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# A characterization of projective special unitary group $U_3(5)$ by nse

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**Abstract.** Let $G$ be a group and $\omega(G)$ be the set of element orders of $G$. Let $k \in \omega(G)$ and $s_k$ be the number of elements of order $k$ in $G$. Let $\mathrm{nse}(G) = \{s_k | k \in \omega(G)\}$. In Khatami et al. and Liu's works $L_3(2)$ and $L_3(4)$ are unique determined by nse($G$). In this paper, we prove that if $G$ is a group such that $\mathrm{nse}(G) = \mathrm{nse}(U_3(5))$, then $G \cong U_3(5)$.

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## 1. Introduction

A finite group $G$ is called a simple $K_4$-group, if $G$ is a simple group with $|\pi(G)| = 4$. In 1987, J.G. Thompson posed a very interesting problem related to algebraic number fields as follows (see [13]).

*Thompson's Problem*. Let $T(G) = \{(n,s_n) | n \in \omega(G)$ and $s_n \in \mathrm{nse}(G)\}$, where $s_n$ is the number of elements with order $n$. Suppose that $T(G) = T(H)$. If $G$ is a finite solvable group, is it true that $H$ is also necessarily solvable?

It is easy to see that if $G$ and $H$ are of the same order type, then $\mathrm{nse}(G) = \mathrm{nse}(H)$ and $|G| = |H|$. It was proved that: Let $G$ be a group and $M$ some simple $K_i$-group, $i = 3,4$, then $G \cong M$ if and only if $|G| = |M|$ and $\mathrm{nse}(G) = \mathrm{nse}(M)$ (see [11,10]). And also the group $A_{12}$ is characterizable by order and nse (see [7]). Recently, all sporadic simple groups are characterizable by nse and order (see [5]).

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Comparing the sizes of elements of same order but disregarding the actual orders of elements in $T(G)$ of the Thompson Problem, in other words, it remains only nse($G$), whether can it characterize finite simple groups? Up to now, some groups especial for $L_2(q)$, where $q \in \{7, 8, 9, 11, 13\}$, can be characterized by only the set nse($G$) (see [6,12]). The author has proved that the group $L_3(4)$ is characterizable by nse (see [8]). In this paper, it is shown that the group $U_3(5)$ also can be characterized by nse($U_3(5)$).

Here we introduce some notations which will be used. If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. Let $G$ be a group. The set of element orders of $G$ is denoted by $\omega(G)$. Let $k \in \omega(G)$ and $s_k$ be the number of elements of order $k$ in $G$. Let nse$(G) = \{s_k | k \in \omega(G)\}$. Let $\pi(G)$ denote the set of prime p such that $G$ contains an element of order $p$. $L_n(q)$ denotes the projective special linear group of degree $n$ over finite fields of order $q$. $U_n(q)$ denotes the projective special unitary group of degree $n$ over finite fields of order $q$. The other notations and notions are standard (See [1]).

## 2. Some lemmas

**Lemma 2.1.** [2]. *Let G be a finite group and m be a positive integer dividing* $|G|$. *If* $L_m(G) = \{g \in G | g^m = 1\}$, *then* $m \mid |L_m(G)|$.

**Lemma 2.2.** [9]. *Let G be a finite group and* $p \in \pi(G)$ *be odd. Suppose that P is a Sylow p-subgroup of G and* $n = p^s m$ *with* $(p,m) = 1$. *If P is not cyclic and* $s > 1$, *then the number of elements of order n is always a multiple of* $p^s$.

**Lemma 2.3.** [12]. *Let G be a group containing more than two elements. If the maximal number s of elements of the same order in G is finite, then G is finite and* $|G| \leqslant s(s^2 - 1)$.

**Lemma 2.4.** [3, Theorem 9.3.1]. *Let G be a finite solvable group and* $|G| = mn$, *where* $m = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$, $(m, n) = 1$. *Let* $\pi = \{p_1, \ldots, p_r\}$ *and* $h_m$ *be the number of Hall* $\pi$*-subgroups of G. Then* $h_m = q_1^{\beta_1} \cdots q_s^{\beta_s}$ *satisfies the following conditions for all* $i \in \{1, 2, \ldots, s\}$:

(1) $q_i^{\beta_i} \equiv 1 \pmod{p_j}$ for some $p_j$.
(2) *The order of some chief factor of G is divided by* $q_i^{\beta_i}$.

To prove $G \cong U_3(5)$, we need the structure of simple $K_4$-groups.

**Lemma 2.5.** [14]. *Let G be a simple* $K_4$*-group. Then G is isomorphic to one of the following groups:*

(1) $A_7$, $A_8$, $A_9$ or $A_{10}$.
(2) $M_{11}$, $M_{12}$ or $J_2$.
(3) *One of the following:*
  (a) $L_2(r)$, where $r$ is a prime and $r^2 - 1 = 2^a \cdot 3^b \cdot v^c$ with $a \geqslant 1$, $b \geqslant 1$, $c \geqslant 1$, and *v is a prime greater than 3.*

(b) $L_2(2^m)$, where $2^m - 1 = u$, $2^m + 1 = 3t^b$ with $m \geqslant 2$, $u$, $t$ are primes, $t > 3$, $b \geqslant 1$.
(c) $L_2(3^m)$, where $3^m + 1 = 4t$, $3^m - 1 = 2u^c$ or $3^m + 1 = 4t^b$, $3^m - 1 = 2u$, with $m \geqslant 2$, $u$, $t$ are odd primes, $b \geqslant 1$, $c \geqslant 1$.

(4) *One of the following 28 simple groups:* $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_8^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_3(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, ${}^2D_4(2)$ or ${}^2F_4(2)'$.

**Lemma 2.6.** *Let G be a simple $K_4$-group and $5^3 \| G \| 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$. Then $G \cong U_3(5)$.*

**Proof.** From Lemma 2.5(1)(2), order consideration rules out this case. So we consider Lemma 2.5(3). We will deal with this with the following cases.

Case 1. $G \cong L_2(r)$, where $r \in \{3, 5, 7\}$.
Let $r = 3$, then $|\pi(r^2 - 1)| = 1$, which contradicts $|\pi(r^2 - 1)| = 3$.
Let r = 5, 7 then $|\pi(r^2 - 1)| = 2$, which contradicts $|\pi(r^2 - 1)| = 3$.
Case 2. $G \cong L_2(2^m)$, where $u \in \{3, 5, 7\}$.
Let $u = 3$, then m = 2 and so $5 = 3t^b$. But the equation has no solution in $N$, a contradiction.
Let $u = 5$, then $2^m - 1 = 5$. But the equation has no solution in $N$.
Let $u = 7$, then $m = 3$, and $2^3 + 1 = 3t^b$. Thus $t = 3$ and $b = 1$. But $t > 3$, a contradiction.
Case 3. $G \cong L_2(3^m)$ We will consider the case by the following two subcases.
Subcase 3.1. $3^m + 1 = 4t$ and $3^m - 1 = 2u^c$.
We can suppose that $t \in \{3, 5, 7\}$.
Let $t = 3, 5, 7$, the equation $3^m + 1 = 4t$ has no solution.
Subcase 3.2. $3^m + 1 = 4t^b$ and $3^m - 1 = 2u$.
We can suppose that $u \in \{3, 5, 7\}$
Let $u = 3, 5, 7$, then the equation $3^m - 1 = 2u$ has no solution in $N$, a contradiction.
In review of Lemma 2.5(4), $G \cong U_3(5)$.

This completes the proof of the Lemma. □

## 3. Main theorem and its proof

Let $G$ be a group such that nse($G$) = nse($U_3(5)$), and $s_n$ be the number of elements of order $n$. By Lemma 2.3 we have $G$ that is finite. We note that $s_n = k\phi(n)$, where $k$ is the number of cyclic subgroups of order $n$. Also we note that if $n > 2$, then $\phi(n)$ is even. If $m \in \omega(G)$, then by Lemma 2.1 and the above discussion, we have

$$\begin{cases} \phi(m) | s_m \\ m | \sum_{d|m} s_d \end{cases} \quad (1)$$

**Theorem 3.1.** *Let G be a group with nse(G) = nse($U_3(5)$) = {1, 525, 3500, 10,500, 12,600, 15,624, 15,750, 31,500, 36,000}, where $U_3(5)$ is the projective special unitary group of degree 3 over field of order 5. Then $G \cong U_3(5)$.*

**Proof.** We prove the theorem by first proving that $\pi(G) \subseteqq \{2, 3, 5, 7\}$, second showing that $|G| = |U_3(5)|$, and so $G \cong U_3(5)$.

By (1), $\pi(G) \subseteqq \{2, 3, 5, 7, 19, 37, 10{,}501, 12{,}601\}$. If $m > 2$, then $\phi(m)$ is even, then $s_2 = 525$, $2 \in \pi(G)$.

In the following, we prove that $19 \notin \pi(G)$. If $19 \in \pi(G)$, then by (1), $s_{19} = 15750$. If $2{\cdot}19 \in \omega(G)$, then by Lemma 2.1, $\phi(2{\cdot}19) \mid s_{2\cdot19}$ and so $s_{2\cdot19} = 12{,}600, 15{,}624, 15{,}750, 31{,}500, 36{,}000$. On the other hand, $2.19 \mid 1 + s_2 + s_{19} + s_{2\cdot19} (= 28{,}876, 31{,}900, 32{,}026, 47{,}776, 52{,}276)$, a contradiction. So $s_{2\cdot19} \notin$ nse($G$). Therefore $2{\cdot}19 \notin \omega(G)$. Now we consider Sylow 19-subgroup $P_{19}$ acts fixed point freely on the set of elements of order 2, then $|P_{19}| \mid s_2$, a contradiction. Similarly we can prove that the primes 37, 10,501 and 12,601 do not belong to $\pi(G)$. Hence we have $\pi(G) \subseteqq \{2, 3, 5, 7\}$. Furthermore, by (1) $s_3 = 3500$, $s_5 = 15{,}624$, and $s_7 = 36{,}000$.

If $2^a \in \omega(G)$, then $\phi(2^a) = 2^{a-1} | s_{2^a}$ and so $0 \leqslant a \leqslant 6$.
If $3^a \in \omega(G)$, then $1 \leqslant a \leqslant 3$.
If $5^a \in \omega(G)$, then $1 \leqslant a \leqslant 4$.
If $7^a \in \omega(G)$, then $1 \leqslant a \leqslant 2$.

Therefore we have that $\{2\} \subseteqq \pi(G) \subseteqq \{2, 3, 5, 7\}$
Since $\exp(P_2) = 2, 4, 8, 16, 32, 64$, then by Lemma 2.1, $|P_2| \,|\, 1 + s_2 + s_{2^2} + \cdots + s_{2^6}$ and so $|P_2| \mid 2^6$.

If $3 \in \pi(G)$ and $\exp(P_3) = 3, 9, 27$, then by Lemma 2.1, $|P_3| \mid 1 + s_3 + s_9 + s_{27}$ and so $|P_3| \mid 3^5$
If $5 \in \pi(G)$ and $\exp(P_5) = 5, 25, 125, 625$, then by Lemma 2.1, $|P_5| \,|\, 1 + s_5 + s_{5^2} + s_{5^3} + s_{5^4}$ and so $|P_5| \mid 5^4$.
If $7 \in \pi(G)$ and $\exp(P_7) = 7, 49$, then by Lemma 2.1, $|P_7| \,|\, 1 + s_7 + s_{7^2}$ and so $|P_7| \mid 7^2$.

Case a. $\pi(G) = \{2\}$. Therefore $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m$, where $k_1, \ldots, k_7$ and $m$ are non-negative integers. We rule out this case since $|\omega(G)| = 7$ and $|nse(G)| = 9$.

Case b. $\pi(G) = \{2, 3\}$. We know that $\exp(P_3) = 3, 9, 27$.
Let $\exp(P_3) = 3$. Then by Lemma 2.1, $|P_3| \mid 1 + s_3$ and so $|P_3| \mid 3^2$. If $|P_3| = 3$, then since $n_3 = s_3/\phi(3)$, $5, 7 \in \pi(G)$, a contradiction. Therefore $|P_3| = 9$ and $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m{\cdot}3^2$, where $k_1, \ldots, k_7$, and $m$ are non-negative integers and $0 \leqslant \sum_{i=1}^{7} k_i \leqslant 3$. Since $126{,}000 \leqslant |G| \leqslant 126{,}000 + 3{\cdot}36000$, then $m = 14$, which is a contradiction since $m$ is at most 7.

Let $\exp(P_3) = 9$. Then by Lemma 2.1, $|P_3| | 1 + s_3 + s_{3^2}$ and so $|P_3| | 3^3$. If $|P_3| = 9$, then $n_3 = s_{3^2}/\phi(3^2)$, it follows that 5 or 7 belongs to $\pi(G)$, a contradiction. Thus $|P_3| = 27$, then $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m \cdot 3^3$, where $k_1, \ldots, k_7$, and $m$ are non-negative integers and $0 \leqslant \sum_{i=1}^{7} k_i \leqslant 11$. Since $126{,}000 \leqslant |G| \leqslant 126{,}000 + 11 \cdot 36000$, then $m = 14, 15$, which is a contradiction since $m$ is at most 7.

Let $\exp(P_3) = 27$. Then by Lemma 2.1, $|P_3| | 1 + s_3 + s_9 + s_{27}$ and so $|P_3| | 3^5$. If $|P_3| = 27$, then $n_3 = s_{27}/\phi(27)$ and so $5, 7 \in \pi(G)$, a contradiction. If $|P_3| = 81$, then $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m \cdot 3^4$, where $k_1, \ldots, k_7$, and $m$ are non-negative integers and $0 \leqslant \sum_{i=1}^{7} k_i \leqslant 16$. Since $126{,}000 \leqslant |G| \leqslant 126{,}000 + 16 \cdot 36000$, then $m = 11, 12, 13$, which is a contradiction since $m$ is at most 7. Similarly if $|P_3| = 3^5$, then $m = 10, 11$, a contradiction.

Case c. $\pi(G) = \{2, 5\}$. We know that $\exp(P_5) = 5, 25, 125, 625$. If $\exp(P_5) = 5$, then by Lemma 2.1, $|P_5| | 1 + s_5$ and so $|P_5| | 5^6$.

If $|P_5| = 5$, then $n_5 = s_5/\phi(5)$ and so $3, 7 \in \pi(G)$, a contradiction.

If $|P_5| = 5^2$, then $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m \cdot 5^2$, where $k_1, k_2, \ldots, k_7, m$ are nonnegative integers and $0 \leqslant \sum_{i=1}^{7} k_i \leqslant 3$. Since $126{,}000 \leqslant |G| \leqslant 126{,}000 + 3 \cdot 36000$, then the equation has no solution.

If $|P_5| = 5^3, 5^4$, then similarly we get the same results.

If $|P_5| = 5^5$, then $m = 6$ or $7$ and so $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m \cdot 5^5$ where $k_1, \ldots k_7$, m are non-negative integers and $0 \leqslant \sum_{i=0}^{7} s_k \leqslant 17$. But the equation has no solution in $N$.

If $|P_5| = 5^6$, then similarly, the equation has no solution in $N$.

If $\exp(P_5) = 5^2$, then by Lemma 2.1, $|P_5| | 1 + s_5 + s_{5^2}$ and so $|P_5| | 5^3$. If $|P_5| = 5^2$, then 3 or $7 \in \pi(G)$ since $n_5 = s_{5^2}/\phi(5^2)$, a contradiction. If $|P_5| = 5^3$, then $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m \cdot 5^3$, where $k_1, k_2, \ldots, k_7, m$ are nonnegative integers and $0 \leqslant \sum_{i=1}^{7} k_i \leqslant 8$. Since $126{,}000 \leqslant |G| \leqslant 126{,}000 + 8 \cdot 36000$, then the equation has no solution in $N$.

If $\exp(P_5) = 5^3$, then by Lemma 2.1, $|P_5| | 1 + s_5 + s_{5^2} + s_{5^3}$ and so $|P_5| | 5^4$. If $|P_5| = 5^3$, then 3 or $7 \in \pi(G)$ since $n_{5^3} = s_{5^3}/\phi(5^3)$, a contradiction. If $|P_5| = 5^4$, then $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m \cdot 5^4$, where $k_1, k_2, \ldots, k_7, m$ are nonnegative integers and $0 \leqslant \sum_{i=1}^{7} k_i \leqslant 13$. Since $126{,}000 \leqslant |G| \leqslant 126{,}000 + 13 \cdot 36000$, then the equation has no solution in $N$

If $\exp(P_5) = 5^4$, then $|P_5| = 5^4$. We also have 3 or $7 \in \pi(G)$, a contradiction.

Case d. $\pi(G) = \{2, 7\}$. We know that $\exp(P_7) = 7, 49$.Let $\exp(P_7) = 7$. Then by Lemma 2.1, $|P_7| | 1 + s_7$ and so $|P_7| = 7$. Since $n_7 = s_7/\phi(7)$, then $3, 5 \in \pi(G)$, a contradiction. Let $\exp(P_7) = 49$. Then by Lemma 2.1, $|P_7| | 1 + s_7 + s_{7^2}$ and so $|P_7| | 7^2$. Since $n_7 = s_{7^2}/\phi(7^2)$, then $3, 5 \in \pi(G)$, a contradiction.

Case e. $\pi(G) = \{2, 3, 5\}$.

From Lemma $2{\cdot}7, 3{\cdot}5 \notin \omega(G)$ Similarly $4{\cdot}5, 4{\cdot}3 \notin \omega(G)$. It follows that the Sylow 5-subgroups of $G$ acts fixed freely on the set of elements of order 3 of $G$ and so $|P_5| \mid s_3 = 15{,}750$ and $|P_5| \mid 5^3$. Similarly $|P_3| \mid 3^2$.

We know that $\exp(P_5) = 5, 25, 125$.

If $\exp(P_5) = 5$, then by Lemma 2.1, $|P_5| \mid 1 + s_5$ and $|P_5| \mid 5^6$. So $|P_5| \mid 5^3$.

If $|P_5| = 5$, then $n_5 = s_5/\phi(5)$ and so $7 \in \pi(G)$, a contradiction.
If $|P_5| = 5^2$, then $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m \cdot 3^n \cdot 5^2$, where $k_1, k_2, \ldots, k_7, m$ and $n$ are nonnegative integers and $0 \leqslant \sum_{i=1}^{7} k_i \leqslant 21$. Since $126{,}000 \leqslant |G| \leqslant 126{,}000 + 21{\cdot}36000$, then $(m, n) = (6, 4), (7, 4), (5, 5), (6, 5), (7, 5)$, but $n$ is at most 3, a contradiction.
If $|P_5| = 5^3$, then similarly $(m, n) = (7, 2)$ but $m$ is at most 6, a contradiction.

If $\exp(P_5) = 5^2$, then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{5^2}$ and so $|P_5| \mid 5^3$.

If $|P_5| = 5^2$, then $7 \in \pi(G)$ since $n_5 = s_{5^2}/\phi(5^2)$ and $s_{5^2} \in \{3500, 10{,}500, 12{,}600, 31{,}500\}$, a contradiction. If $s_{5^2} = 36{,}000$, then $n_5 = 1800$. On the other hand, by Sylow theorem, $n_5 = 5k + 1$ for some integer $k$.
If $|P_5| = 5^3$, then $126{,}000 + 3500k_1 + 10{,}500k_2 + 12{,}600k_3 + 15{,}624k_4 + 15{,}750k_5 + 31{,}500k_6 + 36{,}000k_7 = 2^m \cdot 3^n \cdot 5^3$, where $k_1, k_2, \ldots, k_7, m$ are nonnegative integers and $0 \leqslant \sum_{i=1}^{7} k_i \leqslant 38$. Since $126{,}000 \leqslant |G| \leqslant 126{,}000 + 38{\cdot}36000$, then we also have a contradiction as the case "$\exp(P_5) = 5$ and $|P_5| = 5^3$".

If $\exp(P_5) = 5^3$, then by Lemma 2.1, $|P_5| \mid 1 + s_5 + s_{5^2} + s_{5^3}$ and $|P_5| \mid 5^4$. So $|P_5| = 5^3$. Since $n_5 = s_{5^3}/\phi(5^3)$, then if $s_{5^3} \in \{3500, 10{,}500, 31{,}500, 36{,}000\}$, $7 \in \pi(G)$, a contradiction; if $s_{5^3} = 36{,}000$, then $n_5 = 360$, on the other hand, by Sylow's theorem $n_5 = 5k + 1$ for some integer $k$, but the equation has no solution in $N$.

Case f. $\pi(G) = \{2, 3, 7\}$. We know that $\exp(P_7) = 7, 49$. Let $\exp(P_7) = 7$. Then by Lemma 2.1, $|P_7| \mid 1 + s_7$ and so $|P_7| = 7$. Since $n_7 = s_7/\phi(7)$, then $5 \in \pi(G)$, a contradiction. Let $\exp(P_7) = 49$. Then by Lemma 2.1, $|P_7| \mid 1 + s_7 + s_{7^2}$ and so $|P_7| \mid 7^2$. Since $n_7 = s_{7^2}/\phi(7^2)$, then $5 \in \pi(G)$, also we get a contradiction.

Case g. $\pi(G) = \{2, 5, 7\}$. Let $\exp(P_7) = 7, 49$. Then similarly as the Case f, $3 \in \pi(G)$, a contradiction.

Case h. $\pi(G) = \{2, 3, 5, 7\}$. In the following, we first show that $|G| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ or $|G| = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7$ and second prove that $G \cong U_3(5)$).

Step 1. $|G| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ or $|G| = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7$.

We have known that $|P_2| \mid 2^6$, $|P_3| \mid 3^5$, $|P_5| \mid 5^4$ and $|P_7| \mid 7^2$.

If $2{\cdot}7 \in \omega(G)$, set $P$ and $Q$ are Sylow 7-subgroups of $G$, then $P$ and $Q$ are conjugate in $G$ and so $C_G(P)$ and $C_G(Q)$ are also conjugate in $G$. Therefore we have $s_{2\cdot 7} = \phi$

$(2{\cdot}7) \cdot n_7 \cdot k$, where $k$ is the number of cyclic subgroups of order 2 in $C_G(P_7)$. Since $n_7 = s_7/\phi(7) = 36{,}000/6$, $36{,}000 \mid s_{2\cdot 7}$ and so $s_{2\cdot 7} = 36000$. But by Lemma 2.1, $2{\cdot}7 \mid 1 + s_2 + s_7 + s_{2\cdot 7}$, a contradiction. Therefore $2{\cdot}7 \notin \omega(G)$, it follows that the Sylow 2-subgroups of $G$ acts fixed freely on the set of elements of order 7, $|P_2| \mid s_7$ and so $|P_2| \mid 2^5$. Similarly $3.7 \notin \omega(G)$ and $|P_3| \mid 3^2$; $5{\cdot}7 \notin \omega(G)$, $|P_5| \mid 5^3$ and $|P_7| \mid 7$.

Therefore we can assume that $|G| = 2^m \cdot 3^n \cdot 5^p \cdot 7$. Since $126{,}000 = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7 \leqslant |G| = 2^m \cdot 3^n \cdot 5^p \cdot 7$, then $|G| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7$ or $|G| = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7$.

Step 2. $G \cong U_3(5)$

We first prove that there is no group such that $|G| = 2^5 \cdot 3^2 \cdot 5^3 \cdot 7$ and $\mathrm{nse}(G) = \mathrm{nse}(U_3(5))$. Then by [11], we have $G \cong U_3(5)$.

Let $|G| = 2^5{\cdot}3^2 \cdot 5^3 \cdot 7$ and $\mathrm{nse}(G) = \mathrm{nse}(U_3(5))$.

Let $G$ be soluble. Since $s_7(G) = 36{,}000$, then $n_7(G) = s_7(G)/6 = 6000 = 2^4 \cdot 5^3 \cdot 3$. Thus by Lemma 2.4, $3 \equiv 1 \pmod 7$, a contradiction. So $G$ is insoluble.

Therefore $G$ has a normal series $1 \lhd K \lhd L \lhd G$ such that $L/K$ is isomorphic to a simple $K_i$-group with $i = 3, 4$ as 49 does not divide the order of $G$.

If L/K is isomorphic to a simple $K_3$-group, from [4], $L/K \cong A_5, A_6, L_2(7), L_2(8), U_3(4), U_4(2)$. Let $L/K \cong A_5$. Then $|G/L| \mid 2^3 \cdot 3 \cdot 5^2 \cdot 7$.

Let $A/K := C_{G/K}(L/K)$. Then $A/K \cap L/K = 1$.

We see that $(G/K)/(A/K) \lesssim \mathrm{Aut}(L/K) = S_5$ and so $G/A \lesssim S_5$. Since $A/K$, $L/K \lhd G/K$, $A/K \times L/K \leqslant G/K$. Therefore $|L/K| \mid |G/A|$ and so $G/A \cong A_5$ or $S_5$. i.e., $|A| = 2^3 \cdot 3 \cdot 5^2 \cdot 7$ or $2^2 \cdot 3 \cdot 5^2 \cdot 7$. By Sylow theorem, $n_7(A) = 1, 8, 15, 50, 120$. Since $A \lhd G$, we have that $n_7(A) = n_7(G)$, and so $s_7(G) = 6, 48, 90, 300, 720$, which contradicts $s_7(G) \in \mathrm{nse}(G)$. Similarly we can rule out the other cases "L/K ≅ $A_6$, $L_2(7)$, $L_2(8)$, $U_3(4)$, $U_4(2)$".

Hence $G$ is isomorphic to a simple $K_4$-group, then by Lemma 2.6, L/K ≅ $U_3(5)$. So $G/\mathrm{A} \leqslant \mathrm{Aut}(U_3(5))$. Therefore $G/\mathrm{A} \cong U_3(5)$, $G/\mathrm{A} \cong 2{\cdot}U_3(5)$, $G/\mathrm{A} \cong 3{\cdot}U_3(5)$ or $G/\mathrm{A} \cong S_6 \cdot U_3(5)$.

If $G/\mathrm{A} \cong U_3(5)$, then order consideration $|A| = 2$. It follows that $A$ is a normal subgroup generated by a 2-central element of $G$. So there exists an element of order *2.7*, which is a contradiction. Similarly we can rule out the cases "$G/\mathrm{A} \cong 2 \cdot U_3(5)$, $G/\mathrm{A} \cong 3 \cdot U_3(5)$ or $G/\mathrm{A} \cong S_6 \cdot U_3(5)$".

Therefore $|G| = 2^4 \cdot 3^2 \cdot 5^3 \cdot 7 = |U_3(5)|$ and so by [11], $G \cong U_3(5)$.

This completes the proof of the theorem. □

**Remark 3.2.** From [12,6] and [8], some alternating groups, where $q = 7, 8, 9, 11, 13$, $L_3(4)$, and $U_3(5)$ can be characterized by only nse. But for the other simple groups, whether can it be characterized by nse? So we put forward the following problem:

**Problem 1.** Let $H$ is a simple group. Is a group $G$ isomorphic to $H$ if and only if $\mathrm{nse}(G) = \mathrm{nse}(H)$?

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## References

[1] J.H. Conway, R.T. Curtis, S.P. Norton, R.A. Parker, R.A. Wilson, Atlas of Finite Groups: Maximal Subgroups and Ordinary Characters for Simple Groups, Clarendon Press, New York, 1985.
[2] G. Frobenius, Verallgemeinerung des Sylowschen Satze, Berliner Sitz (1895) 981–993.
[3] M. Hall, The Theory of Groups, Macmilan, New York, 1959.
[4] M. Herzog, Finite simple groups divisible by only three primes, J. Algebra 10 (1968) 383–388.
[5] A.R. Khalili Asboei, S.S. Salehi Amiri, A. Iranmanesh, A. Tehranian, A characterization of sporadic simple groups by NSE and order, J. Algebra Appl. 12 (2013), http://dx.doi.org/10.1142/S0219498812501587.
[6] M. Khatami, B. Khosravi, Z. Akhlaghi, A new characterization for some linear groups, Monatsh. Math. 63 (2011) 39–50.
[7] S. Liu, R. Zhang, A new characterization of $A_{12}$, Math. Sci. 6 (2012) 30, http://dx.doi.org/10.1186/2251-7456-6-30.
[8] S. Liu, A characterization of $L_3(4)$, ScienceAsia (2013) in press.
[9] G. Miller, Addition to a theorem due to Frobenius, Bull. Am. Math. Soc. 11 (1904) 6–7.
[10] C.G. Shao, W.J. Shi, Q.H. Jiang, A new characterization of simple $K_3$-groups, Adv. Math. 38 (2009) 327–330 (in Chinese).
[11] C. Shao, W. Shi, Q. Jiang, A new characterization of simple $K_4$-groups, Front. Math. China 3 (2008) 355–370.
[12] R. Shen, C. Shao, Q. Jiang, W. Shi, V. Mazurov, A new characterization of $A_5$, Monatsh. Math. 160 (2010) 337–341.
[13] W.J. Shi, A new characterization of the sporadic simple groups, Group Theory, in: Proc. of the 1987 Singapore Conf., Walter de Gruyter, 1989, pp. 531–540.
[14] W.J. Shi, On simple $K_4$-group, Chin. Sci. Bull. 36 (1991) 1281–1283 (in Chinese).