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A note for "On the rational recursive sequence $x_{n+1} = \frac{A+\sum_{i=0}^{k}\alpha_i x_{n-i}}{\sum_{i=0}^{k}\beta_i x_{n-i}}$", ☆

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Abstract In this paper, we continue to study some properties, such as the existence of non-oscillatory solution, boundedness and persistence, global asymptotic stability, etc., for the rational difference equation in the title, which has been investigated in the known literature. We first point out some errors for the results in the known literature, then solve some questions existing in the known literature and finally state some new results.

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1. Introduction and preliminaries

Consider the following rational difference equation

$$x_{n+1} = \frac{A + \sum_{i=0}^{k} \alpha_i x_{n-i}}{\sum_{i=0}^{k} \beta_i x_{n-i}}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where k is a positive integer, the coefficients A, α_i, β_i, $i = 0, \ldots, k$ and the initial conditions $x_{-k}, x_{-k+1}, \ldots, x_{-1}, x_0$ are positive real numbers, which ensure that every solution of Eq. (1.1) is positive.

Set $\tilde{a} = \sum_{i=0}^{k} \alpha_i$ and $\tilde{b} = \sum_{i=0}^{k} \beta_i$. Then the equilibria points of Eq. (1.1) are the solutions of the equation

$$\tilde{x} = \frac{A + \tilde{a}\tilde{x}}{\tilde{b}\tilde{x}}, \tag{1.2}$$

from which one can see that $\tilde{x}_{1,2} = \frac{\tilde{a} \pm \sqrt{\tilde{a}^2 + 4A\tilde{b}}}{2\tilde{b}}$.

This demonstrates that Eq. (1.1) has a unique positive equilibrium point $\tilde{x} = \frac{\tilde{a} + \sqrt{\tilde{a}^2 + 4A\tilde{b}}}{2\tilde{b}}$. Because of every solution of Eq. (1.1) being positive, we only consider the unique positive equilibrium in the sequel.

Now, we present some crucial necessities about the equilibrium point of a higher-order difference equation.

Definition 1. Let I be some interval of real numbers and let

$$f : I^{k+1} \to I$$

be a continuously differentiable function.

Then for every set of initial conditions $x_{-k}, \ldots, x_{-1}, x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, 2, \ldots \tag{1.3}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

A point $\tilde{x}$ is called an equilibrium point of Eq. (1.3) if

$$\tilde{x} = f(\tilde{x}, \tilde{x}, \ldots, \tilde{x});$$

that is, $x_n = \tilde{x}$ for $n \geqslant 0$ is a solution of Eq. (1.3), or equivalently $\tilde{x}$ is a fixed point of f.

Let $F : (0,\infty)^{k+1} \to (0,\infty)$ be a continuous function defined by

$$F(u_0, u_1, \ldots, u_k) = \frac{A + \sum_{i=0}^{k} \alpha_i u_i}{\sum_{i=0}^{k} \beta_i u_i}. \tag{1.4}$$

The linearized equation of Eq. (1.1) associated with $\tilde{x}$ is

$$y_{n+1} = \sum_{j=0}^{k} b_j y_{n-j}, \tag{1.5}$$

where

$$b_j = \frac{\partial F(\tilde{x}, \tilde{x}, \ldots, \tilde{x})}{\partial u_j} = \frac{\alpha_j - \beta_j \tilde{x}}{\tilde{b}\tilde{x}}. \tag{1.6}$$

The characteristic equation of the linearized Eq. (1.5) is given by

$$\lambda^{k+1} = \sum_{j=0}^{k} b_j \lambda^{k-j}. \tag{1.7}$$

Recently, the authors of Zayed and El-Moneam [15] studied Eq. (1.1) and obtained some results as follows.

Conclusion 1 (*Theorem 5 of Zayed and El-Moneam [15]*). If all roots of the polynomial Eq. (1.7) lie in the open unit disk $|\lambda| < 1$, then

$$\sum_{j=0}^{k} |b_j| < 1.$$

Conclusion 2 (*Theorem 6 of Zayed and El-Moneam [15]*). Let $\{x_n\}_{n=-k}^{\infty}$ be a positive solution of the difference Eq. (1.1) such that for some $n_0 \geqslant 0$,

either $x_n \geqslant \tilde{x}$ for all $n \geqslant n_0$

or $x_n \leqslant \tilde{x}$ for all $n \geqslant n_0$.

Then $\{x_n\}$ converges to the equilibrium point $\tilde{x}$ as $n \to \infty$.

Conclusion 3 (*Theorem 7 of Zayed and El-Moneam [15]*). If $\{x_n\}_{n=-k}^{\infty}$ is a positive solution of Eq. (1.1) which is monotonic increasing, then it is bounded and persists.

Conclusion 4 (*Theorem 8 of Zayed and El-Moneam [15]*). The positive equilibrium points $\tilde{x}_{1,2}$ of the difference Eq. (1.1) are globally asymptotically stable. Let's evaluate the above Conclusions 1–4. First, we give a counter example to show that the above Conclusion 1 is wrong.

Example 1. Consider the polynomial Eq. (1.7) with $k = 1$ with the form

$$\lambda^2 - \lambda + a = 0, \quad 0 < a \leqslant \tfrac{1}{4}. \tag{1.9}$$

Obviously, the roots of this equation are given by

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4a}}{2}.$$

Since $0 < a \leqslant \frac{1}{4}$, it is clear that

$$0 < \lambda_1 = \frac{1 + \sqrt{1 - 4a}}{2} < 1 \quad \text{and} \quad 0 < \lambda_2 = \frac{1 - \sqrt{1 - 4a}}{2} < 1.$$

Namely, all roots of Eq. (1.9) satisfy $|\lambda| < 1$. But

$$\sum_{i=0}^{1} |b_i| = 1 + a > 1.$$

This example shows that Conclusion 1 is wrong.

Conclusion 2 essentially reads that every non-oscillatory solution of Eq. (1.1) approaches $\tilde{x}$. But a problem naturally rises: Are there non-oscillatory solutions of Eq. (1.1)? This problem is not answered in Zayed and El-Moneam [15]. Similarly, Conclusion 3 actually shows that every monotonic increasing positive solution of Eq. (1.1) is bounded and persists. The problem is: Does Eq. (1.1) have monotonic increasing solution? How about not monotonic increasing solutions of Eq. (1.1)? Bounded or unbounded? Is every positive solution of Eq. (1.1) bounded and persist? All these questions are not solved in Zayed and El-Moneam [15]. In this paper, we will positively answer these questions.

Both Conclusion 4 and its proof are also wrong. In fact, although $\tilde{x}_2 = \frac{\tilde{a} - \sqrt{\tilde{a} + 4A\tilde{b}}}{2\tilde{b}}$ is an equilibrium of Eq. (1.1), it is just a negative equilibrium point of Eq. (1.1). And to consider its global asymptotic stability is meaningless relative to all positive solutions of Eq. (1.1). On the other hand, in the proof of the Theorem 8 of Zayed and El-Moneam [15],

$$\sum_{i=0}^{k} \left| \frac{\beta_i \tilde{x}_j - \alpha_i}{\tilde{b}\tilde{x}_j} \right| = \sum_{i=0}^{k} \frac{\beta_i \tilde{x}_i - \alpha_i}{\tilde{b}\tilde{x}_j}, \quad j = 1, 2$$

is obviously wrong if additional conditions are not added. In addition, the function F of (1.4) ((3) in Zayed and El-Moneam [15]) does not satisfy to be nondecreasing in each of its arguments. Thus, the theorem 3 in Zayed and El-Moneam [15] cannot be used.

Based on the above analysis, the corresponding properties of Eq. (1.1) is worthy further investigating.

The study of rational difference equation such as Eq. (1.1) is quite challenging and rewarding due to the fact that some results of rational difference equations offer prototypes for the development of the basic theory of the global behavior of nonlinear difference equations; moreover, the investigations of rational difference equations are still in its infancy so far. To see this, refer to the monographs [5,6] and the papers [7,2,11,9,10,12,4,13,3,14,8 and the references cited therein].

2. Main results

In this section, we will formulate our main results in this note.

Firstly, we will present the existence of non-oscillatory solutions of Eq. (1.1), which has not been answered in Zayed and El-Moneam [15]. One can derive the following result.

Theorem 2.1. *Assume that*

$$\max_{0\leqslant i\leqslant k-1}\left\{\frac{\alpha_i}{\beta_i}\right\} \leqslant \tilde{x} < \frac{\alpha_k}{\beta_k}.$$

Then Eq. (1.1) possesses non-oscillatory solutions asymptotically approaching its equilibrium point $\tilde{x}$.

The main tool to prove this theorem is to make use of L. Berg' Inclusion Theorem [1]. Now, for the sake of convenience of statement, we first state some preliminaries. Consider a general real nonlinear difference equation of order $l \geqslant 1$ with the form

$$F(x_n, x_{n+1}, \ldots, x_{n+l}) = 0, \tag{2.1}$$

where $F : \mathbb{R}^{l+1} \mapsto \mathbb{R}, n \in \mathbb{N}_0$. Let φ_n and ψ_n be two sequences satisfying $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$. Then, (maybe under certain additional conditions), for any given $\epsilon > 0$, there exist a solution $\{x_n\}_{n=-l}^{\infty}$ of Eq. (2.1) *and an $n_0(\epsilon) \in \mathbb{N}$ such*

$$\varphi_n - \epsilon\psi_n \leqslant x_n \leqslant \varphi_n + \epsilon\psi_n, \quad n \geqslant n_0(\epsilon). \tag{2.2}$$

Denote

$$X(\epsilon) = \{x_n \ : \ \varphi_n - \epsilon\psi_n \leqslant x_n \leqslant \varphi_n + \epsilon\psi_n, \quad n \geqslant n_0(\epsilon)\},$$

which is called an asymptotic stripe. So, if $x_n \in X(\epsilon)$, then it is implied that there exists a real sequence C_n such that $x_n = \varphi_n + C_n\psi_n$ and $|C_n| \leqslant \epsilon$ for $n \geqslant n_0(\epsilon)$.

We now state the inclusion theorem [1].

Lemma 2.2. *Let $F(\omega_0, \omega_1, \ldots, \omega_l)$ be continuously differentiable when $\omega_i = y_{n+i}$, for $i = 0,1,\ldots,l$, and $y_n \in X(1)$. Let the partial derivatives of F satisfy*

$$F_{\omega_i}(y_n, y_{n+1}, \ldots, y_{n+l}) \sim F_{\omega_i}(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+l})$$

as $n \to \infty$ uniformly in C_j for $|C_j| \leqslant 1, n \leqslant j \leqslant n+l$, as far as $F_{\omega_i} \neq 0$. Assume that there exist a sequence $f_n > 0$ and constants $A_0, A_1, \ldots, A_l$ such that both

$$F(\varphi_n, \ldots, \varphi_{n+l}) = o(f_n)$$

and

$$\psi_{n+i}F_{w_i}(\varphi_n, \ldots, \varphi_{n+l}) \sim A_i f_n$$

for $i = 0,1,\ldots,l$ *as* $n \to \infty$, *and suppose there exists an integer s, with* $0 \leqslant s \leqslant l$, *such that*

$$|A_0| + \cdots + |A_{s-1}| + |A_{s+1}| + \cdots + |A_l| < |A_s|.$$

Then, for sufficiently large n, there exists a solution $\{x_n\}_{n=-l}^{\infty}$ *of* Eq. (2.1) *satisfying* Eq. (2.2).

Proof of Theorem 2.1. Put $y_n = x_n - \tilde{x}$. Then Eq. (1.1) is transformed into

$$(y_{n+1} + \tilde{x})\left(\sum_{i=0}^{k} \beta_i y_{n-i} + \tilde{b}\tilde{x}\right) - \left(A + \tilde{a}\tilde{x} + \sum_{i=0}^{k} \alpha_i y_{n-i}\right) = 0, \quad n = 0, 1, \ldots.$$

So, for $n = -k, -k+1, \ldots,$

$$y_{n+k+1}\left(\sum_{i=0}^{k} \beta_i y_{n+k-i} + \tilde{b}\tilde{x}\right) + \sum_{i=0}^{k} (\beta_i \tilde{x} - \alpha_i) y_{n+k-i} = 0. \tag{2.3}$$

An approximate equation of Eq. (2.3) is the equation

$$\tilde{b}\tilde{x} z_{n+k+1} + \sum_{i=0}^{k} (\beta_i \tilde{x} - \alpha_i) z_{n+k-i} = 0, \qquad n = -k, -k+1, \ldots \tag{2.4}$$

provided that $z_n \to 0$ as $n \to \infty$. The general solution to (2.4) is $z_n = \sum_{i=0}^{k} c_i t_i^n$, where c_i, $i = 0,1,\ldots,k$, are complex numbers and t_i, $i = 0,1,\ldots,k$, are the $(k+1)$ zeros of the polynomial

$$P(t) = \tilde{b}\tilde{x} t^{k+1} + \sum_{i=0}^{k} (\beta_i \tilde{x} - \alpha_i) t^{k-i}.$$

Obviously,

$$P(1) = \tilde{b}\tilde{x} + \sum_{i=0}^{k} (\beta_i \tilde{x} - \alpha_i) = \tilde{b}\tilde{x} + \tilde{b}\tilde{x} - \tilde{a} = 2\tilde{b}\tilde{x} - \tilde{a} = \sqrt{\tilde{a}^2 + 4A\tilde{b}} > 0$$

and

$$P(0) = \beta_k \tilde{x} - \alpha_k < 0.$$

So $P(0)P(1) < 0$. Hence, $P(t) = 0$ has a solution $t_0 \in (0,1)$. Now, choose the solution $z_n = t_0^n$ for this $t_0 \in (0,1)$. For some $\lambda \in (1,2)$, define the sequences $\{\varphi_n\}$ and $\{\psi_n\}$ respectively as follows:

$$\varphi_n = t_0^n \quad \text{and} \quad \psi_n = t_0^{\lambda n}. \tag{2.5}$$

Obviously, $\psi_n > 0$ and $\psi_n = o(\varphi_n)$ as $n \to \infty$.

Now, define again the function

$$F(\omega_0, \omega_1, \ldots, \omega_k, \omega_{k+1}) = \omega_{k+1}\left(\sum_{i=0}^{k}\beta_i\omega_{k-i} + \tilde{b}\tilde{x}\right) + \sum_{i=0}^{k}(\beta_i\tilde{x} - \alpha_i)\omega_{k-i}. \quad (2.6)$$

Then the partial derivatives of F w.r.t. $\omega_0, \omega_1, \ldots, \omega_{k+1}$ respectively are

$$\begin{aligned} &F_{\omega_i} = (\omega_{k+1} + \tilde{x})\beta_{k-i} - \alpha_{k-i}, \quad i = 0, 1, \ldots, k, \\ &F_{\omega_{k+1}} = \sum_{i=0}^{k}\beta_i\omega_{k-i} + \tilde{b}\tilde{x}. \end{aligned} \quad (2.7)$$

When $y_n \in X(1), y_n \sim \varphi_n$. So,

$$F_{\omega_i}(y_n, y_{n+1}, \ldots, y_{n+k+1}) \sim F_{\omega_i}(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k+1}), \quad i = 0, 1, \ldots, k+1,$$

as $n \to \infty$ uniformly in C_j for $|C_j| \leqslant 1, n \leqslant j \leqslant n + k + 1$.

Moreover, from the definition (2.6) of the function F and Eq. (2.5), after some calculation, we find

$$\begin{aligned} F(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k+1}) &= \sum_{i=0}^{k}\beta_i t_0^{2n+2k+1-i} + \sum_{i=0}^{k}(\beta_i\tilde{x} - \alpha_i)t_0^{n+k-i} + \tilde{b}\tilde{x}t_0^{n+k+1} \\ &= \sum_{i=0}^{k}\beta_i t_0^{2n+2k+1-i}. \end{aligned}$$

Now, choose $f_n = t_0^{\lambda n}$. Then one can easily see that

$$F(\varphi_n, \ldots, \varphi_{n+k+1}) = o(f_n) \quad \text{as } n \to \infty.$$

Again, from (2.5) and (2.7), one has that

$$\begin{aligned} &\psi_{n+i}F_{\omega_i}(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k+1}) = t_0^{\lambda(n+i)}(\beta_{k-i}t_0^{n+k+1} + \beta_{k-i}\tilde{x} - \alpha_{k-i}), \\ &\qquad i = 0, 1, \ldots, k, \\ &\psi_{n+k+1}F_{\omega_{k+1}}(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{n+k+1}) = t_0^{\lambda(n+k+1)}\left(\tilde{b}\tilde{x} + \sum_{i=0}^{k}\beta_i t_0^{n+k-i}\right). \end{aligned}$$

Hence $\psi_{n+i}F_{\omega_i} \sim A_i f_n, \quad i = 0, 1, \ldots, k+1$, where

$$A_i = t_0^{\lambda i}(\beta_{k-i}\tilde{x} - \alpha_{k-i}), \quad i = 0, 1, \ldots, k, A_{k+1} = \tilde{b}\tilde{x}t_0^{\lambda(k+1)}.$$

Therefore, one has

$$\begin{aligned} |A_1| + \cdots + |A_{k+1}| &= t_0^{\lambda}|\beta_{k-1}\tilde{x} - \alpha_{k-1}| + t_0^{2\lambda}|\beta_{k-2}\tilde{x} - \alpha_{k-2}| + \cdots + t_0^{k\lambda}|\beta_0\tilde{x} - \alpha_0| \\ &\quad + \tilde{b}\tilde{x}t_0^{(k+1)\lambda} \\ &< t_0(\beta_{k-1}\tilde{x} - \alpha_{k-1}) + t_0^2(\beta_{k-2}\tilde{x} - \alpha_{k-2}) + \cdots + t_0^k(\beta_0\tilde{x} - \alpha_0) \\ &\quad + \tilde{b}\tilde{x}t_0^{k+1} \\ &= -(\beta_k\tilde{x} - \alpha_k) = |A_0|. \end{aligned}$$

Up to here, all conditions of Lemma 2 with $l = k + 1$ and $s = 0$ are satisfied. Accordingly, we see that, for arbitrary $\epsilon \in (0,1)$ and for sufficiently large n, say $n \geqslant N_0 \in \mathbb{N}$, Eq. (2.3) has a solution $\{y_n\}_{n=-k}^{\infty}$ in the stripe $\varphi_n - \epsilon\psi_n \leqslant y_n \leqslant \varphi_n + \epsilon\psi_n$, $n \geqslant N_0$, where φ_n and ψ_n are as defined in (2.5). Because $\varphi_n - \epsilon\psi_n > \varphi_n - \psi_n = t_0^n - t_0^{\lambda n} > 0$, $y_n > 0$ for $n \geqslant N_0$. Thus, Eq. (1.1) has a solution $\{x_n\}_{n=-k}^{\infty}$ satisfying $x_n = y_n + \tilde{x} > \tilde{x}$ for $n \geqslant N_0$. Since Eq. (1.1) is an autonomous equation, $\{x_{n+N_0+k}\}_{n=-k}^{\infty}$ still is its solution, which evidently satisfies $x_{n+N_0+k} > \tilde{x}$ for $n \geqslant -k$. Therefore, the proof is complete. □

Remark 2.3. If we take $\varphi_n = -t_0^n$ in (2.5), then $\varphi_n + \epsilon\psi_n < -t_0^n + t_0^{\lambda n} < 0$. At this time, Eq. (1.1) possesses solutions $\{x_n\}_{n=-k}^{\infty}$ which remain below its equilibrium for all $n \geqslant -k$, i.e., Eq. (1.1) has solutions with a single negative semi-cycle.

Remark 2.4. The existence and asymptotic behavior of non-oscillatory solution of Eq. (1.1) has not been found to be considered in the known literatures.

Secondly, we answer the existing problem for Conclusion 3 (i.e., Theorem 7 of Zayed and El-Moneam [15]). The following results are derived.

Theorem 2.5. *Every positive solution of Eq. (1.1) is bounded and persists.*

Proof. From Eq. (1.1) one see that, for $n \geqslant 0$,

$$x_{n+1} = \frac{A + \sum_{i=0}^{k}\alpha_i x_{n-i}}{\sum_{i=0}^{k}\beta_i x_{n-i}} > \frac{\min_{0\leqslant i\leqslant k}\alpha_i \sum_{i=0}^{k} x_{n-i}}{\max_{0\leqslant i\leqslant k}\beta_i \sum_{i=0}^{k} x_{n-i}} = \frac{\min\limits_{0\leqslant i\leqslant k}\alpha_i}{\max\limits_{0\leqslant i\leqslant k}\beta_i} =: P.$$

Then, for $n \geqslant k + 1$,

$$\begin{aligned} x_{n+1} &= \frac{A + \sum_{i=0}^{k}\alpha_i x_{n-i}}{\sum_{i=0}^{k}\beta_i x_{n-i}} < \frac{A}{\sum_{i=0}^{k}\beta_i x_{n-i}} + \frac{\max_{0\leqslant i\leqslant k}\alpha_i \sum_{i=0}^{k} x_{n-i}}{\min_{0\leqslant i\leqslant k}\beta_i \sum_{i=0}^{k} x_{n-i}} \\ &< \frac{A}{\tilde{b}P} + \frac{\max_{0\leqslant i\leqslant k}\alpha_i}{\min_{0\leqslant i\leqslant k}\beta_i}, \end{aligned}$$

from which one knows that every positive solution of Eq. (1.1) is bounded and persists. The proof is completed. □

Remark 2.6. Theorem 2.5 demonstrates that, all positive solutions of Eq. (1.1), regardless of monotonoc inreasing ones or not monotonoc inreasing ones, are bounded and persist. So, with respect to the boundedness and persistence, it is not important whether the solutions of Eq. (1.1) are monotonoc inreasing or not.

Finally, we state the global asymptotical stability for positive equilibrium point $\tilde{x}$ of Eq. (1.1). The result is as follows.

Theorem 2.7. *Assume that $\beta_i = \alpha_i$, $i = 0,1,\ldots,k$. Then the positive equilibrium point $\tilde{x}$ of* Eq. (1.1) *are globally asymptotically stable.*

In order to prove this theorem, one needs the following lemma.

Let us consider the higher order difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-m}), \quad n = 0, 1, \ldots, \tag{2.8}$$

where m is a nonnegative integer and $f : \mathbb{R}^{m+1} \to \mathbb{R}$ is a given function. Assume the following:

There exists r_0, s_0 with $-\infty \leqslant r_0 < s_0 \leqslant \infty$ such that:

(H1) *$f(u_0, u_1, \ldots, u_m)$ is nonincreasing in each $u_0, u_1, \ldots, u_m \in I_0$ where $I_0 = (r_0, s_0]$ if $s_0 < \infty$ and $I_0 = (r_0, \infty)$ otherwise.*
(H2) *$g(u) = f(u, \ldots, u)$ is continuous and strictly decreasing for $u \in I_0$.*
(H3) *There is $r \in [r_0, s_0)$ such that $r < g(r) \leqslant s_0$. If $r_0 = -\infty$ or $\lim_{t\to r_0^+} g(t) = \infty$, then we assume that $r \in (r_0, s_0)$.*
(H4) *and* (H3) *holding implies that* Eq. (2.8) *has a unique fixed point x^* in the open interval $(r, g(r))$.*
(H5) *There is $s \in [r, x^*)$ such that $g^2(u) > u$ for all $u \in (s, x^*)$.*

Lemma 2.8 [6, Theorem 1, P_{111}]. *If (H1)–(H3) and (H5) hold then x^* is stable and attracts all solutions of Eq. (2.7) with initial values in $(s, g(s))$.*

Proof. From the known assumption $\beta_i = \alpha_i$, $i = 0,1,\ldots,k$, one can see that $\tilde{a} = \tilde{b}$ and $\tilde{x} = \frac{\tilde{a}+\sqrt{\tilde{a}^2+4A\tilde{b}}}{2\tilde{b}} > 1$.

The linearized equation of Eq. (1.1) associated with $\tilde{x}$ is Eq. (1.5). Because of

$$\sum_{i=0}^{k}\left|\frac{\beta_i\tilde{x}-\alpha_i}{\tilde{b}\tilde{x}}\right| = \sum_{i=0}^{k}\beta_i\left|\frac{\tilde{x}-1}{\tilde{b}\tilde{x}}\right| = \frac{\tilde{x}-1}{\tilde{x}} < 1,$$

it follows from [2, Remark 1.3.1, P_{12}] that $\tilde{x}$ is locally asymptotically stable. It next suffices to prove that $\tilde{x}$ is globally attractive, which will completed by utilizing Lemma 2.8.

Notice that Eq. (1.1) is now reduced into

$$x_{n+1} = \frac{A}{\sum_{i=0}^{k}\beta_i x_{n-i}} + 1, \quad n = 0, 1, 2 \ldots .$$

So, one can choose $I_0 = (0,\infty)$ in (H1) and $r = 1$ in (H3). Again, $g(u) = f(u, \ldots, u) = \frac{A}{\tilde{b}u} + 1$ is continuous and strictly decreasing for $u \in I_0$; moreover, $g^2(u) = \frac{Au}{A+\tilde{b}u} + 1 > u$ for all $u \in (0, x^*)$, holding evidently. Thus, all conditions of Lemma 2.8 are satisfied. The proof is over. □

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