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A new parameter for Ramanujan's theta-functions and explicit values

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Abstract We define a new parameter $A_{k,n}$ involving Ramanujan's theta-functions $\phi(q)$ and $\psi(q)$ for any positive real numbers k and n and study its several properties. We also prove some general theorems for the explicit evaluations of the parameter $A_{k,n}$ and find many explicit values. Finally, we establish an explicit formula for values of $\psi(e^{-2n\pi})$ for any positive real number n in terms of $A_{k,n}$ and give examples.

1. Introduction

For $q := e^{2\pi iz}$, $\text{Im}(z) > 0$, define Ramanujan's theta-functions $\phi(q)$, $\psi(q)$, and $f(-q)$ as

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$$\phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} = \vartheta_3(0, 2z)$$

$$\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2} = 2^{-1}q^{-1/8}\vartheta_2(0, z),$$

and

$$f(-q) := (q; q)_\infty = q^{-1/24}\eta(z),$$

where ϑ_2, ϑ_3 are classical theta-functions [7, p. 464], η denotes the Dedekind eta-function and

$$(a; q)_\infty := \prod_{k=0}^{\infty}(1 - aq^k).$$

In his first notebook [6, vol. I, p. 248] S. Ramanujan recorded many elementary and non elementary values of $\phi(q)$ and $\psi(q)$. All these values were proved by Berndt [4, p. 325] and Berndt and Chan [5]. They also found new explicit values $\phi(q)$. Recently, Yi [8,9] evaluated many new values of $\phi(q)$ and $f(q)$ using modular identities, transformation formulae for theta-functions and the parameters $r_{k,n}$, $r'_{k,n}$, $h_{k,n}$, and $h'_{k,n}$, defined, respectively, by

$$r_{k,n} := \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}}, \tag{1.1}$$

$$r'_{k,n} := \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(q^k)}, \quad q = e^{-\pi\sqrt{n/k}}, \tag{1.2}$$

$$h_{k,n} := \frac{\phi(q)}{k^{1/4}\phi(q^k)}, \quad q = e^{-\pi\sqrt{n/k}}, \tag{1.3}$$

and

$$h'_{k,n} := \frac{\phi(-q)}{k^{1/4}\phi(-q^k)}, \quad q = e^{-2\pi\sqrt{n/k}}. \tag{1.4}$$

Baruah and Saikia [2] also obtained several new explicit values of the theta-function $\psi(q)$ by finding explicit values of the parameters $g_{k,n}$ and $g'_{k,n}$ which are defined, respectively, by

$$g_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)} \quad q = e^{-\pi\sqrt{n/k}}, \tag{1.5}$$

and

$$g'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \quad q = e^{-\pi\sqrt{n/k}}. \tag{1.6}$$

In sequel to the above work, we now define a new parameter $A_{k,n}$ for any positive real numbers k and n and involving theta-functions $\phi(q)$ and $\psi(q)$ as

$$A_{k,n} = \frac{\phi(-q)}{2k^{1/4}q^{k/4}\psi(q^{2k})}, \quad q = e^{-\pi\sqrt{n/k}}. \tag{1.7}$$

In this paper, we study several properties of $A_{k,n}$ which are analogous to those of $h_{k,n}$ and $g_{k,n}$ and also establish its connections with the parameter $r_{k,n}$ and Ramanujan's class invariants. We prove some general theorems for the explicit evaluations of $A_{k,n}$ by using some new theta-function identities and find many explicit values of $A_{k,n}$. We also offer a general formula for explicit evaluations of $\psi(q)$ in terms of $A_{k,n}$ and find some particular values.

In Section 2, we record some transformation formulae for theta-functions and list the explicit values of $r_{k,n}$ and $h_{k,n}$ from [1] and [8] for ready reference in the later sections.

In Section 3, we prove some new theta-function identities which will be used in the subsequent sections.

In Section 4, we study several properties of $A_{k,n}$ and establish relations connecting $A_{k,n}$ with $r_{k,n}$ and Ramanujan's class invariants.

In Section 5, we prove some general theorems for the explicit evaluations of $A_{k,n}$ and find many explicit values of $A_{k,n}$ by using the results in Sections 2–4.

Finally, in Section 6, we offer a general formula for the evaluations of $\psi(q)$ in terms of $A_{k,n}$ and find some particular values.

To end this introduction, we define Ramanujan's modular equation. Let K, K', L, and L' denote the complete elliptic integrals of the first kind associated with the moduli k, k', l, and l', respectively. Suppose that the equality

$$n\frac{K'}{K} = \frac{L'}{L} \tag{1.8}$$

holds for some positive integer n. Then a modular equation of degree n is a relation between the moduli k and l which is implied by (1.8). Ramanujan recorded his modular equations in terms of α and β, where $\alpha = k^2$ and $\beta = l^2$. We say that β has degree n over α. By denoting $z_r = \phi^2(q^r)$, where $q = exp(-\pi K'/K)$, $|q| < 1$, the multiplier m connecting α and β is defined by $m = z_1/z_n$.

2. Preliminary results

Lemma 2.1 [3, p. 43, Entry 27(ii)]. *If α and β are such that the modulus of each exponential argument is less than 1 and $\alpha\beta = \pi$, then*

$$2\sqrt{\alpha}\psi(e^{-2\alpha^2}) = \sqrt{\beta}e^{\alpha^2/4}\phi(-e^{-\beta^2}). \tag{2.1}$$

Lemma 2.2 [3, p. 122, Entry 10(i), (ii), (iii)]. *We have*

$$\phi(q) = \sqrt{z_1}, \tag{2.2}$$

$$\phi(-q) = \sqrt{z_1}(1-\alpha)^{1/4}, \tag{2.3}$$

$$\phi(-q^2) = \sqrt{z_1}(1-\alpha)^{1/8}. \tag{2.4}$$

Lemma 2.3. ([3, p. 123, Entry 11(iii), (iv), (v)]). *We have*

$$\psi(q^2) = \frac{\sqrt{z_1}\alpha^{1/4}}{2q^{1/4}}, \tag{2.5}$$

$$\psi(q^4) = \frac{\sqrt{z_1}\{1-\sqrt{1-\alpha}\}^{1/2}}{2\sqrt{2}q^{1/2}}, \tag{2.6}$$

$$\psi(q^8) = \frac{\sqrt{z_1}\{1-(1-\alpha)^{1/4}\}}{4q}. \tag{2.7}$$

We also note that if we replace q by q^n in the Lemmas 2.2 and 2.3, then z_1 and α will be replaced by z_n and β, respectively, where β has degree n over α.

Lemma 2.4. ([3, p. 233, (5.2), (5.5)]). *If $m = z_1/z_3$ and β has degree 3 over α, then*

$$1-\alpha = \frac{(m+1)(3-m)^3}{16m^3} \quad and \quad \beta = \frac{(m-1)^3(3+m)}{16m}. \tag{2.8}$$

Lemma 2.5. ([3, p. 231, Entry 5(x)]). *If β has degree 3 over α, then*

$$\begin{aligned} m(1-\alpha)^{1/2} + (1-\beta)^{1/2} &= \frac{3}{m}(1-\beta)^{1/2} - (1-\alpha)^{1/2} \\ &= 2\{(1-\alpha)(1-\beta)\}^{1/8}. \end{aligned} \tag{2.9}$$

Lemma 2.6. ([1,8]). *If $r_{k,n}$ is as defined in (1.1), then*

$$r_{2,3} = (1+\sqrt{2})^{1/4}, \quad r_{2,12} = 2^{5/24}(2(1+\sqrt{2}+\sqrt{6}))^{1/8}, \quad r_{4,4} = 2^{5/16}(1+\sqrt{2})^{1/4},$$

$$r_{4,16} = 2^{3/8}(1+\sqrt{2})^{1/2}(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4})^{1/8}, \quad r_{5,5} = \sqrt{\frac{5+\sqrt{5}}{2}},$$

$$r_{5,20} = \frac{\sqrt{5+\sqrt{5}}}{5^{1/4}-1}, \quad r_{2,5} = \sqrt{\frac{1+\sqrt{5}}{2}}, \quad r_{2,20} = \frac{(1+\sqrt{5})^{5/8}(2+3\sqrt{2}+\sqrt{5})^{1/8}}{\sqrt{2}},$$

$$r_{2,9} = (\sqrt{2}+\sqrt{3})^{1/3}, \quad r_{2,36} = \frac{\{2(1+35\sqrt{2}-28\sqrt{3})\}^{1/8}}{(\sqrt{3}-\sqrt{2})^{2/3}}, \quad r_{2,8} = 2^{3/16}(1+\sqrt{2})^{1/4},$$

$$r_{2,32} = 2^{3/16}(1+\sqrt{2})^{1/4}(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4})^{1/8}, \quad r_{2,3/2} = 2^{-7/24}(1+\sqrt{3})^{1/4},$$

$$r_{2,6} = 2^{1/24}(1+\sqrt{3})^{1/4}, \quad r_{2,5/2} = \frac{\left(\sqrt{\sqrt{5}+1}+\sqrt{2}\right)^{1/4}}{2^{1/4}},$$

$$r_{2,10} = \left(\frac{1}{2}(1+\sqrt{5})\left(\sqrt{\sqrt{5}+1}+\sqrt{2}\right)\right)^{1/4}, \quad r_{2,10} = \left(\frac{1}{2}(1+\sqrt{5})(\sqrt{\sqrt{5}+1}+\sqrt{2})\right)^{1/4},$$

$$r_{2,25/2} = \frac{5^{1/4}+1}{2^{5/8}}, \quad r_{2,50} = \frac{2^{5/8}}{5^{1/4}-1}, \quad r_{4,2} = 2^{1/8}(1+\sqrt{2})^{1/8}, \quad r_{4,3} = (2+\sqrt{3})^{1/4},$$

$$r_{4,5}=\frac{1}{\sqrt{2}}\left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)^{1/2},\ r_{4,7}=(8+3\sqrt{7})^{1/4},$$
$$r_{4,8}=2^{1/4}(1+\sqrt{2})^{3/8}\left(4+\sqrt{2+10\sqrt{2}}\right)^{1/8},\ r_{4,9}=\frac{1}{2}+\frac{3^{1/4}}{\sqrt{2}}+\frac{\sqrt{3}}{2},$$
$$r_{4,25}=\frac{1}{2}\left(3+\sqrt[4]{5}+\sqrt{5}+\sqrt[4]{5^3}\right),\ r_{4,49}=\frac{1}{4}\left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right)^2,$$
$$r_{4,6}=(1+\sqrt{2})^{3/8}\left(2(1+\sqrt{2}+\sqrt{6})\right)^{1/8},\ r_{4,10}=\frac{(1+\sqrt{5})^{9/8}(2+3\sqrt{2}+\sqrt{5})^{1/8}}{2},$$
$$r_{4,18}=2^{1/8}(\sqrt{3}+\sqrt{2})(1+35\sqrt{2}-28\sqrt{3})^{1/8},\ r_{2,18}=2^{-11/24}(1+\sqrt{3})^{1/3}(1+\sqrt{3}+\sqrt{2}3^{3/4})^{1/3},$$
$$r_{2,72}=2^{-13/48}(\sqrt{2}-1)^{-5/12}(\sqrt{2}+\sqrt{3})^{1/3}(4-\sqrt{2}+2\sqrt{3}+3^{3/4}(\sqrt{3}+1))^{1/3}.$$

Baruah and Saikia [1] corrected Yi's incorrect value of $r_{2,72}$.

From [8, p. 12, Theorem 2.1.2(i)–(iii)], we note that

$$r_{k,1}=1, r_{k,n}=r_{n,k} \text{ and } r_{k,1/n}=1/r_{k,n}. \tag{2.10}$$

Lemma 2.7. ([8]). *If $h_{k,n}$ is as defined in (1.3), then*

$$h_{3,3}=(2\sqrt{3}-3)^{1/4},\quad h_{3,9}=\frac{1-2^{1/3}+4^{1/3}}{\sqrt{3}},\quad h_{3,4}=\frac{2-\sqrt{2}}{\sqrt{3}-1},$$
$$h_{3,5}=\sqrt{\frac{\sqrt{5}-1}{2}}.$$

3. Theta-function identities

In this section, we prove some new theta-function identities which will be used in the subsequent sections.

Theorem 3.1. *If $P=\dfrac{\phi(-q)}{q^{1/2}\psi(q^4)}$ and $Q=\dfrac{\phi(-q^2)}{q\psi(q^8)}$*

then $P^2+\dfrac{32}{Q^2}+\left(\dfrac{4P^2}{Q^2}-\dfrac{Q^2}{P^2}\right)+8=0.$

Proof. Transcribing P by using (2.3) and (2.6) and simplifying, we obtain

$$\sqrt{1-\alpha}=\frac{P^2}{8+P^2}. \tag{3.1}$$

Similarly, transcribing Q by using (2.4) and (2.7) and simplifying, we arrive at

$$Q^4(1+\sqrt{1-\alpha})^2-(16+2Q^2)^2\sqrt{1-\alpha}=0. \tag{3.2}$$

Employing (3.1) in (3.2) and simplifying, we obtain

$$32P^2 + 4P^4 + 8P^2Q^2 + P^4Q^2 - Q^4 = 0. \tag{3.3}$$

Dividing (3.3) by P^2Q^2 and rearranging the terms, we complete the proof. □

Theorem 3.2. *If* $P = \dfrac{\phi(-q)}{q^{1/2}\psi(q^4)}$ *and* $Q = \dfrac{\phi(-q^3)}{q^{3/2}\psi(q^{12})}$

then $$\left(\frac{P^2}{Q^2} - \frac{Q^2}{P^2}\right) + \left(PQ + \frac{32}{PQ}\right) + 6\left(\frac{P}{Q} + \frac{Q}{P}\right) = 0.$$

Proof. Transcribing P and Q using (2.3) and (2.6) and simplifying, we obtain

$$x := \sqrt{1-\alpha} = \frac{P^2}{8+P^2} \quad \text{and} \quad y := \sqrt{1-\beta} = \frac{Q^2}{8+Q^2}, \tag{3.4}$$

where β has degree 3 over α.

Employing (3.4) in Lemma 2.5, we obtain

$$mx + y = 2(xy)^{1/4} \tag{3.5}$$

and

$$\frac{3y}{m} - x = 2(xy)^{1/4}. \tag{3.6}$$

Eliminating m between (3.5) and (3.6) and then simplifying, we arrive at

$$2\sqrt{xy}(\sqrt{xy} - 1) = (x - y)(xy)^{1/4}. \tag{3.7}$$

Squaring (3.7) and simplifying, we obtain

$$4\sqrt{xy}(xy + 1) = x^2 + y^2 + 6xy. \tag{3.8}$$

Squaring (3.8) and then factorizing by employing (3.4), we deduce that

$$(P^4 - 32PQ - 6P^3Q - 6PQ^3 - P^3Q^3 - Q^4)$$
$$(P^4 + 32PQ + 6P^3Q + 6PQ^3 + P^3Q^3 - Q^4) = 0. \tag{3.9}$$

It is examined that there exists a neighborhood about origin, where the first factor of (3.9) is not zero. Then the second factor is zero in this neighborhood. By the identity theorem the second factor is identically zero. Thus, we conclude that

$$P^4 + 32PQ + 6P^3Q + 6PQ^3 + P^3Q^3 - Q^4 = 0. \tag{3.10}$$

Dividing (3.10) by P^2Q^2 and rearranging the terms, we complete the proof. □

Theorem 3.3. *If* $P = \dfrac{\phi(-q)}{q^{3/4}\psi(q^6)}$ *and* $Q = \dfrac{\phi(q)}{\phi(q^3)}$

then $$P^4 = \frac{432 - 288Q^4 + 128Q^6 - 16Q^8}{Q^8 - 6Q^4 + 8Q^2 - 3}.$$

Proof. Transcribing P by using (2.3) and (2.5) and Q by (2.2), we obtain

$$P = 2\sqrt{m}\left(\frac{1-\alpha}{\beta}\right)^{1/4} \quad \text{and} \quad Q = \sqrt{m}, \tag{3.11}$$

where β has degree 3 over α and $m = z_1/z_3$.

From (3.11), we deduce that

$$P^4 = 16Q^4\left(\frac{1-\alpha}{\beta}\right). \tag{3.12}$$

Employing Lemma 2.4 in (3.12) and simplifying, we find that

$$P^4(m-1)^3(3+m)m^2 - 16Q^4(m+1)(3-m)^3 = 0. \tag{3.13}$$

Substituting for m from (3.11) in (3.13) and then simplifying, we complete the proof. □

4. Properties of $A_{k,n}$

Theorem 4.1. *For all positive real numbers k and n, we have*

(i) $A_{k,1} = 1$,
(ii) $A_{k,1/n} = 1/A_{k,n}$.

Proof. Using the definition of $A_{k,n}$ and Lemma 2.1, we easily arrive at (i). Replacing n by $1/n$ in $A_{k,n}$ and using Lemma 2.1, we find that $A_{k,n}A_{k,1/n} = 1$ which completes the proof of (ii). □

Remarks 4.2. By using the definitions of $\phi(q)$, $\psi(q)$ and $A_{k,n}$, it can be seen that $A_{k,n}$ has positive real value and that the values of $A_{k,n}$ increase as n increases when $k > 1$. Thus, by Theorem 4.1(i), $A_{k,n} > 1$ for all $n > 1$ if $k > 1$.

Theorem 4.3. *For all positive real numbers k, m, and n, we have*

$$A_{k,n/m} = \frac{A_{mk,n}}{A_{nk,m}}.$$

Proof. Using the definition of $A_{k,n}$, we obtain

$$\frac{A_{mk,n}}{A_{nk,m}} = \frac{n^{1/4}\phi(-e^{-2\pi\sqrt{n/mk}})}{m^{1/4}\phi(-e^{-2\pi\sqrt{m/nk}})}. \tag{4.1}$$

Using Lemma 2.1 in the denominator of right hand side of (4.1) and simplifying, we complete the proof.

Corollary 4.4. *For all positive real numbers k and n, we have*

$$A_{k^2,n} = A_{nk,n}A_{k,n/k}.$$

Proof. Setting $k = n$ in Theorem 4.3 and simplifying using Theorem 4.1(ii), we obtain

$$A_{k^2,m} = A_{mk,k}A_{k,m/k}. \tag{4.2}$$

Replacing m by n, we complete the proof. □

Theorem 4.5. *Let k, a, b, c, and d be positive real numbers such that ab = cd. Then*

$$A_{a,b}A_{kc,kd} = A_{ka,kb}A_{c,d}.$$

Proof. From the definition of $A_{k,n}$, we deduce that, for positive real numbers k, a, b, c, and d,

$$A_{ka,kb}A_{a,b}^{-1} = \frac{e^{\pi(k-1)\sqrt{ab}/4}\psi(e^{-2\pi\sqrt{ab}})}{k^{1/4}\psi(e^{-2k\pi\sqrt{ab}})} \tag{4.3}$$

and

$$A_{kc,kd}A_{c,d}^{-1} = \frac{e^{\pi(k-1)\sqrt{cd}/4}\psi(e^{-2\pi\sqrt{cd}})}{k^{1/4}\psi(e^{-2k\pi\sqrt{cd}})}. \tag{4.4}$$

Now the result follows readily from (4.3), (4.4) and the hypothesis $ab = cd$. □

Corollary 4.6. *For any positive real numbers n and p, we have*

$$A_{np,np} = A_{np^2,n}A_{p,p}.$$

Proof. The result follows immediately from Theorem 4.5 with $a = p^2$, $b = 1, c = d = p$ and $k = n$. □

Now, we give some relations connecting the parameter $A_{k,n}$ with $r_{k,n}$ and Ramanujan's class invariants.

Theorem 4.7. *Let k and n be any positive real numbers. Then*

(i) $A_{k,n} = \dfrac{r_{4k,n}^2}{r_{k,n}}$,

(ii) $A_{n,k} = \dfrac{r_{4n,k}^2}{r_{4k,n}^2} A_{k,n}$.

Proof. Let $q = e^{-\pi\sqrt{n/k}}$. Using the results $\phi(-q) = \dfrac{f^2(-q)}{f(-q^2)}$ and $\psi(q) = \dfrac{f^2(-q^2)}{f(-q)}$ from [3, p. 39] in the definition of $A_{k,n}$, we find that

$$A_{k,n} = \frac{f(-q^{2k})f^2(-q)}{2k^{1/4}q^{k/4}f(-q^2)f^2(-q^{4k})}. \tag{4.5}$$

Employing the definition of $r_{k,n}$ from (1.1) in (4.5), we complete the proof of (i).
Proof of (ii) follows easily from (i). □

Corollary 4.8. *For all positive real numbers n and p, we have*

(i) $A_{1,n} = r_{4,n}^2$,

(ii) $A_{n,n} = \dfrac{r_{4n,n}^2}{r_{n,n}}$,

(iii) $A_{np,np} = A_{n,np^2} A_{p,p} r_{4,p^2}^{-2}$.

Proof. To prove (i), we set $k = 1$ in Theorem 4.7(i) and use the result $r_{k,1} = 1$ from (2.10). Proof of (ii) follows from Theorem 4.7(i) with $k = n$. To prove (iii), we set $a = 1, b = p^2$, $c = d = p$, and $k = n$ in Theorem 4.5 and use part (i).

Theorem 4.9. *For all positive real number n, we have*

(i) $A_{n,2} = \dfrac{g_{8n}^2}{g_{2n}}$,

(ii) $A_{n/2,2} = 2^{1/2} g_n G_n^2$.

Proof. (i) From [8, p. 18, Theorem 2.3.3(i)], we note that

$$g_n = r_{2,n/2}, \tag{4.6}$$

where the class invariant g_n is given by

$$g_n = 2^{-1/2} q^{-1/24} \chi(-q),$$

where $q := e^{-\pi\sqrt{n}}$, n is a positive real number, and $\chi(q) = (-q;q^2)_\infty$.
Setting $n = 2$ and replacing k by n, we get

$$A_{n,2} = \frac{r_{2,4n}^2}{r_{2,n}}, \tag{4.7}$$

where we used the result $r_{k,n} = r_{n,k}$ from (2.10).

Using the result (4.6) in (4.7), we complete the proof.

To prove (ii), we replace n by $n/2$ in part (i) and use the result $g_{4n} = 2^{1/4} g_n G_n$ from [4, p. 187, Entry 2.1], where the class invariant G_n is given by $G_n = 2^{-1/2} q^{-1/24} \chi(q)$. □

5. Explicit evaluations of $A_{k,n}$

In this section, we prove some general theorems on $A_{k,n}$ and then use these theorems to find explicit values of $A_{k,n}$.

Theorem 5.1. *We have*

$$2^{5/2}\left(A_{2,n}^2 + \frac{1}{A_{2,4n}^2}\right) + \left(\frac{2A_{2,n}}{A_{2,4n}}\right)^2 - \left(\frac{A_{2,4n}}{A_{2,n}}\right)^2 + 8 = 0.$$

Proof. We use the definition of $A_{k,n}$ in Theorem 3.1 to prove the theorem. □

Theorem 5.2. *We have*

(i) $A_{2,2} = \sqrt{2+\sqrt{2}}$,

(ii) $A_{2,4} = \sqrt{2\left(2+\sqrt{2}+\sqrt{7+5\sqrt{2}}\right)}$,

(iii) $A_{2,1/2} = 1/\sqrt{2+\sqrt{2}} = \sqrt{\frac{2-\sqrt{2}}{2}}$,

(iv) $A_{2,1/4} = 1/\sqrt{2\left(2+\sqrt{2}+\sqrt{7+5\sqrt{2}}\right)} = \sqrt{\left((\sqrt{2}-1)\sqrt{7+5\sqrt{2}}-\sqrt{2}\right)/2}.$

Proof. Setting $n = 1/2$ in Theorem 5.1 and using Theorem 4.1(ii), we obtain

$$\frac{2^{7/2}}{A_{2,2}^2} + \left(\frac{4}{A_{2,2}^4} - A_{2,2}^4\right) + 8 = 0. \tag{5.1}$$

Solving (5.1) and using the fact in Remark 4.2, we complete the proof of (i).

Again setting $n = 1$ in Theorem 5.1 and using Theorem 4.1(i), we obtain

$$2^{5/2}\left(1 + \frac{1}{A_{2,4}^2}\right) + \left(\frac{4}{A_{2,4}^2} - A_{2,4}^2\right) + 8 = 0. \tag{5.2}$$

Solving (5.2) and using the fact in Remark 4.2, we prove (ii).

Proofs of (iii) and (iv) easily follow from parts (i) and (ii), respectively and the Theorem 4.1(ii). □

Theorem 5.3. *We have*

$$\left(\frac{A_{2,n}}{A_{2,9n}}\right)^2 - \left(\frac{A_{2,9n}}{A_{2,n}}\right)^2 + 2^{5/2}\left(A_{2,n}A_{2,9n} + \frac{1}{A_{2,n}A_{2,9n}}\right) + 6\left(\frac{A_{2,n}}{A_{2,9n}} + \frac{A_{2,9n}}{A_{2,n}}\right) = 0.$$

Proof. Proof follows from Theorem 3.2 and the definition of $A_{k,n}$. □

Theorem 5.4. *We have*

(i) $A_{2,3} = \sqrt{2+\sqrt{2}+\sqrt{9+6\sqrt{2}}}$,
(ii) $A_{2,9} = 3+2\sqrt{2}+\sqrt{2(9+6\sqrt{2})}$,
(iii) $A_{2,1/3} = (3-2\sqrt{2})\sqrt{9+6\sqrt{2}}+\sqrt{2}-2$,
(iv) $A_{2,1/9} = \sqrt{2(9+6\sqrt{2})}-(3+2\sqrt{2})$.

Proof. Setting $n = 1/3$ in Theorem 5.3 and using Theorem 4.1(ii), we obtain

$$\left(\frac{1}{A_{2,3}^4} - A_{2,3}^4\right) + 6\left(\frac{1}{A_{2,3}^2} + A_{2,3}^2\right) + 2^{7/2} = 0. \tag{5.3}$$

Solving (5.3) and using the fact in Remark 4.2, we arrive at (i).

Again setting $n = 1$ in Theorem 5.3 and using Theorem 4.1(i), we obtain

$$\left(\frac{1}{A_{2,9}^2} - A_{2,9}^2\right) + (6+2^{5/2})\left(\frac{1}{A_{2,9}} + A_{2,9}\right) = 0. \tag{5.4}$$

Solving (5.4) and using the fact in Remark 4.2, we complete the proof of (ii).

Noting $A_{2,1/3} = 1/A_{2,3}$ and $A_{2,1/9} = 1/A_{2,9}$ from Theorem 4.1(ii), we prove (iii) and (iv), respectively. □

Theorem 5.5. *We have*

(i) $A_{3,2} = \left((2+2\sqrt{2})(1+\sqrt{2}+\sqrt{6})\right)^{1/4}$,
(ii) $A_{4,4} = 2^{1/4}\left(1+\sqrt{2})^{3/4}(16+152^{1/4}+12\sqrt{2}+92^{3/4}\right)^{1/4}$,
(iii) $A_{5,5} = 2^{1/2}(5+\sqrt{5})^{1/2}(5^{1/4}-1)^{-2} = 2^{-7/2}(5+\sqrt{5})^{1/2}(5^{1/4}+1)^2(\sqrt{5}+1)^2$,
(iv) $A_{5,2} = 2^{-1/2}(1+\sqrt{5})^{3/4}(2+3\sqrt{2}+\sqrt{5})^{1/4}$,
(v) $A_{9,2} = 2^{1/4}(\sqrt{3}+\sqrt{2})(1+35\sqrt{2}-28\sqrt{3})^{1/4}$,
(vi) $A_{8,2} = 2^{3/16}(1+\sqrt{2})^{1/4}(16+152^{1/4}+12\sqrt{2}+2^{3/4})^{1/4}$,
(vii) $A_{3/2,2} = 2^{3/4}(1+\sqrt{3})^{1/4}$,

(viii) $A_{5/2,2} = 2^{-1/4}(1+\sqrt{5})^{1/2}(\sqrt{\sqrt{5}+1}+\sqrt{2})^{1/4}$,
(ix) $A_{25/2,2} = 2^{-17/8}(\sqrt{5}+1)^2(5^{1/4}+1)$.

Proof. The proof of the theorem follows from Theorem 4.7(i) and the corresponding values of $r_{k,n}$ from Lemma 2.6. □

One can easily find values of $A_{3,1/2}$, $A_{4,1/4}$, $A_{5,1/5}$, $A_{5,1/2}$, $A_{9,1/2}$, $A_{8,1/2}$, $A_{3/2,1/2}$, $A_{5/2,1/2}$, and $A_{25/2,1/2}$ by using Theorems 5.5 and 4.1(ii).

Theorem 5.6. *We have*

(i) $A_{1,1} = 1$,
(ii) $A_{1,2} = 2^{1/4}\left(1+\sqrt{2}\right)^{1/4}$,
(iii) $A_{1,3} = \left(2+\sqrt{3}\right)^{1/2}$,
(iv) $A_{1,4} = 2^{5/8}\left(1+\sqrt{2}\right)^{1/2}$,
(v) $A_{1,5} = \left(1+\sqrt{5}+\sqrt{2(1+\sqrt{5})}\right)\Big/2$,
(vi) $A_{1,6} = \left(1+\sqrt{2}\right)^{3/4}\left(2(1+\sqrt{2}+\sqrt{6})\right)1/4$
(vii) $A_{1,7} = \left(8+3\sqrt{7}\right)^{1/2}$,
(viii) $A_{1,8} = 2^{1/2}\left(1+\sqrt{2}\right)^{3/4}\left(4+\sqrt{2+10\sqrt{2}}\right)1/4$,
(ix) $A_{1,9} = \left(\frac{1}{2}+\frac{3^{1/4}}{\sqrt{2}}+\frac{\sqrt{3}}{2}\right)^2$,
(x) $A_{1,10} = \left(1+\sqrt{5}\right)^{9/4}\left(2+3\sqrt{2}+\sqrt{5}\right)1/4/4$,
(xi) $A_{1,16} = 2^{3/4}\left(1+\sqrt{2}\right)\left(16+15\cdot 2^{1/4}+12\sqrt{2}+9\cdot 2^{3/4}\right)1/4$,
(xii) $A_{1,18} = 2^{1/4}\left(\sqrt{3}+\sqrt{2}\right)^2\left(1+35\sqrt{2}-28\sqrt{3}\right)1/4$,
(xiii) $A_{1,25} = \left(3+\sqrt[4]{5}+\sqrt{5}+\sqrt[4]{5^3}\right)^2/4$,
(xiv) $A_{1,49} = \left(\sqrt{4+\sqrt{7}+\sqrt{21+8\sqrt{7}}}+\sqrt{\sqrt{7}+\sqrt{21+8\sqrt{7}}}\right)4\Big/16$.

Proof. The proof of the theorem follows from Corollary 4.8(i) and the corresponding values of $r_{4,n}$ from Lemma 2.6. □

The values of $A_{1,1/n}$ for n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 16, 18, 25, and 49 can easily be calculated by applying Theorems 5.6 and 4.1(ii).

Theorem 5.7. *We have*

$$A_{3,n}^4 = \frac{9-18h_{3,n}^4+8\sqrt{3}h_{3,n}^6-3h_{3,n}^8}{9h_{3,n}^8-18h_{3,n}^4+8\sqrt{3}h_{3,n}^2-3}.$$

Proof. A proof of the theorem follows directly from Theorem 3.3 and the definitions of $h_{k,n}$ and $A_{k,n}$ from (1.3) and (1.7), respectively. □

Theorem 5.8. *We have*

(i) $A_{3,3} = 3^{1/8}\left(-3+2\sqrt{3}\right)^{3/8}\left(30-18\sqrt{3}+\sqrt{-9+6\sqrt{3}}\right)^{-1/4}$,

(ii) $A_{3,4} = \sqrt{2}\left(-84+69\sqrt{2}+36\sqrt{3}-31\sqrt{6}\right)^{1/4}\left(105-90\sqrt{2}+126\sqrt{3}-80\sqrt{6}\right)^{-1/4}$,

(iii) $A_{3,5} = \left(-57-32\sqrt{3}+27\sqrt{5}+16\sqrt{15}\right)^{1/4}\left(3-8\sqrt{3}-9\sqrt{5}+8\sqrt{15}\right)^{-1/4}$,

(iv) $A_{3,9} = (-1+2^{1/3})^{1/2}\left(9+6\sqrt{3}\right)^{1/4}\left(-39+182^{1/3}+92^{2/3}-2\sqrt{3}+22^{2/3}\sqrt{3}\right)^{-1/4}$.

Proof. The proof of the theorem follows from Theorem 5.7 and the corresponding values of $h_{3,n}$ from Lemma 2.7. □

The values of $A_{3,1/n}$ for $n = 3$, 4, 5, and 9 can easily be found by applying Theorems 5.8 and 4.1(ii).

Theorem 5.9. *We have*

$$\begin{aligned} A_{6,6} = {} & 2^{-1/12}3^{1/8}\left(\sqrt{2}-1\right)^{-5/6}\left(1+\sqrt{3}\right)^{-1/3}\left(\sqrt{2}+\sqrt{3}\right)^{2/3}\left(-3+2\sqrt{3}\right)^{3/8} \\ & \times\left(1+\sqrt{3}+\sqrt{2}3^{3/4}\right)^{-1/3}\left(4-\sqrt{2}+2\sqrt{3}+3^{3/4}(\sqrt{3}+1)\right)^{2/3} \\ & \times\left(30-18\sqrt{3}+\sqrt{-9+6\sqrt{3}}\right)^{-1/4}. \end{aligned}$$

Proof. Setting $n = 2$ and $p = 3$ in Corollary 4.6, we get

$$A_{6,6} = A_{18,2}A_{3,3}. \tag{5.5}$$

Again setting $k = 18$ and $n = 2$ in Theorem 4.7(i), we find that

$$A_{18,2} = r_{72,2}^2 r_{18,2}^{-1} = r_{2,72}^2 r_{2,18}^{-1}, \tag{5.6}$$

where we used the result $r_{k,n} = r_{n,k}$ from (2.10).

Employing the values of $r_{2,72}$ and $r_{2,18}$ from Lemma 2.6 in (5.6), we obtain

$$\begin{aligned} A_{18,2} = {} & 2^{-1/12}\left(\sqrt{2}-1\right)^{-5/6}\left(1+\sqrt{3}\right)^{-1/3}\left(\sqrt{2}+\sqrt{3}\right)^{2/3}\left(1+\sqrt{3}+\sqrt{2}3^{3/4}\right)^{-1/3} \\ & \times\left(4-\sqrt{2}+2\sqrt{3}+3^{3/4}(\sqrt{3}+1)\right)^{2/3}. \end{aligned} \tag{5.7}$$

Employing the values of $A_{18,2}$ from (5.7) and $A_{3,3}$ from Theorem 5.8(i) in (5.5), we complete the proof. □

The value of $A_{6,1/6}$ easily follows from Theorems 5.9 and 4.1(ii).

6. Explicit evaluations of $\psi(q)$

In this section, we establish an explicit formula for $\psi(e^{-2n\pi})$, for positive real number n and give some examples.

Lemma 6.1. *Let $a = \pi^{1/4}/\Gamma(3/4)$. Then*

$$\phi(-e^{-\pi}) = a2^{-1/4}.$$

For proof, see Entry 1(ii) in Chapter 35 of [4, p. 325].

Theorem 6.2. *For every positive real number n, we have*

$$\psi(e^{-2n\pi}) = \frac{2^{-5/4}ae^{n\pi/4}}{n^{1/4}A_{n,n}}.$$

Proof. Using the definition of $A_{n,n}$ and Lemma 6.1, we complete the proof. □

Theorem 6.3. *We have*

(i) $\psi(e^{-2\pi}) = a2^{-5/4}e^{\pi/4}$,

(ii) $\psi(e^{-4\pi}) = a2^{-2}e^{\pi/2}\left(2-\sqrt{2}\right)^{1/2}$,

(iii) $\psi(e^{-6\pi}) = a2^{-5/4}3^{-3/8}e^{3\pi/4}\left(30-18\sqrt{3}+\sqrt{-9+6\sqrt{3}}\right)^{1/4}\left(-3+2\sqrt{3}\right)^{-3/8}$,

(iv) $\psi(e^{-8\pi}) = a2^{-2}e^{\pi}\left(1+\sqrt{2}\right)^{-3/4}\left(16+152^{1/4}+12\sqrt{2}+92^{3/4}\right)^{-1/4}$,

(v) $\psi(e^{-10\pi}) = a2^{-7/4}5^{-1/4}e^{5\pi/4}\left(5^{1/4}-1\right)^{2}\left(5+\sqrt{5}\right)^{-1/2}$,

(vi) $\psi(e^{-12\pi}) = ae^{3\pi/2}2^{-17/12}3^{-3/8}\left(\sqrt{2}-1\right)^{5/6}\left(1+\sqrt{3}\right)^{1/3}\left(\sqrt{2}+\sqrt{3}\right)^{-2/3}$
$\left(-3+2\sqrt{3}\right)^{-3/8}\times\left(1+\sqrt{3}+\sqrt{2}3^{3/4}\right)^{1/3}\left(4-\sqrt{2}+2\sqrt{3}+3^{3/4}(\sqrt{3}+1)\right)^{-2/3}$
$\left(30-18\sqrt{3}+\sqrt{-9+6\sqrt{3}}\right)^{1/4}$.

Proof. The proof of the theorem follows from Theorem 6.2 and the corresponding values of $A_{n,n}$ from Theorem 5.6(i), Theorem 5.2(i), Theorem 5.8(i), Theorem 5.5(ii) and (iii), and Theorem 5.9. □

The values of $\psi(e^{-n\pi})$ for $n = 2$, 4, and 8 can also be found in [4, p. 325].

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