contradiction.

Case (ii): $(x_i^* \in \Omega^+ \cap \overline{\Omega}^N)$

$$\begin{aligned} 0 < P^N(Z+\mu s)(x_i^*) = &-\varepsilon\delta^2(Z+\mu^* s)(x_i^*) + a_2(x_i^*)D^+(Z+\mu^* s)(x_i^*) \\ &+ b(x_i^*)(Z+\mu^* s)^I(x_i^*) \le 0. \end{aligned}$$

It is a contradiction.

Case (iii): $(x_i^* = x_{N/2})$

$$0 \le [D(Z+\mu^* s)](x_{N/2}) = [DZ](x_{N/2}) + \mu^*[Ds](x_{N/2}) < 0.$$

It is a contradiction. Hence the proof of the theorem. □

Lemma 6.2. *For any mesh function $U(x_i)$ we have*

$$|U(x_i)| \le C\max\big\{|U(x_0)|,\ |U(x_N)|,\ \max_{j\in I_N\setminus\{0,N/2,N\}} P^N U(x_j)\big\}, \quad x_i \in \overline{\Omega}^N.$$

Proof. One can easily prove this lemma by using Lemma 6.1 and the discrete barrier function $\varphi^\pm(x_i) = CC_1 s(x_i) \pm U(x_i),\ x_i \in \overline{\Omega}^N$, where $C_1 = \max\big\{|U(x_0)|, |U(x_N)|, \max_{j\in I_N\setminus\{0,N/2,N\}} P^N U(x_j)\big\}$. □

Analogous to the continuous function u, we decompose the numerical solution $U(x_i)$ defined by (6.1)–(6.3) as $U(x_i) = V(x_i) + W(x_i)$, where $V(x_i)$ and $W(x_i)$ satisfy the following:

$$\begin{cases} P^N V(x_i) = f^*(x_i), \quad i \in I_N\setminus\{0,N/2,N\}, \\ V(x_0) = v(0), \qquad [D]V(x_{N/2}) = [v'](1), \qquad V(x_N) = v(2), \end{cases} \tag{6.4}$$

and

$$\begin{cases} P^N W(x_i) = 0, \quad i \in I_N\setminus\{0,N/2,N\}, \\ W(x_0) = w(0), \qquad W(x_N) = w(2), \qquad [D]W(x_{N/2}) = -[D]V(x_{N/2}). \end{cases} \tag{6.5}$$

Theorem 6.3. *Let $U(x_i)$ be the numerical solution of (2.2) defined by (6.1)–(6.3) and further let $V(x_i)$ be the numerical solution of (5.7) given by (6.4). Then,*

$$|U(x_i) - V(x_i)| \le C\begin{cases} N^{-1}, \quad i \in I_N\setminus\{N/4+1,\dots,5N/8-1\} \\ N^{-1} + |U(x_{N/2}) - V(x_{N/2})|, \quad \textit{otherwise}. \end{cases}$$

Proof. Consider a mesh function $\varphi^\pm(x_i) = C_1N^{-1}[s(x_i)+\eta(x_i)] + C_1\psi(x_i)|U(x_{N/2}) - V(x_{N/2})| \pm (U(x_i) - V(x_i)),\ i \in I_N$ where

$$s(x_i) = \begin{cases} \dfrac{3}{2} + \dfrac{x_i}{2}, & x_i \in \Omega^-\cap\bar{\Omega}^N, \\ 3 - x_i, & x_i \in \Omega^+\cap\bar{\Omega}^N, \end{cases}, \qquad \eta(x_i) = \begin{cases} 2 + x_i, & x_i \in \Omega^-\cap\bar{\Omega}^N, \\ 2 - x_i, & x_i \in \Omega^+\cap\bar{\Omega}^N, \end{cases}$$

$$\psi(x_i) = \begin{cases} 0, & i \in I_N\setminus\{N/4+1,\dots,5N/8-1\} \\ \eta(x_i)|U(x_{N/2}) - V(x_{N/2})|, & i = N/4+1,\dots,5N/8-1. \end{cases}$$

It is easy to see that, $\varphi^{\pm}(x_0) \geq 0$ and $\varphi^{\pm}(x_N) \geq 0$ for a suitable $C_1 > 0$. Further,

$$\begin{aligned}
P^N\varphi^{\pm}(x_i) =& C_1\left[a(x_i)N^{-1}\left(1+\frac{s(x_i)-s(x_{i-1})}{x_i-x_{i-1}}\right)\right] \\
&+\ C_1[a(x_i)D^{-}\psi(x_i)] \pm P^N(U(x_i)-V(x_i)), \quad x_i \in \Omega^{-}\cap\bar{\Omega}^N. \\
P^N\varphi^{\pm}(x_i) =& C_1\left[a(x_i)N^{-1}\left(-1+\frac{s(x_{i+1})-s(x_i)}{x_{i+1}-x_i}\right)+b(x_i)N^{-1}[s(x_i)+\eta(x_i)]^I\right] \\
&+\ C_1|U(x_{N/2})-V(x_{N/2})|[a(x_i)D^{+}\psi(x_i)+b(x_i)\psi(x_i)^I] \\
&\pm\ P^N(U(x_i)-V(x_i)), \quad x_i \in \Omega^{+}\cap\bar{\Omega}^N.
\end{aligned}$$

Note that, for $i \in I_N \setminus \{0,\ N/2,\ N\}$, we have $P^N(U(x_i)-V(x_i)) = 0$.

Hence $P^N\varphi^{\pm}(x_i) \geq 0,\ i \in I_N\setminus\{0, N/2, N\}$ by a proper choice of C_1.

Let $x_i = x_{N/2}$, then $[D]\varphi^{\pm}(x_i) = -C_1\frac{7N^{-1}}{2} - C_1 2|U(x_{N/2})-V(x_{N/2})| \pm [[D]U(x_i) - [D]V(x_i)] \leq 0$, by (6.2), (6.4) and $(x_{i+1}-x_{i-1})|\delta^2 v(x_i)| \leq \max_{[x_{i-1},x_{i+1}]}|v''(x)|N^{-1}$ [15, page 52]. Then by Lemma 6.1, we have $\varphi^{\pm}(x_i) \geq 0, \forall i \in I_N$. Hence the proof. □

7. Error analysis

In this section we derive an error estimate for the numerical solution obtained by the scheme (6.1)–(6.3) for the problem (2.1).

Lemma 7.1. *Let v be the solution of the problem (5.7) and let $V(x_i)$ be its numerical solution defined by (6.4). Then, $|v(x_i)-V(x_i)| \leq CN^{-1},\ i \in I_N$.*

Proof. Now,

$$\begin{aligned}
P^N(v(x_i)-V(x_i)) =& -\varepsilon\left(\delta^2-\frac{d^2}{dx^2}\right)v(x_i) + a(x_i)\left(D^{-}-\frac{d}{dx}\right)v(x_i) \\
&+\quad b(x_i)\begin{cases}0, & i=1,2,\dots,N/2-1,\\ v^I(x_i)-v(x_i-1), & i=N/2+1,\dots,N-1.\end{cases}
\end{aligned}$$

Since $|v^I(x_i)-v(x_i-1)| \leq CN^{-2}$ [11], then $|P^N(v(x_i)-V(x_i))| \leq CN^{-1},\ i \in I_N\setminus\{0,N/2,N\}$. Then by Lemma 6.2, we have $|v(x_i)-V(x_i)| \leq CN^{-1},\ i \in I_N$. Hence the proof. □

Lemma 7.2. *Let w be the solution to the problem (5.8) and let $W(x_i)$ be its numerical solution defined by (6.5). If $\varepsilon \leq CN^{-1}$, then we have $|w(x_i)-W(x_i)| \leq CN^{-1}\ln^2 N$, $i \in I_N$.*

Proof. Note that $|w(x_i)-W(x_i)| \leq |u(x_i)-U(x_i)| + |v(x_i)-V(x_i)|$. Then by Eq. (5.9), Theorem 6.3 and Lemma 7.1, we have

$$\begin{aligned}
&|u(x_i)-U(x_i)| \leq |U(x_i)-V(x_i)| + |v(x_i)-V(x_i)| + |u(x_i)-v(x_i)| \\
&\leq C\begin{cases}
N^{-1}+\exp(-\alpha\tau_1/\varepsilon), \quad i=0,1,\dots,N/4,\\
N^{-1}+|U(x_{N/2})-V(x_{N/2})|+\exp(-\alpha(1-x_i)/\varepsilon),\\
\quad i=N/4+1,\dots,N/2,\\
N^{-1}+|U(x_{N/2})-V(x_{N/2})|+\exp(-\alpha(x_i-1)/\varepsilon),\\
\quad i=N/2+1,\dots,5N/8,\\
N^{-1}+\exp(-\alpha\tau_2/\varepsilon), \quad i=5N/8+1,\dots,N,
\end{cases}
\end{aligned}$$

$$\leq C\begin{cases} N^{-1}, \quad i=0,1,\ldots,N/4,\\ N^{-1}+|U(x_{N/2})-V(x_{N/2})|+\exp(-\alpha(1-x_i)/\varepsilon),\\ \quad i=N/4+1,\ldots,N/2,\\ N^{-1}+|U(x_{N/2})-V(x_{N/2})|+\exp(-\alpha(x_i-1)/\varepsilon),\\ \quad i=N/2+1,\ldots,5N/8,\\ N^{-1}, \quad i=5N/8+1,\ldots,N.\end{cases}$$

Therefore

$$\begin{aligned}|w(x_i)-W(x_i)| &\leq |u(x_i)-U(x_i)|+|v(x_i)-V(x_i)|\\ &\leq CN^{-1}, \quad i=0,1,\ldots,N/4,\ 5N/8,\ldots,\ N. \end{aligned} \tag{7.1}$$

Now consider a mesh function

$$\varphi^{\pm}(x_i)=\begin{cases} C_1N^{-1}\left[[2+x_i]+\dfrac{\tau}{\varepsilon^2}[x_i-1+\tau_1]\right]\pm(w(x_i)-W(x_i)),\\ \quad x_i\in[1-\tau_1,1)\cap\overline{\Omega}^N,\\ C_1N^{-1}\left[[2-x_i]+\dfrac{\tau}{\varepsilon^2}[1+\tau_2-x_i]\right]\pm(w(x_i)-W(x_i)),\\ \quad x_i\in[1,1+\tau_2]\cap\overline{\Omega}^N,\end{cases}$$

where $\tau=\min\{\tau_1,\ \tau_2\}$. From (7.1), it is clear that $\varphi^{\pm}(x_{N/4})\geq 0$ and $\varphi^{\pm}(x_{5N/8})\geq 0$ for a suitable choice of $C_1>0$.

$$P^N\varphi^{\pm}(x_i)=\begin{cases} C_1N^{-1}a_1\left[1+\dfrac{\tau}{\varepsilon^2}\right]\pm P^N(w(x_i)-W(x_i)),\\ \quad x_i\in[1-\tau_1,1)\cap\overline{\Omega}^N,\\ C_1N^{-1}\left[a_2\left[-1-\dfrac{\tau}{\varepsilon^2}\right]+b(x_i)(2-x_i)^I+\dfrac{\tau}{\varepsilon^2}[1+\tau_2-x_i]^I\right]\\ \quad \pm P^N(w(x_i)-W(x_i)), \quad x_i\in(1,1+\tau_2]\cap\overline{\Omega}^N\end{cases}$$

$$\geq\begin{cases} C_1N^{-1}\alpha\left[1+\dfrac{\tau}{\varepsilon^2}\right]\pm P^N(w(x_i)-W(x_i)),\\ \quad x_i\in[1-\tau_1,1)\cap\overline{\Omega}^N,\\ C_1N^{-1}[\alpha+2\beta_0]\left[1+\dfrac{\tau}{\varepsilon^2}\right]\pm P^N(w(x_i)-W(x_i)),\\ \quad x_i\in(1,1+\tau_2]\cap\overline{\Omega}^N.\end{cases}$$

Note that,

$$\begin{aligned}P^N(w(x_i)-W(x_i)) &= P^Nw(x_i)-P^NW(x_i)\\ &=\begin{cases} -\varepsilon\left(\delta^2-\dfrac{d^2}{dx^2}\right)w(x_i)+a_1(x_i)\left(D^- -\dfrac{d}{dx}\right)w(x_i),\\ \quad x_i\in[1-\tau_1,1)\cap\overline{\Omega}^N,\\ -\varepsilon\left(\delta^2-\dfrac{d^2}{dx^2}\right)w(x_i)+a_2(x_i)\left(D^+ -\dfrac{d}{dx}\right)w(x_i)\\ \quad +b(x_i)[w^I(x_i)-w(x_i-1)], \quad x_i\in(1,1+\tau_2]\cap\overline{\Omega}^N.\end{cases}\end{aligned}$$

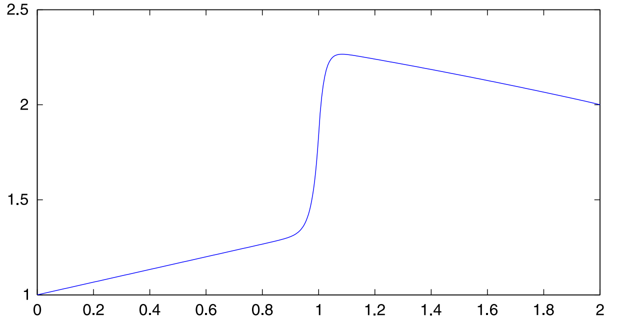

Fig. 1. Numerical solution of the problem stated in Example 8.1.

Also note that, $|w^I(x_i) - w(x_i - 1)| \leq CN^{-1}$ [11]. Further, $|P^N(w(x_i) - W(x_i))| \leq C_2\varepsilon^{-2}N^{-1}$, where $C_2 > 0$ a constant independent of ε and N.

Therefore, $P^N\varphi^{\pm}(x_i) \geq 0,\ i \in \{N/4+1, \dots, N/2-1, N/2+1, \dots, 5N/8-1\}$. Then by the Lemma 6.1, we have $|w(x_i) - W(x_i)| \leq CN^{-1}\ln^2 N,\ i = N/4+1, \dots, 5N/8-1$. Hence the proof. $\square$

Theorem 7.3. *Let u be the solution of the problem (2.2), $U(x_i)$ be its numerical solution defined by (6.1)–(6.3). Then $|u(x_i) - U(x_i)| \leq CN^{-1}\ln^2 N,\ i \in I_N$.*

Proof. The desired estimate follows from the fact that $u = v + w,\ U = V + W$ and from the Lemmas 7.1 and 7.2. $\square$

8. NUMERICAL EXAMPLES

In this section, three examples are given to illustrate the numerical method discussed in this paper. We use the double mesh principle to estimate the error and compute the experiment rate of convergence in our computed solutions for all problems. For this we put $D_\varepsilon^M = \max_{0\leq i\leq M}|U_i^M - U_{2i}^{2M}|$, where U_i^M and U_{2i}^{2M} are the ith components of the numerical solutions on meshes of M and $2M$ points respectively. We compute the uniform error and rate of convergence as $D^M = \max_\varepsilon D_\varepsilon^M$ and $p^M = \log_2\left(\frac{D^M}{D^{2M}}\right)$. For the following examples the numerical results are presented for the values of perturbation parameter $\varepsilon \in \{2^{-27},\ 2^{-12},\ \cdots,\ 2^{-6}\}$.

Example 8.1.

$$\begin{cases} -\varepsilon u''(x) + 3u'(x) - u(x-1) = 0, & x \in \Omega^- \\ -\varepsilon u''(x) - 4u'(x) - u(x-1) = 0, & x \in \Omega^+ \\ u(x) = 1, \quad x \in [-1, 0], \quad u(2) = 2. \end{cases} \tag{8.1}$$

Table 1 presents the values of D^N and p^N for this problem. Figs. 1 and 2 represent the numerical solution and the maximum point wise error for this problem, respectively.

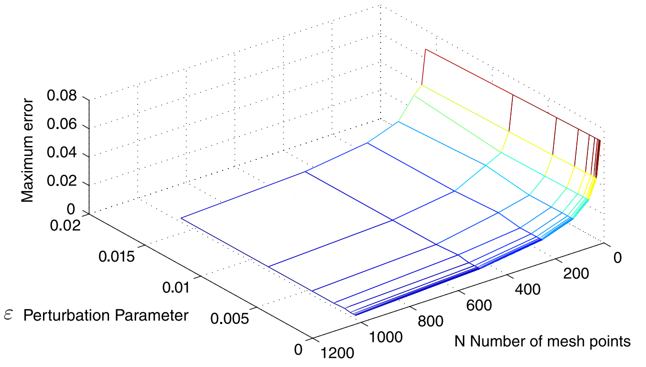

Fig. 2. Maximum point wise error for the problem stated in Example 8.1.

Table 1
Numerical results for the problem stated in Example 8.1.

ε ↓	N (Number of mesh points) 16	32	64	128	256	512	1024
2^{-6}	6.9037e−2	4.4673e−2	3.9152e−2	2.4284e−2	1.6478e−2	1.0510e−2	6.3683e−3
2^{-7}	7.1024e−2	4.5980e−2	3.3034e−2	2.7566e−2	1.6621e−2	1.0508e−2	6.3483e−3
2^{-8}	7.2001e−2	4.6628e−2	3.3467e−2	2.3775e−2	1.8534e−2	1.0523e−2	6.3774e−3
2^{-9}	7.2486e−2	4.6951e−2	3.3683e−2	2.3943e−2	1.6407e−2	1.1663e−2	6.3716e−3
2^{-10}	7.2727e−2	4.7112e−2	3.3790e−2	2.4026e−2	1.6488e−2	1.0447e−2	6.9885e−3
2^{-11}	7.2847e−2	4.7192e−2	3.3844e−2	2.4068e−2	1.6528e−2	1.0484e−2	6.3334e−3
2^{-12}	7.2907e−2	4.7232e−2	3.3871e−2	2.4089e−2	1.6548e−2	1.0502e−2	6.3508e−3
2^{-13}	7.2937e−2	4.7253e−2	3.3884e−2	2.4099e−2	1.6558e−2	1.0511e−2	6.3595e−3
2^{-14}	7.2952e−2	4.7263e−2	3.3891e−2	2.4104e−2	1.6563e−2	1.0516e−2	6.3638e−3
2^{-15}	7.2960e−2	4.7268e−2	3.3894e−2	2.4107e−2	1.6566e−2	1.0518e−2	6.3660e−3
2^{-16}	7.2964e−2	4.7270e−2	3.3896e−2	2.4108e−2	1.6567e−2	1.0519e−2	6.3671e−3
2^{-17}	7.2966e−2	4.7271e−2	3.3897e−2	2.4109e−2	1.6567e−2	1.0520e−2	6.3676e−3
2^{-18}	7.2967e−2	4.7272e−2	3.3897e−2	2.4109e−2	1.6568e−2	1.0520e−2	6.3679e−3
2^{-19}	7.2967e−2	4.7272e−2	3.3897e−2	2.4109e−2	1.6568e−2	1.0520e−2	6.3680e−3
2^{-20}	7.2967e−2	4.7272e−2	3.3898e−2	2.4109e−2	1.6568e−2	1.0520e−2	6.3681e−3
2^{-21}	7.2967e−2	4.7273e−2	3.3898e−2	2.4109e−2	1.6568e−2	1.0520e−2	6.3681e−3
2^{-22}	7.2967e−2	4.7273e−2	3.3898e−2	2.4109e−2	1.6568e−2	1.0520e−2	6.3682e−3
2^{-23}	7.2967e−2	4.7273e−2	3.3898e−2	2.4109e−2	1.6568e−2	1.0520e−2	6.3682e−3
2^{-24}	7.2967e−2	4.7273e−2	3.3898e−2	2.4109e−2	1.6568e−2	1.0520e−2	6.3682e−3
2^{-25}	7.2967e−2	4.7273e−2	3.3898e−2	2.4109e−2	1.6568e−2	1.0520e−2	6.3682e−3
2^{-26}	7.2967e−2	4.7273e−2	3.3898e−2	2.4109e−2	1.6568e−2	1.0520e−2	6.3681e−3
2^{-27}	7.2967e−2	4.7273e−2	3.3898e−2	2.4110e−2	1.6568e−2	1.0521e−2	6.3682e−3
D^N	7.2967e−2	4.7273e−2	3.9152e−2	2.7566e−2	1.8534e−2	1.1663e−2	6.9885e−3
p^N	6.2625e−1	2.7192e−1	5.0620e−1	5.7270e−1	6.6825e−1	7.3887e−1	–

Example 8.2.

$$\begin{cases} -\varepsilon u''(x) + (3 + x^2)u'(x) - u(x-1) = 1, & x \in \Omega^- \\ -\varepsilon u''(x) - (4 + x)u'(x) - u(x-1) = -1, & x \in \Omega^+ \\ u(x) = 1, \quad x \in [-1, 0], & u(2) = 2. \end{cases} \tag{8.2}$$

Table 2
Numerical results for the problem stated in Example 8.2.

ε ↓	N (Number of mesh points) 16	32	64	128	256	512	1024
2^{-6}	4.4334e−2	2.6901e−2	2.4542e−2	1.4287e−2	9.6205e−3	6.1830e−3	3.7665e−3
2^{-7}	4.5728e−2	2.7859e−2	1.9444e−2	1.6827e−2	9.7456e−3	6.1893e−3	3.7555e−3
2^{-8}	4.6412e−2	2.8332e−2	1.9796e−2	1.3927e−2	1.1337e−2	6.2057e−3	3.7820e−3
2^{-9}	4.6751e−2	2.8566e−2	1.9970e−2	1.4069e−2	9.5858e−3	7.1177e−3	3.7784e−3
2^{-10}	4.6920e−2	2.8683e−2	2.0057e−2	1.4140e−2	9.6544e−3	6.1463e−3	4.2869e−3
2^{-11}	4.7004e−2	2.8742e−2	2.0101e−2	1.4176e−2	9.6887e−3	6.1776e−3	3.7477e−3
2^{-12}	4.7046e−2	2.8771e−2	2.0122e−2	1.4193e−2	9.7058e−3	6.1932e−3	3.7624e−3
2^{-13}	4.7067e−2	2.8785e−2	2.0133e−2	1.4202e−2	9.7144e−3	6.2010e−3	3.7697e−3
2^{-14}	4.7077e−2	2.8792e−2	2.0139e−2	1.4207e−2	9.7186e−3	6.2049e−3	3.7734e−3
2^{-15}	4.7083e−2	2.8796e−2	2.0141e−2	1.4209e−2	9.7208e−3	6.2068e−3	3.7752e−3
2^{-16}	4.7085e−2	2.8798e−2	2.0143e−2	1.4210e−2	9.7219e−3	6.2078e−3	3.7761e−3
2^{-17}	4.7087e−2	2.8799e−2	2.0143e−2	1.4211e−2	9.7224e−3	6.2083e−3	3.7766e−3
2^{-18}	4.7087e−2	2.8799e−2	2.0144e−2	1.4211e−2	9.7227e−3	6.2086e−3	3.7768e−3
2^{-19}	4.7088e−2	2.8800e−2	2.0144e−2	1.4211e−2	9.7228e−3	6.2087e−3	3.7769e−3
2^{-20}	4.7088e−2	2.8800e−2	2.0144e−2	1.4211e−2	9.7229e−3	6.2087e−3	3.7770e−3
2^{-21}	4.7088e−2	2.8800e−2	2.0144e−2	1.4211e−2	9.7229e−3	6.2088e−3	3.7770e−3
2^{-22}	4.7088e−2	2.8800e−2	2.0144e−2	1.4211e−2	9.7229e−3	6.2088e−3	3.7770e−3
2^{-23}	4.7088e−2	2.8800e−2	2.0144e−2	1.4211e−2	9.7229e−3	6.2088e−3	3.7770e−3
2^{-24}	4.7088e−2	2.8800e−2	2.0144e−2	1.4211e−2	9.7229e−3	6.2088e−3	3.7770e−3
2^{-25}	4.7088e−2	2.8800e−2	2.0144e−2	1.4211e−2	9.7229e−3	6.2088e−3	3.7770e−3
2^{-26}	4.7088e−2	2.8800e−2	2.0144e−2	1.4211e−2	9.7229e−3	6.2088e−3	3.7770e−3
2^{-27}	4.7088e−2	2.8800e−2	2.0144e−2	1.4211e−2	9.7227e−3	6.2090e−3	3.7772e−3
D^N	4.7088e−2	2.8800e−2	2.4542e−2	1.6827e−2	1.1337e−2	7.1177e−3	4.2869e−3
p^N	7.0929e−1	2.3083e−1	5.4448e−1	5.6974e−1	6.7152e−1	7.3149e−1	–

Table 2 presents the values of D^N and p^N for this problem. Figs. 3 and 4 represent the numerical solution and the maximum point wise error for this problem, respectively.

Example 8.3.

$$\begin{cases} -\varepsilon u''(x) + (\exp(x) + x^2)u'(x) - u(x-1) = \exp(x^2), \quad x \in \Omega^- \\ -\varepsilon u''(x) - (4 + \exp(-x))u'(x) - u(x-1) = 0, \quad x \in \Omega^+ \\ u(x) = 1, \quad x \in [-1, 0], \quad u(2) = 2. \end{cases} \tag{8.3}$$

Table 3 presents the values of D^N and p^N for this problem.

9. Discussion

A BVP for one type of SPDDEs is considered. To obtain an approximate solution to this type of problem, an upwind finite difference scheme with piecewise linear interpolation on Shishkin mesh is presented. The method is shown to be of almost first order convergence. This is very much reflected on the numerical results (Tables 1–3). Also Figs. 1 and 3 represent that the model problems stated in Examples 8.1 and 8.2 exhibit strong interior layers at $x = 1$ and a weak boundary layer at $x = 2$. Figs. 2 and 4 represent the maximum point wise error for the numerical solutions. Further these Figs. 2 and 4 represent the uniform convergence of the

Table 3
Numerical results for the problem stated in Example 8.3.

ε	N (Number of mesh points)						
$\downarrow$	16	32	64	128	256	512	1024
2^{-6}	6.8150e−2	3.7767e−2	2.8472e−2	1.5380e−2	9.2391e−3	5.2438e−3	2.9335e−3
2^{-7}	6.9961e−2	3.8976e−2	2.3702e−2	1.7956e−2	9.3354e−3	5.3363e−3	3.0515e−3
2^{-8}	7.0863e−2	3.9595e−2	2.4127e−2	1.4829e−2	1.0949e−2	5.3842e−3	3.0680e−3
2^{-9}	7.1312e−2	3.9913e−2	2.4339e−2	1.4992e−2	9.2106e−3	6.2932e−3	3.0871e−3
2^{-10}	7.1537e−2	4.0072e−2	2.4446e−2	1.5074e−2	9.2781e−3	5.3106e−3	3.5991e−3
2^{-11}	7.1649e−2	4.0151e−2	2.4499e−2	1.5115e−2	9.3119e−3	5.3405e−3	3.0583e−3
2^{-12}	7.1705e−2	4.0191e−2	2.4525e−2	1.5135e−2	9.3288e−3	5.3554e−3	3.0725e−3
2^{-13}	7.1733e−2	4.0211e−2	2.4539e−2	1.5145e−2	9.3372e−3	5.3629e−3	3.0795e−3
2^{-14}	7.1747e−2	4.0221e−2	2.4545e−2	1.5150e−2	9.3415e−3	5.3666e−3	3.0831e−3
2^{-15}	7.1754e−2	4.0226e−2	2.4549e−2	1.5153e−2	9.3436e−3	5.3685e−3	3.0848e−3
2^{-16}	7.1758e−2	4.0228e−2	2.4550e−2	1.5154e−2	9.3446e−3	5.3694e−3	3.0857e−3
2^{-17}	7.1760e−2	4.0229e−2	2.4551e−2	1.5155e−2	9.3452e−3	5.3699e−3	3.0862e−3
2^{-18}	7.1760e−2	4.0230e−2	2.4552e−2	1.5155e−2	9.3454e−3	5.3701e−3	3.0864e−3
2^{-19}	7.1761e−2	4.0230e−2	2.4552e−2	1.5155e−2	9.3456e−3	5.3702e−3	3.0865e−3
2^{-20}	7.1761e−2	4.0230e−2	2.4552e−2	1.5155e−2	9.3456e−3	5.3703e−3	3.0865e−3
2^{-21}	7.1761e−2	4.0231e−2	2.4552e−2	1.5155e−2	9.3457e−3	5.3703e−3	3.0866e−3
2^{-22}	7.1761e−2	4.0231e−2	2.4552e−2	1.5155e−2	9.3457e−3	5.3703e−3	3.0866e−3
2^{-23}	7.1761e−2	4.0231e−2	2.4552e−2	1.5155e−2	9.3457e−3	5.3703e−3	3.0866e−3
2^{-24}	7.1761e−2	4.0231e−2	2.4552e−2	1.5155e−2	9.3457e−3	5.3704e−3	3.0866e−3
2^{-25}	7.1761e−2	4.0231e−2	2.4552e−2	1.5155e−2	9.3457e−3	5.3704e−3	3.0866e−3
2^{-26}	7.1761e−2	4.0231e−2	2.4552e−2	1.5155e−2	9.3457e−3	5.3703e−3	3.0866e−3
2^{-27}	7.1761e−2	4.0231e−2	2.4552e−2	1.5155e−2	9.3455e−3	5.3703e−3	3.0868e−3
D^N	7.1761e−2	4.0231e−2	2.8472e−2	1.7956e−2	1.0949e−2	6.2932e−3	3.5991e−3
p^N	8.3491e−1	4.9876e−1	6.6508e−1	7.1360e−1	7.9898e−1	8.0616e−1	–

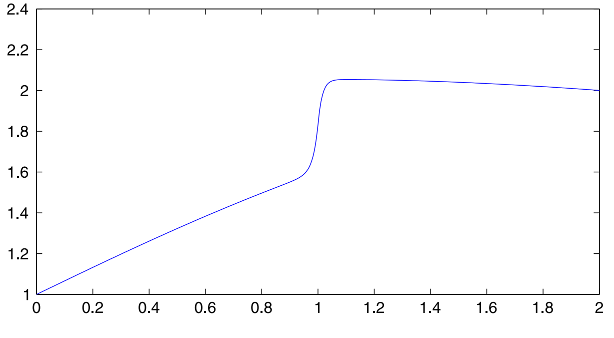

Fig. 3. Numerical solution of the problem stated in Example 8.2.

numerical method presented in this paper. The authors of [7] have considered second order ordinary differential equations with discontinuous convection coefficient with different signs on different subdomains. The solution to the problem considered in [7] exhibits strong interior layers at an interior point. Whereas the problem considered in this paper exhibits strong interior layers at $x = 1$ and weak boundary layer at $x = 2$ (see Theorem 5.2). This is due to the presence of the delay term with the differential equation. Therefore, to accommodate these interior layers and boundary layer in numerical solution, the Shishkin mesh $\bar{\Omega}^N$ has been

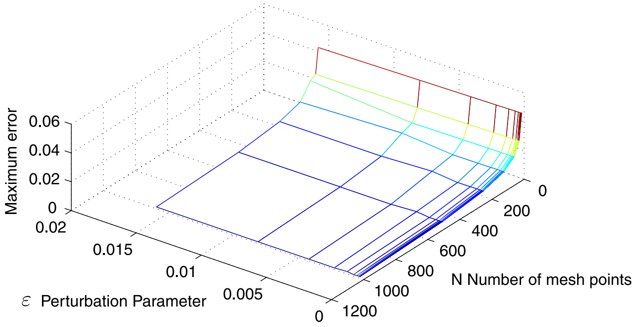

Fig. 4. Maximum point wise error for the problem stated in Example 8.2.

constructed in Section 6.1. In [7], the authors have suggested a uniform numerical method without interpolation, whereas the finite difference method with interpolation is needed in this paper, since the point $x_i - 1,\ i > N/2$ need not be a mesh point.

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