



ORIGINAL ARTICLE

A generalization of the Meir–Keeler type contraction

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Received 4 June 2011; revised 22 January 2012; accepted 3 March 2012

Available online 15 March 2012

KEYWORDS

Fixed point theorems;
The Meir–Keeler
contraction;
 T -contraction

Abstract In this paper, we prove a fixed point theorem which has applications on maps called T -contractions which include a class that satisfies the Meir–Keeler type contractive condition. We also present an example that illustrates that T -contractions are a natural extension of the Meir–Keeler type contraction.

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1. Introduction and preliminaries

It is a fact that the fixed point theory has applications not only in many areas of Mathematics but also in many branches of quantitative sciences such as Economics and Computer Sciences. The most famous result in this field is known as the Banach Contraction Principle [3] which states that each contraction T on a

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Peer review under responsibility of King Saud University.

<http://dx.doi.org/10.1016/j.ajmsc.2012.03.001>



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complete metric space (X, d) has a unique fixed point. Here d denotes a given metric on X . A self-mapping $T: X \rightarrow X$ is called a contraction if there exists a constant $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)$.

In the literature one of the elegant generalizations of the Banach Contraction Principle is called the Meir–Keeler Contraction Principle [8]. Meir–Keeler contraction has many extensions studied by many authors in the area (see [1,2,7,9,11]). In this article, we introduce new extensions from the view point of T -contractions which are extensively developed in [4,5,10].

We now state the theorem of Ćirić [6] which is more general and therefore more suitable than Meir–Keeler’s theorem [8] for our purposes.

Theorem 1. *Let T be a self-mapping on a complete metric space X*

$$\begin{aligned} & \text{Given } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ & \varepsilon \leq d(x, y) < \varepsilon + \delta \text{ implies that } d(Tx, Ty) < \varepsilon. \end{aligned} \tag{1.1}$$

Then T has a unique fixed point $z \in X$.

Definition 2 (see e.g. [5,10]). Let T and S be two self-mappings on a metric space (X, d) . A mapping S is said to be a T -contraction if there exists $k \in (0, 1)$ such that

$$d(TSx, TSy) \leq kd(Tx, Ty) \tag{2}$$

for each $x, y \in X$.

It is clear that if we choose $Tx = x$ for all $x \in X$ then a T -contraction becomes a contraction. We would like to present an example for Definition 2.

Example 3. Let $X = [1, \infty)$ with the usual metric $d(x, y) = |x - y|$ induced by (\mathbb{R}, d) . Consider the following self-mappings $T(x) = 2 - \frac{1}{x}$ and $Sx = 6x$ on X . It is clear that S is not a contraction. On the contrary,

$$d(TSx, TSy) = \left| 2 - \frac{1}{6x} - 2 + \frac{1}{6y} \right| = \frac{1}{6} \left| \frac{1}{y} - \frac{1}{x} \right| = \frac{1}{6} \left| 2 - \frac{1}{x} - 2 + \frac{1}{y} \right| = \frac{1}{6} d(Tx, Ty).$$

Definition 4 (see e.g. [4,10]). Let (X, d) be a metric space. A mapping $T: X \rightarrow X$ is called *sequentially convergent* if the statement $\{Ty_n\}$ is convergent implies that $\{y_n\}$ is a convergent sequence for every sequence $\{y_n\}$.

2. Main results

We start this section with the first of our main theorems.

Theorem 5. *Let (X, d) be a complete metric space. Let $T: X \rightarrow X$ be an injective, continuous and sequentially convergent mapping. If a self-mapping S of X satisfies the condition:*

$$\begin{aligned} & \text{Given } \varepsilon > 0 \text{ there exists } \delta > 0 \text{ such that} \\ & \varepsilon \leq d(Tx, Ty) < \varepsilon + \delta \text{ implies that } d(TSx, TSy) < \varepsilon, \end{aligned} \tag{2.1}$$

then S has a fixed point $z \in X$ and $\lim_{n \rightarrow \infty} T^n x = z$ for every $x \in X$.

Proof. Let T and S be the maps defined in Theorem 5. Let $x_0 \in X$ be an arbitrary point. We construct two iterative sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$x_{n+1} = Sx_n = S^{n+1}x_0 \text{ and } y_n = Tx_n \tag{2.2}$$

for $n = 0, 1, 2, \dots$. Notice that if $y_{n_0+1} = y_{n_0}$ holds for some $n_0 \in \mathbb{N}$ then $Tx_{n_0+1} = Tx_{n_0}$. Since T is an injective mapping, $x_{n_0+1} = x_{n_0}$ if and only if $Sx_{n_0} = x_{n_0}$. Thus, x_{n_0} is a fixed point of S . Therefore, we suppose that

$$y_{n+1} \neq y_n \tag{2.3}$$

for $n \in \mathbb{N}$. Regarding (2.3), we have

$$d(y_n, y_{n+1}) > 0 \tag{2.4}$$

for $n = 0, 1, 2, \dots$. Due to (2.1), the sequence $K = \{d(y_n, y_{n+1})\} = \{d(Tx_n, Tx_{n+1})\}$ of real numbers is non-increasing and is bounded below by 0. Hence, K converges to $\varepsilon_0 \geq 0$, the greatest lower bound of K , that is,

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = \varepsilon_0.$$

We assert that $\varepsilon_0 = 0$. Assume on the contrary that $\varepsilon_0 > 0$. Then, there exists $\delta_0 = \delta(\varepsilon_0)$ and there exists some $m \in \mathbb{N}$ such that

$$\varepsilon_0 \leq d(y_m, y_{m+1}) = d(Tx_m, Tx_{m+1}) < \varepsilon_0 + \delta_0.$$

By (2.1), this implies that

$$d(TSx_m, TSx_{m+1}) = d(Tx_{m+1}, Tx_{m+2}) = d(y_{m+1}, y_{m+2}) < \varepsilon_0$$

which contradicts the fact that ε_0 is the greatest lower bound of S . Thus, we obtain that

$$\lim_{n \rightarrow \infty} d(y_n, y_{n+1}) = \lim_{n \rightarrow \infty} d(Tx_n, Tx_{n+1}) = 0. \tag{2.5}$$

We show that $\{y_n\}$ is a Cauchy sequence. Take $\varepsilon > 0$ and choose $\delta = \delta(\varepsilon)$ in such a way that $\delta \leq \varepsilon$. Regarding (2.5), there exists some positive integer M such that

$$d(y_{n-1}, y_n) = d(Tx_{n-1}, Tx_n) < \delta \text{ for all } n > M. \tag{2.6}$$

Now, let us fix $n > M$. To conclude that $\{y_n\}$ is a Cauchy sequence, it is sufficient to show that

$$d(y_n, y_{n+p}) = d(Tx_n, Tx_{n+p}) \leq \varepsilon \tag{2.7}$$

for $p = 1, 2, \dots$. We prove (2.7) by induction. Since $\delta \leq \varepsilon$, the inequality (2.7) for the case $p = 1$ follows from (2.6) and (2.1). Now, suppose that (2.7) holds for some fixed $p \in \mathbb{N}$. Then by (2.6) and the assumption we have,

$$d(Tx_{n-1}, Tx_{n+p}) = d(y_{n-1}, y_{n+p}) \leq d(y_{n-1}, y_n) + d(y_n, y_{n+p}) < \delta + \varepsilon.$$

Thus, by (2.1), we get

$$d(TSx_{n-1}, TSx_{n+p}) = d(Tx_n, Tx_{n+p+1}) = d(y_n, y_{n+p+1}) \leq \varepsilon. \tag{2.8}$$

Thus we proved that (2.7) holds for all $p \in \mathbb{N}$ and hence $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a $w \in X$ such that

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} Tx_n = \lim_{n \rightarrow \infty} TS^n x_0 = w. \tag{2.9}$$

Since T is sequentially convergent, $\{x_n\} = \{S^n x_0\}$ converges to some point in X , say z . By the continuity of T , we have $Tz = w$. Also, since TS is continuous, we have

$$w = \lim_{n \rightarrow \infty} y_{n+1} = \lim_{n \rightarrow \infty} TSx_n = TSz.$$

We obtain $TSz = Tz$. Since T is injective, we get $Sz = z$. To conclude the proof, let us show that z is a unique fixed point of S . Assume the contrary, that is, there exists $w \in X$ such that $w \neq z$ and $Sw = w$. Thus, $d(z, w) > 0$ and since T is injective $d(Tz, Tw) > 0$. Given $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon)$ such that $\varepsilon \leq d(Tz, Tw) < \varepsilon + \delta$. Due to (2.1), we get $d(TSz, TSw) = d(Tz, Tw) < \varepsilon$ which is a contradiction. \square

The following theorem is an extension of the main result of [1].

Theorem 6. *Let (X, d) be a complete metric space and $F: X \rightarrow X$ be a continuous map. Let $T: X \rightarrow X$ be an injective, continuous and sequentially convergent mapping. Suppose for each $\varepsilon > 0$ there exists $\delta > 0$ such that for $x, y \in X$ we have that*

$$\varepsilon \leq M_T(x, y) < \varepsilon + \delta \text{ implies } d(TFx, TFy) < \varepsilon, \tag{2.10}$$

where

$$M_T(x, y) = \max \left\{ d(Tx, Ty), d(Tx, TFx), d(Ty, TFy), \frac{1}{2}[d(Tx, TFy) + d(Ty, TFx)] \right\}$$

Then there exists a unique $x \in X$ with $x = Fx$.

Proof. Let $x_0 \in X$ be an arbitrary point. We construct two iterative sequences $\{x_n\}$ and $\{y_n\}$ in the following way:

$$x_{n+1} = Fx_n = F^{n+1}x_0 \text{ and } y_n = Tx_n \text{ for } n = 0, 1, 2, \dots \tag{2.11}$$

Notice that if $y_{n_0+1} = y_{n_0}$ holds for some $n_0 \in \mathbb{N}$ then $Tx_{n_0+1} = Tx_{n_0}$. Since T is an injective mapping, $x_{n_0+1} = x_{n_0}$ if and only if $Fx_{n_0} = x_{n_0}$. Thus, x_{n_0} is a fixed point of F . Therefore, we suppose that

$$y_{n+1} \neq y_n \tag{2.12}$$

for every $n \in \mathbb{N}$. Hence $\delta_n = d(y_{n+1}, y_n) > 0$ for all $n \in N$. Now, we claim that $\{y_n\}$ is a Cauchy sequence. If $\delta_{n-1} < \delta_n$ for some $n \in N$ then we have

$$\begin{aligned} M_T(x_{n-1}, x_n) &= \max \left\{ d(Tx_{n-1}, Tx_n), d(Tx_{n-1}, TFx_{n-1}), d(Tx_n, TFx_n), \right. \\ &\quad \left. \frac{1}{2} [d(Tx_{n-1}, TFx_n) + d(Tx_n, TFx_{n-1})] \right\} \\ &= \max \left\{ d(y_{n-1}, y_n), d(y_{n-1}, y_n), d(y_n, y_{n+1}), \frac{1}{2} d(y_{n-1}, y_{n+1}) \right\} \\ &\leq \max \left\{ d(y_n, y_{n+1}), d(y_{n-1}, y_n), \frac{1}{2} d(y_{n-1}, y_n) + d(y_n, y_{n+1}) \right\} \\ &< d(y_n, y_{n+1}) + d(y_{n-1}, y_n) = \delta_n + \delta_{n-1} \end{aligned} \tag{2.13}$$

and $M_T(x_{n-1}, x_n) \geq d(y_{n-1}, y_n) = \delta_{n-1}$. We are able to calculate a concrete value of M_T . Namely, $M_T(x_{n-1}, x_n)$ is equal to δ_n , what account of (2.10) leads to

$$\delta_n = d(y_n, y_{n+1}) = d(TFx_{n-1}, TFx_n) < \delta_n,$$

a contradiction. Therefore $\delta_n \leq \delta_{n-1}$ for all n . Thus $\{\delta_n\}$ is a non-increasing sequence of positive real numbers and there exists $r > 0$ such that $\lim_{n \rightarrow \infty} \delta_n = \inf_n \delta_n = r$. We assert that $r = 0$. If $r > 0$ then there exists $\delta > 0$ such that

$$r \leq M_T(x, y) < r + \delta \text{ implies } d(TFx, TFy) < r.$$

Since $\lim_{n \rightarrow \infty} \delta_n = \inf_n \delta_n = r$, there exists a positive integer N such that

$$r < \delta_n < r + \delta$$

for every $n \geq N$. It follows from (2.13) and $d(y_n, y_{n+1}) \leq d(y_{n-1}, y_n)$ that

$$M_T(x_{n-1}, x_n) = d(y_{n-1}, y_n) = \delta_{n-1}.$$

Hence, if $n \geq N + 1$ then

$$r < M_T(x_{n-1}, x_n) < r + \delta$$

and this implies that $d(TFx_{n-1}, TFx_n) = d(y_n, y_{n+1}) < r$. We get a contradiction. Therefore $\lim_{n \rightarrow \infty} \delta_n = \inf_n \delta_n = 0$. Next, we show that $\{y_n\}$ is a Cauchy sequence.

Suppose not. Then there is an $\varepsilon > 0$ and a subsequence $\{y_{n(k)}\}$ of $\{y_n\}$ such that $d(y_{n(k)}, y_{n(k+1)}) > 2\varepsilon$. For this ε , there exists δ such that

$$\varepsilon \leq M_T(x, y) < \varepsilon + \delta \text{ implies } d(TFx, TFy) < \varepsilon.$$

Set $\delta_0 = \min\{\varepsilon, \delta\}$. Since $\lim_{n \rightarrow \infty} \delta_n = \inf_n \delta_n = 0$, there exists $N \in \mathbb{N}$ such that $\delta_m < \frac{\delta_0}{4}$ for $m \geq N$. Let $n(k) \geq N$. If $d(y_{n(k)}, y_{n(k+1)-1}) \leq \varepsilon + \frac{\delta}{2}$ then

$$d(y_{n(k)}, y_{n(k+1)}) \leq d(y_{n(k)}, y_{n(k+1)-1}) + d(y_{n(k+1)-1}, y_{n(k+1)}) \leq \varepsilon + \frac{\delta_0}{2} + \frac{\delta_0}{4} < 2\varepsilon.$$

We achieve a contradiction with $d(y_{n(k)}, y_{n(k+1)}) > 2\varepsilon$. Hence, there are integers l with $n(k) \leq l \leq n(k + 1)$ such that

$$d(y_{n(k)}, y_l) > \varepsilon + \frac{\delta_0}{2}.$$

Let l be the smallest integers with $l \geq n(k)$ such that

$$d(y_{n(k)}, y_l) \geq \varepsilon + \frac{\delta_0}{2}.$$

Then we obtain

$$d(y_{n(k)}, y_{l-1}) < \varepsilon + \frac{\delta_0}{2}.$$

We have

$$d(y_{n(k)}, y_l) \leq d(y_{n(k)}, y_{l-1}) + d(y_{l-1}, y_l) < \varepsilon + \frac{\delta_0}{2} + \frac{\delta_0}{4} = \varepsilon + \frac{3\delta_0}{4}.$$

Thus, there exists an integers l with $n(k) \leq l \leq n(k + 1)$ such that

$$\varepsilon + \frac{\delta_0}{2} \leq d(y_{n(k)}, y_l) < \varepsilon + \frac{3\delta_0}{4}.$$

Since

$$d(y_{n(k)}, y_l) < \varepsilon + \frac{3\delta_0}{4} < \varepsilon + \delta_0$$

$$d(y_{n(k)}, y_{n(k+1)}) = \delta_{n(k)} < \frac{\delta_0}{4} < \varepsilon + \delta_0$$

$$d(x_l, x_{l+1}) \leq \delta_l < \frac{\delta_0}{4} < \varepsilon + \delta_0$$

and

$$\begin{aligned} \frac{1}{2}[d(y_{n(k)}, y_{l+1}) + d(y_{n(k)+1}, y_l)] &\leq \frac{1}{2}[d(y_{n(k)}, y_l) + d(y_l, y_{l+1}) + d(y_{n(k)+1}, y_{n(k)}) + d(y_{n(k)}, y_l)] \\ &= d(y_{n(k)}, y_l) + \frac{1}{2}[d(y_l, y_{l+1}) + d(y_{n(k)+1}, y_{n(k)})] \\ &\leq \varepsilon + \frac{3\delta_0}{4} + \frac{\delta_0}{4} = \varepsilon + \delta_0 \end{aligned}$$

we have

$$M_T(x_{n(k)}, x_l) < \varepsilon + \delta_0.$$

It follows that

$$d(y_{n(k)+1}, y_{l+1}) = d(TFx_{n(k)}, TFx_l) < \varepsilon.$$

On the other hand

$$d(y_{n(k)+1}, y_{l+1}) \geq d(y_{n(k)}, y_l) - d(y_{n(k)}, y_{n(k)+1}) - d(y_l, y_{l+1}) > \varepsilon + \frac{\delta_0}{2} - \frac{\delta_0}{4} - \frac{\delta_0}{4} = \varepsilon,$$

a contradiction. Hence $\{y_n\}$ is a Cauchy sequence. Since (X, d) is complete, there exists $y \in X$ such that

$$\lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} Tx_n = y.$$

Since T is sequentially convergent, we deduce that $\{x_n\}$ converges to $x \in X$. By continuity of F , we have $x = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} Fx_n = Fx$. It remains to prove the uniqueness. If $y = Fy$ with $y \neq x$ then

$$M_T(x, y) = d(Tx, Ty) > 0.$$

By the condition (2.10), we have

$$d(Tx, Ty) = d(TFx, TFy) < d(Tx, Ty).$$

This is a contradiction. \square

If we choose $Tx = x$ for all $x \in X$ then Theorem 5 implies Theorem 1. The following example is an illustration for our extension.

Example 7. Let $X = [1, +\infty)$ with metric induced by $\mathbb{R} : d(x, y) = |x - y|$. Then X is a complete metric space. Consider the function $Sx = \frac{8}{\sqrt{x}}$ for all $x \in X$. It is easy to see that $x = 4$ is the unique fixed point of S . We will claim that S do not satisfy the Theorem 1. Indeed, for each $\varepsilon > 0$ and $x, y \in X$ satisfying

$$\varepsilon \leq d(x, y) = |x - y| < \varepsilon + \delta$$

for some $\delta > 0$, we have

$$d(Sx, Sy) = \left| \frac{8}{\sqrt{x}} - \frac{8}{\sqrt{y}} \right| = 8 \frac{|x - y|}{\sqrt{xy}(\sqrt{x} + \sqrt{y})} \geq \varepsilon$$

if

$$\sqrt{xy}(\sqrt{x} + \sqrt{y}) \leq 8.$$

Hence, the condition (1.1) does not hold for any δ . Now, we will show that S satisfies Theorem 5. To do this, we consider the map $T: [1, +\infty) \rightarrow [1, +\infty)$ defined by

$$Tx = \ln(x + 1), \quad x \in X.$$

Observe that T is continuous, sequentially convergent, injective. For each $\varepsilon > 0$, if we choose $\delta = \frac{\varepsilon}{3}$ and

$$\varepsilon < d(Tx, Ty) = |\ln x - \ln y| < \varepsilon + \delta = \frac{4}{3}\varepsilon$$

then

$$d(TSx, TSy) = \left| \ln \frac{8}{\sqrt{x}} - \ln \frac{8}{\sqrt{y}} \right| = \frac{1}{2} |\ln x - \ln y| < \frac{2}{3}\varepsilon < \varepsilon.$$

This shows that S satisfies Theorem 5.

Acknowledgement

The authors would like to extend their gratitude to the anonymous referees for their constructive and useful remarks and suggestions.

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