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A note on commutativity of rings with additive mappings

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Abstract. We investigate commutativity of the ring $\boldsymbol{R}$ involving some additive mapping with necessary torsion restrictions on commutators. We give counter examples which show that the hypotheses of our theorems are not superfluous.

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1. Introduction

This research is inspired by the work of Ashraf and Quadri [1,2]. Throughout this paper R will denote an associative ring with the identity 1. A ring R is said to be n-torsion free if $nx = 0$ implies $x = 0$ for all $x \in R$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$. An additive mapping $d: R \to R$ is said to be a derivation of R if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. We say that a map $f: R \to R$ preserves commutativity if $[f(x), f(y)] = 0$ whenever $[x, y] = 0$ for $x, y \in R$. In [3], Bell and Daif investigated a certain kind of commutativity preserving maps as follows: Let S be a subset of R. A map $f: S \to R$ is called strong commutativity preserving (SCP) on S if $[f(x), f(y)] = [x, y]$ for all $x, y \in S$. Precisely, they proved that if a semiprime ring R admits a derivation which is SCP on a right ideal ρ, then $\rho \subseteq Z(R)$. In particular, R is

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commutative if $\rho = R$. In [4], Deng and Ashraf proved that if there exist a derivation d of a semiprime ring R and a map f: $I \rightarrow R$ defined on a nonzero ideal I of R such that $[f(x),d(y)] = [x,y]$ for all $x,y \in I$, then R contains a nonzero central ideal. In particular, they showed that R is commutative if $I = R$.

The literature includes several papers on commutativity in rings with commutator constraints involving elements of the ring and images of elements under suitable maps (see [1,5–8]). In this paper, our intent is to investigate the commutativity of rings satisfying certain identities involving additive mapping on the ring R. Throughout this paper we will denote $\max\{m,n\}$ by $(m \vee n)$, where m and n are positive integers.

2. The main results

We begin with the following lemma which is essential for developing the proof of our main results.

Lemma 2.1. *If there is a positive integer n such that $[x,y^n] = 0$ for all $x,y \in R$ and commutators in R are $n!$-torsion free, then R is commutative.*

Proof. By the hypothesis, we have

$$[x, y^n] = 0 \text{ for all } x, y \in R. \tag{2.1}$$

Replacing y by $1 + y$ in (2.1), we obtain

$$\binom{n}{1}[x, y] + \binom{n}{2}[x, y^2] + \ldots + \binom{n}{n-1}[x, y^{n-1}] = 0 \text{ for all } x, y \in R. \tag{2.2}$$

Substituting qy for y in (2.2), where $q = 1, 2, \ldots, (n-1)$, we get

$$q\binom{n}{1}[x, y] + q^2\binom{n}{2}[x, y^2] + \ldots + q^{n-1}\binom{n}{n-1}[x, y^{n-1}] = 0 \text{ for all } x, y \in R. \tag{2.3}$$

The above equation produces the system of $(n-1)$ homogeneous equations, the coefficient matrix of this system is Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 2 & 2^2 & \ldots & 2^{n-1} \\ \vdots & \vdots & \vdots & \vdots \\ (n-1) & (n-1)^2 & \ldots & (n-1)^{n-1} \end{pmatrix}.$$

Since the determinant of the matrix is equal to a product of positive integers, each of which is less than n, it follows that $\binom{n}{r}[x, y^r] = 0$ for all $x,y \in R$ and $r = 1, 2, \ldots, (n-1)$. In particular, for $r = 1$, we have $n[x,y] = 0$ for all $x,y \in R$. Since commutators in R are n-torsion free, the last expression implies that $[x, y] = 0$ for all $x,y \in R$. Hence, R is commutative. This proves the lemma. □

Theorem 2.2. *Let $f: R \to R$ be any mapping. Then the following are equivalent:*

(i) *There is an integer $n > 1$ such that $(f[x,y])^n = \pm[x^n,y^n]$ for all $x,y \in R$ and commutators in R are $n!$-torsion free.*
(ii) *There is an integer $n > 1$ such that $f([x,y]^n) = \pm[x^n,y^n]$ for all $x,y \in R$ and commutators in R are $n!$-torsion free.*
(iii) *There are positive integers m and n with $m + n > 2$ such that $(f^m[x,y])^n = \pm[x^m,y^n]$ for all $x,y \in R$ or $(f^m[x,y])^n = \pm[x^n,y^m]$ for all $x,y \in R$ and commutators in R are $(m \vee n)!$-torsion free.*
(iv) *There are positive integers m and n with $m + n > 2$ such that $f^m([x,y]^n) = \pm[x^m,y^n]$ for all $x,y \in R$ or $f^m([x,y]^n) = \pm[x^n,y^m]$ for all $x,y \in R$ and commutators in R are $(m \vee n)!$-torsion free.*
(v) *R is commutative.*

Proof. It is immediate that commutativity of R implies each of the conditions (i) through (iv). Now, we show that each of the conditions implies commutativity of R.(i) $\Rightarrow$ (v).
We assume that

$$(f[x,y])^n = \pm[x^n,y^n] \text{ for all } x,y \in R. \tag{2.4}$$

Substituting x by $1 + x$ in (2.4), we get

$$(f[x,y])^n = \pm\binom{n}{1}[x,y^n] \pm \binom{n}{2}[x^2,y^n] \pm \ldots \pm \binom{n}{n-1}[x^{n-1},y^n] \pm [x^n,y^n] \text{ for all } x,y \in R.$$

Application of (2.4) yields that

$$\binom{n}{1}[x,y^n] + \binom{n}{2}[x^2,y^n] + \ldots + \binom{n}{n-1}[x^{n-1},y^n] = 0 \text{ for all } x,y \in R. \tag{2.5}$$

Now, using the same techniques as we have used in Lemma 2.1, we get $[x,y^n] = 0$ for all $x,y \in R$. Thus, in view of Lemma 2.1, we conclude that R is commutative.
(ii) $\Rightarrow$ (v) is similar to (i) $\Rightarrow$ (v).

(iii) $\Rightarrow$ (v). First we consider the case

$$(f^m[x,y])^n = \pm[x^m,y^n] \text{ for all } x,y \in R. \tag{2.6}$$

Replacing x by $1 + x$ in (2.6), we get

$$\binom{m}{1}[x,y^n] + \binom{m}{2}[x^2,y^n] + \ldots + \binom{m}{m-1}[x^{m-1},y^n] = 0 \text{ for all } x,y \in R. \tag{2.7}$$

Using the same arguments as we have used to prove (i) $\Rightarrow$ (v), we conclude that $\binom{m}{r}[x^r,y^n] = 0$ for all $x,y \in R$ and $r = 1, 2, \ldots, (m-1)$. In particular, for $r = 1$ we have $m[x,y^n] = 0$ for all $x,y \in R$. The last expression implies that $[x,y^n] = 0$ for all $x,y \in R$, since commutators in R are m-torsion free. Thus, by Lemma 2.1, R is commutative.

Similar conclusion holds for the case $(f^m[x,y])^n = \pm[x^n,y^m]$ for all $x,y \in R$.

(iv) $\Rightarrow$ (v). Use the parallel arguments as we have used in the proof of (iii) $\Rightarrow$ (v). This completes the proof of the theorem. $\square$

Theorem 2.3. *Let $f,g{:}R \to R$ be two mappings such that f is additive and $f(1) = 1$. Then the following are equivalent:*

(i) *There is an integer $n > 1$ such that $[f(x),g(y)]^n = \pm[x^n,y^n]$ for all $x,y \in R$, and commutators in R are $n!$-torsion free.*
(ii) *There are positive integers m and n with $m + n > 2$ such that $[f^m(x),g^n(y)] = \pm[x^m,y^n]$ for all $x,y \in R$ or $[f^m(x),g^n(y)] = \pm[x^n,y^m]$ for all $x,y \in R$, and commutators in R are $(m \vee n)!$-torsion free.*
(iii) *There are positive integers m and n with $m + n > 2$ such that $[f^m(x),g^m(y)]^n = \pm[x^m,y^n]$ for all $x,y \in R$ or $[f^m(x),g^m(y)]^n = \pm[x^n,y^m]$ for all $x,y \in R$, and commutators in R are $(m \vee n)!$-torsion free.*
(iv) *R is commutative.*

Proof. Clearly, (iv) $\Rightarrow$ (i), (iv) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (iii). Now, we will prove that

(i) $\Rightarrow$ (iv). By the assumption, we have

$$[f(x), g(y)]^n = \pm[x^n, y^n] \text{ for all } x, y \in R. \tag{2.8}$$

Replacing x by $1 + x$ in (2.8), we get

$$[f(1)+f(x),g(y)]^n = \pm\binom{n}{1}[x,y^n] \pm \binom{n}{2}[x^2,y^n] \pm \ldots \pm \binom{n}{n-1}[x^{n-1},y^n] \pm [x^n,y^n]$$

for all $x,y \in R$. Using (2.8) and the fact that image of identity is identity under f, we conclude that

$$\binom{n}{1}[x,y^n] + \binom{n}{2}[x^2,y^n] + \ldots + \binom{n}{n-1}[x^{n-1},y^n] = 0 \text{ for all } x,y \in R.$$

Using parallel arguments as we have used to prove (i) $\Rightarrow$ (v) in Theorem 2.2, we find that $\binom{n}{r}[x^r,y^n] = 0$ for all $x,y \in R$ and $r = 1, 2, \ldots, (n-1)$. In particular, we have $n[x,y^n] = 0$ for all $x,y \in R$. Since commutators in R are n-torsion free, the last expression yields that $[x,y^n] = 0$ for all $x,y \in R$. Hence, R is commutative by Lemma 2.1.(ii) $\Rightarrow$ (iv).

First we assume that

$$[f^m(x), g^n(y)] = \pm[x^m, y^n] \text{ for all } x, y \in R. \tag{2.9}$$

Replacing x by $1 + x$ in (2.9), we obtain

$$[f^m(1+x),g^n(y)] = \pm\binom{m}{1}[x,y^n] \pm \binom{m}{2}[x^2,y^n] \pm \ldots \pm \binom{m}{m-1}[x^{m-1},y^n] \pm [x^m,y^n]$$

for all $x,y \in R$. Using (2.9) and the fact that the image of identity is identity under f, we get

$$\binom{m}{1}[x, y^n] + \binom{m}{2}[x^2, y^n] + \ldots + \binom{m}{m-1}[x^{m-1}, y^n] = 0 \text{ for all } x, y \in R.$$

The above expression is similar to the relation (2.7) and henceforth using the same approach as we have used to obtain commutativity of R from the expression (2.7) in the proof of Theorem 2.2, we get the required result.

Similarly, we can prove the result for the case $[f^m(x),g^n(y)] = \pm[x^n,y^m]$ for all $x,y \in R$.

(iii) $\Rightarrow$ (iv). It can be proved by using the same techniques with necessary variations. Thereby, the proof is completed. □

The next theorem is motivated by [4, Theorem 1].

Theorem 2.4. *Let $d:R \to R$ be a derivation of R and g be any mapping of R. If there are positive integers m and n with $m + n > 2$ such that $[d(x)^m,g(y)^n] = [x^m,y^n]$ for all $x,y \in R$ and commutators in R are $(m \vee n)!$-torsion free, then R is commutative.*

Proof. By the assumption, we have

$$[d(x)^m, g(y)^n] = [x^m, y^n] \text{ for all } x, y \in R. \tag{2.10}$$

Replacing x by $1 + x$ in (2.10) and using the fact that $d(1) = 0$, we get

$$\binom{m}{1}[x, y^n] + \binom{m}{2}[x^2, y^n] + \ldots + \binom{m}{m-1}[x^{m-1}, y^n] = 0 \text{ for all } x, y \in R.$$

The above expression is same as the relation in (2.7) and henceforth using the same approach as we have used to obtain commutativity of R from the expression (2.7) in the proof of Theorem 2.2, we get the required result. This proves the theorem. □

At the end, let us write two examples which show that the restriction in our results are not superfluous.

Example 2.1. Let S be any noncommutative ring and

$$R = \left\{ \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix} \middle| a, b \in S \right\}.$$

Obviously, R is a ring without identity. Also, it can be easily seen that for any integer $n > 1$, the identity $[x,y^n] = 0$ holds for all $x,y \in R$, but R is not commutative. Hence, in Lemma 2.1 identity element is necessary. Further, define a mapping $f:R \to R$ such that

$$f\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ a & b & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b & 0 \end{pmatrix}, \qquad a, b \in S.$$

It is easy to see that f satisfies all the requirements of Theorem 2.2. However, R is not commutative.

Example 2.2. Let

$$R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \middle| \, a, b, c \in \mathbb{Z} \right\}.$$

Clearly, R is a ring without identity. Consider the mappings $f,g{:}R \to R$ such that

$$f\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad g\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \quad a, b, c \in \mathbb{Z}.$$

It is straightforward to check that f, and g satisfy all the requirements of Theorem 2.3, but R is not commutative.

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