Arab J. Math. Sc.
Volume 6, Number 2, December 2000
pp. 99-115

TESTING NBUFR AND NBAFR CLASSES OF LIFE DISTRIBUTIONS USING KERNEL METHODS

M.I. HENDI, H. AL-NACHAWATI AND M.N. AL-GRAIAN

ABSTRACT. The new better than used failure rate (NBUFR) and new better than average failure rate (NBAFR) classes of life distributions, which are generalizations of the new better than used (NBU) class, and their dual classes the new worse than used failure rate (NWUFR) and the new worse than average failure rate (NWAFR), have been considered in the literature as natural weakenings of NBU (NWU) property. The paper considers testing exponentiality against strictly NBUFR (NBAFR) alternatives, or their duals, based on kernel methods. The percentiles of these test statistics are tabulated for sample size $n = 5(1)50$. Pitman's asymptotic efficiency of the tests are calculated for both classes and compared. An example of 40 patients suffering from blood cancer disease (Leukemia) demonstrates a practical application of the proposed tests in the Medical Sciences.

1. INTRODUCTION AND DEFINITIONS

Ever since the works of Barlow *et al.* (1963) and Bryson and Siddiqui (1969), various classes of life distributions have been introduced in reliability. Currently the applications of these classes of life distribution can be seen in engineering, social and biological sciences, maintenance and biometrics. Therefore, statisticians and reliability analysts have shown a growing interest in modeling survival data using classification of life distributions based on some aspects of aging, see for example Barlow and Proschan (1981) and Zacks (1992).

Among these aspects are increasing failure rate (IFR), increasing failure rate average (IFRA), new better than used (NBU), new better than used in expectation (NBUE), decreasing mean remaining life (DMRL), harmonic new better than used in expectation (HNBUE), new better than used failure rate (NBUFR) and new better than averaged failure rate (NBAFR). The implications among these classes of life distributions are as follows:

$$\begin{array}{lcccccccccc}
 & \longrightarrow & \text{DMRL} & \longrightarrow & & \longrightarrow & & \longrightarrow & & & \\
\text{IFR} & & & & & & & & \searrow & & \\
 & \longrightarrow & \text{IFRA} & \longrightarrow & \text{NBU} & & \longrightarrow & & \text{NBUE} & \longrightarrow & \text{HNBUE} \\
 & & & & \downarrow & & & & & & \\
 & & & & \text{NBUFR} & & \longrightarrow & & \text{NBAFR} & &
\end{array}$$

For the classes IFR, IFRA, NBU, NBUE and DMRL, we refer to Barlow and Proschan (1981), for the class HNBUE to Rolski (1975) and Klefsjo (1981 and 1982), for the class NBUFR to Deshpande et al. (1986) and Abouammoh and Ahmed (1988) and for the class NBAFR to Loh (1984).

The problem of testing exponentiality against various classes (like IFR, IFRA, DMRL, NBU, NBUFR, NBAFR, NBUE and HNBUE) of life distributions has got a good deal of attention in the literature, for example see Proschan and Pyke (1967), Ahmad (1976 and 1995), Abouammoh et al. (1988 and 1994), Abouammoh and Newby (1989) and Hendi et al. (1998) among others.

For the sake of continuity in exposition we give the following definitions.

Definition 1.1. Let X be nonnegative random variable (r.v.) with (life) distribution $F(x) = P(X \leq x)$ where $F(x) = 0$ for $x < 0$ and $F(0)$ may be zero. The corresponding survival distribution is $\bar{F}(x) = 1 - F(x)$, for $x \geq 0$ and the density function (if exists) is $f(x) = \frac{d}{dx}F(x)$. The instantaneous conditional failure rate, or simply the failure rate, at time x is defined by

$$r(x) = \lim_{\delta x \to 0} \frac{1}{\delta x} \frac{P(x < X \leq x + \delta x)}{P(X > x)}$$

i.e.

$$r(x) = f(x)/\bar{F}(x) \qquad x \geq 0 \tag{1.1}$$

$r(x)$ in (1.1) is also known as the hazard rate, force of mortality or intensity.

Definition 1.2. F is new better than used (NBU) (and is new worse than used (NWU)) if

$$\bar{F}(x+y) \leq (\geq)\bar{F}(x)\bar{F}(y) \qquad \forall x, y \geq 0 \tag{1.2}$$

By relation (1.2) one has

$$\bar{F}(x) - \bar{F}(x+y) \geq (\leq)\bar{F}(x)\{1 - \bar{F}(y)\}.$$

By dividing the above formula by y and taking the limit as y tends to zero, we get

$$\frac{1}{\bar{F}(x)} \lim_{y\to 0} \frac{F(x+y) - F(x)}{y} \geq (\leq) \lim_{y\to 0} \frac{F(y)}{y}$$

i.e.

$$\frac{f(x)}{\bar{F}(x)} \geq (\leq) f(0), \qquad x \geq 0. \tag{1.3}$$

If $f(0) > 0$ we have the following:

Definition 1.3 . F has the new better than used failure rate (NBUFR) (and has the new worse than used failure rate (NWUFR)) if

$$r(x) \geq (\leq) r(0), \qquad x \geq 0 \tag{1.4}$$

where $r(0) = \frac{f(0)}{\bar{F}(0)} = f(0)$, where $\bar{F}(0) = 1$.

This concept was introduced by Deshpande *et al.* (1986) and Abouammoh and Ahmed (1988). Integrating both sides of (1.4) implies that

$$x^{-1} \int_0^x r(u)du \geq (\leq) r(0), \qquad x > 0 \tag{1.5}$$

i.e

$$-x^{-1} \ln \bar{F}(x) \geq (\leq) r(0), \qquad x > 0.$$

Definition 1.4. F has the new better average failure rate (NBAFR) (and the new worse average failure rate (NWAFR)) if

$$-x^{-1} ln\bar{F}(x) \geq (\leq) r(0), \qquad x > 0 \tag{1.6}$$

i.e.

$$\bar{F}(x) \le (\ge) e^{-f(0)x}, \qquad x > 0. \tag{1.7}$$

Relations (1.6) and (1.7) were introduced by Loh (1984). From above we can write the following implications:

$$NBU \Longrightarrow NBUFR \Longrightarrow NBAFR.$$

Similar implications hold for the corresponding dual classes NWU, NWUFR and NWAFR, viz.,

$$NWU \Longrightarrow NWUFR \Longrightarrow NWAFR.$$

This article proposes test statistics for testing exponentiality against strict NBUFR and NBAFR alternatives. This seems to be the first such procedures for these new classes of life distribution. The approaches of testing exponentiality against a class of life distribution is based on defining a measure of departure from H_0 in favor of $H_1^{(i)}$, $i = 1, 2$ for each of the two classes, respectively and then estimating this measure empirically by using an approach based on these measures of departure from H_0 that depend on the underlying pdf $f(x)$. The empirical versions of these measures require estimating $f(x)$, and thus one may use the celebrated "kernel method". For background materials on this method, we refer the readers to Prakasa Rao (1983), Silverman (1986), Scott (1992), and Jones and Wand (1995). In fact, using the kernel methods in reliability is not new since estimating the hazard (failure) rate using kernel methods appears in early work of Watson and Leadbetter (1964), Ahmad (1976), Ahmed and Lin (1977) among others, while using the kernel method for testing IFR,IFRA are pioneered by Ahmad (1996), cf. Ahmad, Hendi and Al-Nachawati (1999).

The exponential distribution is the only distribution where equality is attained in (1.4) and (1.5). Hence we test H_0 : F is exponential (μ) against $H_1^{(1)}$: F is NBUFR and not exponential or $H_1^{(2)}$: F is NBAFR and not exponential.

In order to test H_0 against $H_1^{(1)}$ we use the following measure of departure from H_0,

$$\delta_F^{(1)} = \int_0^\infty \{f(x) - f(0)\bar{F}(x)\} f(x)dx$$

i.e.

(1.8) $$\delta_F^{(1)} = \int_0^\infty f(x)dF(x) - \frac{1}{2}f(0).$$

Note that $\delta_F^{(1)} = 0$ under H_0 and is strictly positive under $H_1^{(1)}$. To estimate $\delta_F^{(1)}$ let $X_1, \ldots, X_n$ be a random sample from F, let $F_n(x) = \frac{1}{n}\sum_{j=1}^n I(X_j < x)$ denote the empirical distribution of cdf F, $dF_n(x) = \frac{1}{n}$ and pdf $f(x)$ is estimated by $\hat{f}_n = \frac{1}{na_n}\sum_{j=1}^n k(\frac{x-X_j}{a_n})$, where $k(.)$ be a known pdf, symmetric and bounded with 0 mean and variance $\sigma_k^2 > 0$. Symmetric uniform, normal, double exponential are examples of such pdf. Let $\{a_n\}$ be a sequence of reals such that $a_n \to 0$ and $na_n \to \infty$ as $n \to \infty$. Other conditions on k and a_n will be stated when needed. We propose to estimate $\delta_F^{(1)}$ by.

(1.9) $$\hat{\delta}_{F_n}^{(1)} = \int_0^\infty \hat{f}_n(x)dF_n(x) - \frac{1}{2}\hat{f}_n(0),$$

i.e.

$$\hat{\delta}_{F_n}^{(1)} = \int_0^\infty \frac{1}{na_n}\sum_{j=1}^n k(\frac{x-X_j}{a_n})dF_n(x) - \frac{1}{2na_n}\sum_{j=1}^n k(\frac{-X_j}{a_n}),$$

i.e.

(1.10) $$\hat{\delta}_{F_n}^{(1)} = \frac{1}{n^2a_n}\sum_{i=1}^n\sum_{j=1}^n \left\{k(\frac{X_i - X_j}{a_n}) - \frac{1}{2}k(\frac{-X_j}{a_n})\right\}.$$

Let us rewrite (1.10) as

(1.11) $$\hat{\delta}_{F_n}^{(1)} = \frac{1}{n(n-1)}\sum^n\sum^n_{i\neq j} \phi_{1n}(X_i, X_j).$$

Set

$$\phi_{1n}(X_1, X_2) = \frac{1}{a_n}k(\frac{X_1 - X_2}{a_n}) - \frac{1}{2a_n}k(\frac{-X_2}{a_n})$$

and define the symmetric kernel

$$\xi(X_1, X_2) = \frac{1}{2!}\sum_R \phi_{1n}(X_{i_1}, X_{i_2}),$$

where the sum over all arrangements of X_1 and X_2. Then $\hat{\delta}_{F_n}^{(1)}$ is equivalent to the U-statistic.

In Section 2, conditions under which $\sqrt{n}(\hat{\delta}_{F_n}^{(1)} - \delta_F^{(1)})$ is asymptotically normal are given and the null and non-null variances are obtained. The test based on $\hat{\delta}_{F_n}^{(1)}$ is shown to be consistent and its Pitman asymptotic efficiency are given for some well-known alternatives. Small sample Monte Carlo critical values of the test statistic are given.

To test against $H_1^{(2)}$ we use the following measure of departure from H_0:

$$\delta_F^{(2)} = \int_0^\infty \{e^{-f(0)x} - \bar{F}(x)\} f(x) dF(x). \tag{1.12}$$

To estimate $\delta_F^{(2)}$ we propose

$$\hat{\delta}_{F_n}^{(2)} = \int_0^\infty \{e^{-\hat{f}_n(0)x} - \bar{F}_n(x)\} \hat{f}_n(x) dF_n(x) \tag{1.13}$$

i.e.

$$\hat{\delta}_{F_n}^{(2)} = (n(n-1)(n-2)a_n)^{-1} \sum^n \sum^n \sum_{i \neq j \neq l}^n k(\frac{X_i - X_j}{a_n}) \left\{e^{-\hat{f}_n(0)X_i} - I(X_l > X_i)\right\} \tag{1.14}$$

where $\hat{f}_n(0) = \frac{1}{na_n} \sum_1^n k(\frac{-X_j}{a_n})$, $\bar{F}_n(x) = \frac{1}{n} \sum_{l=1}^n I(X_l > X_i)$ and

$$I(X_l > X_i) = \begin{cases} 0 & \text{if } X_l - X_i \leq 0 \\ 1 & \text{if } X_l - X_i > 0 \end{cases}.$$

In Section 3, we establish the asymptotic normality of $\hat{\delta}_{F_n}^{(2)}$ and obtain its null and non-null variance and compare its efficiency to the test $\hat{\delta}_{F_n}^{(1)}$. In Section 4, an example using field data blood cancer disease (Leukemia) is used as an application in medical sciences.

2. TESTING AGAINST NBUFR ALTERNATIVES

2.1 The Test Procedure

The "kernel method" was used recently for testing exponentiality against some classes of life distributions cf. Ahmad *et al.* (1999) and Hendi (1999).

In this section, we derive a kernel-test for testing $H_0 : F$ is exponential (μ) against $H_1^{(1)} : F$ is NBUFR and not exponential. First we prove the following:

Theorem 2.1. *If $na_n^4 \to 0$ as $n \to \infty$, if F has bounded second derivative and if $V(\psi_{1n}(X_1)) < \infty$, where $\psi_{1n}(X_1)$ is given in (2.7), then $\sqrt{n}(\hat{\delta}_{F_n}^{(1)} - \delta_F^{(1)})$ is asymptotically normal with mean 0 and variance $\lim_n V(\psi_{1n}(X_1))$. Under H_0, the variance is $\frac{1}{3}$.*

Before we prove this theorem, we need the following lemma

Lemma 2.1. *Let $\theta_n^{(1)} = E(\hat{\delta}_{F_n}^{(1)})$, then*

(2.1) $$\theta_n^{(1)} = \int_0^\infty E(\hat{f}_n(x))dF(x) - \frac{1}{2}E(\hat{f}_n(0)).$$

Proof. Note that
$$E(\hat{f}_n(x)) = \frac{1}{a_n}\int k(\frac{x-y}{a_n})f(y)dy,$$
and
$$E(\hat{f}_n(0)) = \frac{1}{a_n}\int k(\frac{-y}{a_n})f(y)dy.$$
Set $g_n(x) = E(\hat{f}_n(x))$ and $g_n(\theta) = E(\hat{f}_n(0))$, thus
$$E(\hat{\delta}_{F_n}) = \theta_n^{(1)} = E(\phi_{1n}(X_1, X_2))$$

where,

(2.2) $$\phi_{1n}(X_1, X_2) = \frac{1}{a_n}k(\frac{X_1 - X_2}{a_n}) - \frac{1}{2a_n}k(\frac{-X_2}{a_n}).$$
$$\theta_n^{(1)} = E(g_n(x)) - \frac{1}{2}E(g_n(0))$$

i.e.

$$\begin{aligned}\theta_n^{(1)} &= \int g_n(x)dF(x) - \frac{1}{2}\int g_n(0)dF(x)\\ &= \int E(\hat{f}_n)(x)dF(x) - \frac{1}{2}g_n(0).\end{aligned}$$

Then
$$\theta_n^{(1)} = \int_0^\infty E(\hat{f}_{F_n}(x))dF(x) - \frac{1}{2}E(\hat{f}_n(0)).$$

Proof of Theorem 2.1. Note that

$$\sqrt{n}(\hat{\delta}^{(1)}_{F_n} - \delta^{(1)}_F) = \sqrt{n}(\hat{\delta}^{(1)}_{F_n} - \theta^{(1)}_n) + \sqrt{n}(\theta^{(1)}_n - \delta^{(1)}_F). \tag{2.3}$$

But

$$\begin{aligned} E(\hat{f}_n(x)) &= a_n^{-1}\int k(\frac{x-y}{a_n})f(y)dy = \int k(w)f(x-a_nw)dw \\ &\simeq f(x) + \frac{a_n^2\sigma_k^2}{2}f''(x); \text{under the conditions assumed on } k \\ &\quad \text{and using Taylor expansion,} \end{aligned}$$

and

$$\begin{aligned} E(\hat{f}_n(0)) &= a_n^{-1}\int k(\frac{-y}{a_n})f(y)dy = \int k(w)f(-aw)dw \\ &\simeq f(0) + \frac{a_n^2\sigma_k^2}{2}f''(0); \text{under the same conditions on } k \\ &\quad \text{and using Taylor expansion.} \end{aligned}$$

Hence (2.1) can be written as

$$\theta^{(1)}_n \simeq \delta^{(1)}_F + \frac{a_n^2\sigma_k^2}{2}\{\int_0^\infty f''(x)dF(x) - \frac{1}{2}f''(0)\} \tag{2.4}$$

which leads to

$$\sqrt{n}(\theta^{(1)}_n - \delta^{(1)}_F) = O(\sqrt{n}a_n^2) = o(1), \text{by assumptions.} \tag{2.5}$$

Note also that $\hat{\delta}^{(1)}_{F_n})$ is unbiased estimate of $\theta^{(1)}_n$ i.e. $\theta^{(1)}_n = E\hat{\delta}_{F_n}$ and also is asymptotically unbiased for $\delta^{(1)}_F$. Next write

$$\sqrt{n}(\hat{\delta}^{(1)}_{F_n} - \theta^{(1)}_n) = \sqrt{n}(\frac{1}{n}\sum_{i=1}^{n}\psi_{1n}(X_i) + \frac{1}{n(n-1)}\sum\sum_{i\neq j}\xi_{1n}(X_i,X_j)), \tag{2.6}$$

where

$$\psi_{1n}(X_1) = E[\phi_{1n}(X_1,X_2)|X_1] + E[\phi_{1n}(X_2,X_1)|X_1] - 2\theta^{(1)}_n, \tag{2.7}$$

and

$$\xi_{1n}(X_1,X_2) = \phi_{1n}(X_1,X_2) - \psi_{1n}(X_1) - \theta^{(1)}_n. \tag{2.8}$$

Now, by Layaponouff's central limit theorem, the first term in the right hand side of (2.6) is asymptotically normal if

$L_n = E(\psi_{1n}(X_1))^{2+\delta}/n^{\frac{1}{2}}.\ [V(\psi_{1n}(X_1)]^{1+\delta/2} \to 0$ as $n \to \infty$. Now, using (2.4) it is not difficult to see that for large n,

$$E[\phi_{1n}(X_1, X_2)|X_1] = f(X_1) - \frac{1}{2}f(0). \tag{2.9}$$

Observe that $E[\phi_{1n}(X_2, X_1)|X_1]$ has the same representation as in (2.9). Set $\eta_1(X_1)$ to be the sum of twice the right hand side of (2.9). Thus $\eta_1(X_1) = 2f(X_1) - f(0)$, and

$$\psi_{1n}(X_1) = \eta_1(X_1) + O_p(a_n^2), \text{ say.} \tag{2.10}$$

Hence $V(\psi_{1n}(X_1)) = V(\eta_1(X_1)) + O(a_n^2)$ and for $p > 2$, $E|\psi_{1n}(X_1)|^p \leq C_p.\ E|\eta_1(X_1)|^p = O(1)$. Hence $L_n \to 0$ as $n \to \infty$ provided that $na_n^4 \to 0$ as $n \to \infty$.

Next, look at

$$\begin{aligned} & E\{\frac{\sqrt{n}}{n(n-1)}\sum\sum_{i\neq j} \xi_{1n}(X_i, X_j)\}^2 \\ = & \frac{1}{n(n-1)^2}\sum\sum_{i\neq j}\sum\sum_{i^*\neq j^*} E\{\xi_{1n}(X_i, X_j,).\xi_{1n}(X_{i^*}, X_{j^*})\} \\ = & \frac{1}{n-1}E\xi_{1n}^2(X_1, X_2) = O((na_n)^{-1}) = o(1). \end{aligned} \tag{2.11}$$

Under H_0, $\bar{F}(x) = e^{-x}$ (the test is scale invariant) and

$$\eta_1(X_1) = 2e^{-X_1} - 1 + O(a_n^2). \tag{2.12}$$

Thus $E_0\eta_1(X_1) = 0$ and $\sigma_0^2 = V(\eta_1(X_1)) = \frac{1}{3}$ by direct calculations. The theorem is thus proved □.

Thus to conduct the test, calculate $\sqrt{3n}\hat{\delta}_{F_n}^{(1)}$ and reject H_0 if this value exceeds Z_α, the standard normal variate at level α.

2.2 Asymptotic Relative Efficiency

Since the above test is new and no other tests are known for NBUFR we compare our test to the test $\hat{\delta}_{F_n}^{(2)}$ of NBAFR class in section (3). The Pitman asymptotic relative efficiency is defined. Let $\hat{\delta}_{F_n}^{(1)}$ and $\hat{\delta}_{F_n}^{(2)}$ be two test statistics for testing $H_0 : F\epsilon\{F_{\otimes_n}\}$, $\theta_n = \theta + cn^{-1/2}$ with c an arbitrary constant, the asymptotic relative efficiency of $\hat{\delta}_{F_n}^{(1)}$ relative to $\hat{\delta}_{F_n}^{(2)}$ is defined by

$$e(\hat{\delta}_{F_n}^{(1)}, \hat{\delta}_{F_n}^{(2)}) = [\mu_1'(\theta_0)/\sigma_1(\theta_0)]/[\mu_2'(\theta_0)/\sigma_2(\theta_0)],$$

where $\mu_i'(\theta_0) = \{\lim_{n\to\infty}(\delta/\delta\theta)E(\hat{\delta}_{F_n}^{(i)})]\}_{\theta\to\theta_0}$
and $\sigma_i^2(\theta_0) = \lim_{n\to\infty} Var_0(\hat{\delta}_{F_n}^{(i)}), i = 1, 2$ is the null variance.

We choose the following two alternatives:

(i) The Linear Failure Rate family: $\bar{F}_1(x) = \exp(-x - \theta\frac{x^2}{2}), x \geq 0, \theta \geq 0.$

(ii) The Makeham family: $\bar{F}_2(x) = \exp(-x - \theta(x + e^{-x} - 1)), x \geq 0, \theta \geq 0.$

Note that H_0 (the exponential) is attained at $\theta = 0$ in (i) and (ii).

For the above two alternatives, direct calculations give efficiencies as follows

Efficiency	Linear failure rate	Makeham
$\frac{\mu_1'(\theta_0)}{\sigma_1(\theta_0)} = \frac{\partial\delta_F^{((1)}}{\partial\theta}\vert\theta \to \theta_0/\sigma_1(\theta_0)$	0.433	0.289

2.3. Monte Carlo Null Distribution Critical Points

For samples 5(1)50 and using 5000 replications, the upper percentiles of the statistic $\hat{\delta}_{F_n}^{(1)}$ are given in table (2.1). Note that since the above procedure is independent of choosing a_n or k, we select k to be the standard normal and those a_n by the normal scale rule (cf. Jones and Wand (1995) p. 60).

Table (2.1): Critical values of $\hat{\delta}^{(1)}_{F_n}$.

n	% 90	% 95	% 99
6	0.11726	0.16461	0.35330
7	0.13468	0.18345	0.34640
8	0.14707	0.19774	0.35187
9	0.15634	0.19878	0.34925
10	0.16446	0.20623	0.33791
11	0.16948	0.21096	0.33029
12	0.17772	0.21766	0.32887
13	0.17792	0.21329	0.31336
14	0.17939	0.21701	0.31641
15	0.18184	0.21879	0.31443
16	0.18405	0.21677	0.30159
17	0.18952	0.22880	0.30897
18	0.19012	0.22129	0.29867
19	0.18881	0.22007	0.29624
20	0.18920	0.21967	0.30529
21	0.19112	0.22044	0.29741
22	0.18987	0.22112	0.29422
23	0.19235	0.22215	0.29100
24	0.19165	0.21988	0.28052
25	0.19278	0.21916	0.28866
26	0.19448	0.22284	0.29163
27	0.19289	0.21711	0.28310
28	0.19251	0.21766	0.27705
29	0.19120	0.21679	0.27970
30	0.19273	0.21706	0.27861
31	0.19537	0.21921	0.27923
32	0.19163	0.21495	0.26883
33	0.19153	0.21455	0.26875
34	0.19173	0.21603	0.26621
35	0.19026	0.21229	0.26830
36	0.19254	0.21461	0.26773
37	0.19120	0.21466	0.26266
38	0.19205	0.21381	0.26022
39	0.19031	0.21413	0.26538
40	0.19006	0.20886	0.25304
41	0.19013	0.21062	0.25644
42	0.18923	0.20961	0.25448
43	0.18915	0.20774	0.25389
44	0.18858	0.20900	0.25802
45	0.18855	0.20839	0.25164
46	0.18824	0.20848	0.25019
47	0.18967	0.20904	0.25257
48	0.18879	0.20819	0.24972
49	0.18820	0.20758	0.25072
50	0.18665	0.20633	0.24826

3. TESTING AGAINST NBAFR ALTERNATIVES

3.1 The Test Procedure

Here we want to test H_0 : F is exponential (μ) against $H_1^{(2)}$: F is NBAFR and not exponential. We state and prove the following result.

Theorem 3.1. *If $na_n^4 \to 0$ as $n \to \infty$, f is bounded with bounded second derivative and if $V(\psi_{2n}(X_1)) < \infty$, where $\psi_{2n}(X_1)$ is as in (3.4), then $\sqrt{n}(\hat{\delta}^{(2)}_{F_n} - \delta^{(2)}_F)$ is asymptotically normal with mean 0 and variance $V(\eta_2(X_1))$. Under H_0, the variance is $\frac{1}{45}$.*

Proof. Set $\delta_F^{(2)} = \delta_F^{(2)}(f(0))$ and $\hat{\delta}_{F_n}^{(2)} = \hat{\delta}_{F_n}^{(2)}(\hat{f}(0))$. Thus we can write

$$\hat{\delta}_{F_n}^{(2)}(\hat{f}_n(0)) = \int_0^\infty (\hat{f}_n(x)e^{-\hat{f}_n(0)x} - \hat{f}_n(x)\bar{F}_n(x))dF_n(x)$$

i.e.

$$\hat{\delta}_{F_n}^{(2)}(\hat{f}_n(0)) = [n(n-1)(n-2)a_n]^{-1}\sum\sum\sum_{i\neq j\neq l} k(\frac{X_i - X_j}{a_n})\{e^{-\hat{f}_n(0)X_i} - I(X_l > X_i)\}. \tag{3.1}$$

First, we prove the result for $\hat{\delta}_{F_n}^{(2)}(f(0))$ where

$$\hat{\delta}_{F_n}^{(2)}(f(0)) = [n(n-1)(n-2)a_n]^{-1}\sum\sum\sum_{i\neq j\neq l} k(\frac{X_i - X_j}{a_n})\{e^{-f(0)X_i} - I(X_l > X_i)\}. \tag{3.2}$$

Set

$$\phi_{2n}[(X_1, X_2, X_3)|X_1] = \frac{1}{a_n}k(\frac{X_1 - X_2}{a_n})\{e^{f(0)X_1} - I(X_3 > X_1)\}. \tag{3.3}$$

Understanding the methodology of Theorem 2.1, we need only evaluate $\eta_2(X_1)$, where $\eta_2(X_1) = \lim_{n\to\infty}\psi_{2n}(X_1)$ and

$$\begin{aligned}\psi_{2n}(X_1) &= E[\phi_{2n}(X_1, X_2, X_3)|X_1] + E[\phi_{2n}(X_2, X_1, X_3)|X_2] \\ &\quad + E[\phi_{2n}(X_2, X_3, X_1)|X_3].\end{aligned} \tag{3.4}$$

But for large n,

$$E[\phi_{2n}(X_1, X_2, X_3)|X_1] = e^{-f(0)X_1}f(X_1) - f(X_1)\bar{F}(X_1) + O_p(a_n^2),$$

$$E[\phi_{2n}(X_2, X_1, X_3)|X_1] = e^{-f(0)X_1}f(X_1) - f(X_1)\int_{X_1}^\infty f(y)dy + O_p(a_n^2),$$

and

$$E[\phi_{2n}(X_3, X_2, X_1)|X_1] = \int_0^\infty e^{-f(0)y}f^2(y)dy - \int_0^{X_1} f^2(y)dy + O_p(a_n^2). \tag{3.5}$$

Hence

$$\begin{aligned}\eta_2(X_1) &= 2e^{-f(0)X_1}f(X_1) - f(X_1)\int_{X_1}^\infty f(y)dy + \int_0^\infty e^{-f(0)y}f^2(y)dy \\ &\quad - \int_0^{X_1} f^2(y)dy.\end{aligned} \tag{3.6}$$

Now, since as in Theorem 2.1, $\sqrt{n}(E\hat{\delta}^{(2)}_{F_n}(f(0)) - \delta^{(2)}_F) = o(1)$ using same argument we have that $\sqrt{n}(\hat{\delta}^{(2)}_{F_n}(f(0)) - \delta^{(2)}_F(f(0)))$ is asymptotically normal with mean 0 and variance $V(\eta_2(X_1))$. Under H_0, it is not difficult to show that $\eta_2(X_1)) = -\frac{1}{6} + \frac{1}{2}e^{-2X_1}$. Thus by direct calculations we get that $\sigma_0^2 = 1/45$. Theorem (3.1) is proved for $\hat{\delta}^{(2)}_{F_n}(f(0))$. Now, to extends it to $\hat{\delta}^{(2)}_{F_n}(\hat{f}_n(0))$, we use Theorem 2.13 of Randles (1982). Let $\gamma \epsilon k(f(0))$ and define $L(\gamma, d) = (\gamma - d, \gamma + d)$ to be an interval inside $k(f(0))$. Now, let $h(X_1, X_2, f(0)) = \frac{1}{a_n}e^{-f(0)X_1}k(\frac{X_1-X_2}{a_n})$. Hence we have, $|h(X_1, X_2, \gamma) - h(X_1 + X_2, f(0))| < f_1(0)$ for all γ in some neighborhood of $f(0)$. Next, it easily follows from the definition of h that

$$(3.7) \qquad E[\sup_{\gamma' \epsilon (\gamma - d, \gamma + d)} |h(X_1, X_2, \gamma^*) - h(X_1 + X_2, \gamma)| \leq k_1 d.$$

Thus Theorem 2.13 of Randles(1992) applies since $\sqrt{n}(\hat{f}_n(0) - f(0)) = O_p(1)$ as $n \to \infty$ and the theorem is now proved.

To perform the above test, calculate $\sqrt{45n}\hat{\delta}^{(2)}_{F_n}$ and reject if this value exceeds Z_α the standard normal variate.

3.2 Asymptotic Relative Efficiency

Since the above test is also new and no other tests are known for NBAFR we compare our test to the test $\hat{\delta}^{(1)}_{F_n}$ of NBUFR class in section (2). For the above two alternative in section (2), direct calculation give efficiencies and the relative efficiencies are as follows:

Efficiency	Linear failure rate	Makeham
$\frac{\mu_1'(\theta_0)}{\sigma_{01}} = \frac{\partial \delta_F^{((1)}}{\partial \theta}\vert\theta \to \theta_0/\sigma_{01}$	0.433	0.289
$\frac{\mu_2'(\theta_0)}{\sigma_{02}} = \frac{\partial \delta_F^{((2)}}{\partial \theta}\vert\theta \to \theta_0/\sigma_{02}$	0.249	0.186
$e(\hat{\delta}^{(1)}_{F_n}, \hat{\delta}^{(2)}_{F_n})$	1.739	1.5

3.3 Monte Carlo Null Distribution Critical Points

For samples 5(1)50 and using 5000 replications, the upper percentiles of the statistic $\hat{\delta}^{(2)}_{F_n}$ are given in Table (3.1).

Table (3.1): Critical values of $\hat{\delta}^{(2)}_{F_n}$.

n	% 90	% 95	% 99
6	0.08991	0.11389	0.18423
7	0.09137	0.11622	0.17939
8	0.08786	0.10958	0.15312
9	0.08950	0.10775	0.15337
10	0.08522	0.10276	0.14129
11	0.08540	0.10155	0.13693
12	0.08336	0.09871	0.13648
13	0.08265	0.09758	0.12834
14	0.08216	0.09686	0.12656
15	0.07831	0.09216	0.12517
16	0.07998	0.09311	0.11662
17	0.07839	0.09084	0.11536
18	0.07526	0.08798	0.11267
19	0.07704	0.08742	0.11230
20	0.07535	0.08679	0.11044
21	0.07548	0.08714	0.10756
22	0.07396	0.08547	0.10691
23	0.07261	0.08244	0.10472
24	0.07077	0.08071	0.10103
25	0.07008	0.08028	0.10051
26	0.06970	0.08066	0.10110
27	0.06747	0.07656	0.09674
28	0.06643	0.07612	0.09459
29	0.06667	0.07696	0.09595
30	0.06476	0.07586	0.09429
31	0.06396	0.07407	0.09025
32	0.06247	0.07254	0.09159
33	0.06256	0.07366	0.09127
34	0.06188	0.07171	0.08831
35	0.06045	0.07089	0.08938
36	0.06001	0.07071	0.08783
37	0.05834	0.06876	0.08666
38	0.05723	0.06656	0.08466
39	0.05522	0.06596	0.08144
40	0.05507	0.06508	0.08232
41	0.05439	0.06479	0.08321
42	0.05394	0.06535	0.08152
43	0.05310	0.06441	0.08100
44	0.05161	0.06139	0.07841
45	0.05134	0.06246	0.07988
46	0.04928	0.06044	0.07661
47	0.04861	0.06138	0.08018
48	0.04630	0.05909	0.07429
49	0.04719	0.05890	0.07643
50	0.04666	0.05851	0.07570

4. AN APPLICATION OF THE $\hat{\delta}^{(1)}_{F_n}$ AND $\hat{\delta}^{(2)}_{F_n}$ TESTS

Consider the data in Abouammoh *et al.* (1994). These data represent 40 patients suffering from blood cancer (Leukemia) from one of Ministry of Health Hospitals in Saudi Arabia and ordered life times (in days) are 115, 181, 255, 418, 441, 461, 516, 739, 743, 789, 807, 865, 924, 983, 1024, 1062, 1063, 1165, 1191, 1222, 1222, 1251, 1277, 1290, 1357, 1369, 1408, 1455, 1478, 1549, 1578, 1599, 1603, 1605, 1696, 1735, 1799, 1815, 1852.

From the above set of data, we have the following:

(i) Computing $\hat{\delta}^{(1)}_{F_n}$ from (1.10), we get $\hat{\delta}^{(1)}_{F_n} = 0.21278$ which exceeds the critical value in Table (2.1) at 95% upper percentile. Thus we accept $H_1^{(1)}$ which states that the data set have NBUFR property.

(ii) Computing $\hat{\delta}^{(2)}_{F_n}$ from (1.14), we get $\hat{\delta}^{(2)}_{F_n} = 0.066721$ which exceeds the critical value in Table (3.1) at 95% upper percentile. Thus we accept $H_1^{(2)}$ which states that the data set have NBAFR property.

Acknowledgement. The authors would like to thank Professor Ibrahim Ahmad for his constructive comments during the preparation of the paper. They are also grateful to referee for his useful remarks and corrections.

REFERENCES

1. Abouammoh, A. M. and Ahmed, A. N. *The new better than used failure rate class of life distributions.* Adv. Appl. Prob., **20**(1988), 237-240.

2. Abouammoh, A. M., Abdulghani, S. A. and Qamber, I. S. (1994). *On partial orderings and testing of new better than renewal used classes.* Reliability Eng. Syst. Safety, **43**(1994), 37-41.

3. Abouammoh, A. M. and Newby, M. J. *On partial ordering and tests of the generalized new better than used classes of life distributions.* Reliability Eng. Syst. Safety, **25**(1989), 207-217.

4. Abouammoh, A. M., Hendi, M. I. and Ahmad, A. N. *Shock models with NBUFR and NBAFR survivals.* Trab. de Statist-, **3**(1988), 97-113.

5. Ahmad, I. A. *Nonparametric testing of class of life distributions derived from a variability ordering.* Parisankhyan Samikkha, **2**(1995), 13-18.

6. Ahmad, I. A. (1996). *Testing positive aging using kernel methods.* Sankhya, Series A, **forth coming**(1996).

7. Ahmad, I. A., Hendi, M.I., and Al-Nachawati, H. *Testing new better than used classes of life distribution derived from a convex ordering using kernel methods.* J. Nonparametric Statistics, **11**(1999), 393-411.

8. Ahmad, I. A. *Uniform strong consistency of a generalized failure rate function estimate.* Bull. Math. Statist., **41**(1976), 141-149.

9. Ahmad, I. A. and Lin, P. E. *Nonparametric estimation of a vector-valued bivariate failure rate function.* Ann. Statist., **5**(1977), 1027-1038.

10. Barlow, R. E., Marshall, A. W., and Proschan, F. *Properties of probability distributions with monotone hazard rate.* Ann. Math. Statist., **34**(1963), 375-389.

11. Barlow, R. E. and Proschan, F. it Statistical Theory of Reliability and Life Testing: Probability Models. To Begin With, Silver-Spring, MD, (1981).

12. Bryson, M. C. and Siddiqui, M. M. *Some criteria for aging.* J. Amer. Statist. Assoc., **64**(1969), 1472-1483.

13. Deshpande, J. V., Kochar, S. C. and Singh, H. *Aspects of positive aging.* J. Appl. Prob., **28**(1986), 773-479.

14. Hendi, M. I. *Testing general harmonic new better than used in expectation using kernel methods.* Egypt. Computer Sc. J., **21**(1999), 1-16.

15. Hendi, M. I. and Al-Graian, M. N. *On partial ordering and test NBAFR (NWAFR) class of life distributions based on the total time on transform.* Egypt. Computer Sc. J., **20**(1998), 30-42.

16. Hendi, M. I., Al-Nachawati, H., Montassar, M. and Alwasel, I. A. *An exact test for HNBUE class of life distributions.* J. Statist. Computation and Simulation, **60**(1998), 261-275.

17. Jone, M. C. and Wand, M. P. *Kernel Smothing.* Chapman and Hall, New York, NY, 1995.

18. Klefsjo, B. *HNBUE survival under some shock models.* Scand. J. Statist., **8**(1981), 34-47.

19. Klefsjo, B. *The HNBUE and HNWUE classes of life distributions.* Naval Res. Logistics Quartly, **24**(1982), 331-344.

20. Loh, W. Y. *A new generalization of the class of NBU distribution.* IEEE Trans. Reli., **R-33**(1984), 419-422.

21. Prakasa Rao, B. L. S. *Nonparametric Function Estimation.* Academic Press, Orlando, FL, 1983.

22. Proschan, F. and Pyke, R. *Tests for monotine failure rate.* Proc. 5th Berkeley Symp., (1967) 3293-3312.

23. Randles, R. H. *On asymptotic normality of statistics with estimated parameters.* Ann. Statist., **18**(1982), 462-474.

24. Rolski, T. *Mean residual life.* Z. Waltseheinlichkeits Verw Geb., **33**(1975), 714-718.

25. Scott, D. W. *Multivariate Density Estimation.* Wiley and Sons, New York, NY, 1992.

26. Silverman, B. W. *Density Estimation for Statistics and Data Analysis.* Chapman and Hall, New York, NY, 1986.

27. Watson, G. S. and Leadbetter, M. R. *Hazard Analysis II.* Sankhya, **24**(1964), 101-116.

28. Zacks, S. *Introduction to Reliability Analysis Probability Models and Statistical Methods.* Springer-Verlag New York, Inc, 1992.

Department of Statistics and Operations Research, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia
e-mail: mmihendi@ksu.edu.sa

Date received May 1, 1999.