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COSMOLOGY OF GENERAL RELATIVITY WITHOUT ENERGY CONSERVATION

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ABSTRACT. A modified version of the field equations of general relativity is obtained on relaxing the covariant energy-conservation condition. This introduces a single arbitrary constant and does not upset any of the successes of general relativity in or outside cosmoloy. The matter-dominated cosmological model, based on the generalized field equations, is discussed. It is shown to provide more room for consistency with the observational data.

1. INTRODUCTION AND MOTIVATION

Our motivation in considering this investigation is the desire to seek a solution to the cosmologicalentropy problem [1] within standard Friedmann-Robertson-Walker cosmology. This problem follows from the constancy of entropy during cosmic evolution. It is a puzzle that the universe initially possesses the large entropy that we presently observe in the thermal background radiation. But constancy of the entropy directly originates from conservation of the energy-momentum tensor

$$T^{\mu v}; \mu = 0. \tag{1.1}$$

It is our purpose to seek a solution to this puzzle, within the classical standard cosmological model, by relaxing this condition. We find that it is indeed possible to relax (1.1) without upsetting the successes of general relativity (GR) both in and outside cosmology. One is in effect proposing a modified version of general relativity (MGR) in which (1.1) is not implied by the field equations, but may be imposed as part of "the equation of state" for systems which satisfy the condition. When (1.1) is imposed, the field equations of MGR reduce to those of standard GR.

Before we proceed towards this goal we digress to comment on the definition and conservation of the energy-momentum tensor. As is well-known in a Lonetz- invariant field theory, with $\mathcal{L} = \mathcal{L}(\phi_a, \phi_{a,\mu},)$, $T^{\mu v}$ is defined by

$$T^{\mu v} = g^{\sigma v} \frac{\partial \mathcal{L}}{\partial \phi_{a,\mu}} \phi_{a,\sigma} - g^{\mu v} \mathcal{L}. \tag{1.2}$$

The field equations would then yield the conservation condition, $T^{\mu,v}; \mu = 0$ provided that $\mathcal{L}$ has no explicit x-dependence. One notes that conservation does not hold identically and that $T^{\mu v}$ is not guaranteed to be symmetric. In contrast, $T^{\mu v}$ in GR is defined by the response of the action S to an infinitesimal change of coordinates $x^\mu \to \bar{x}^\mu = x^\mu + \xi^\mu(x)$. This induces a change $\delta g^{\mu v}$ in the metric tensor and $T^{\mu v}$ is defined by

$$\delta S = \frac{1}{2} \int \sqrt{-g} \delta g_{\mu v} T^{\mu v} d^4 x. \tag{1.3}$$

That this $T^{\mu v}$ must satisfy the conservation condition (1.1), follows from the fact that S is invariant under co-ordinate transformations i.e. $\delta S = 0$ for arbitrary ξ^μ. One also notes that (1.3) yields a symmetric $T^{\mu v}$.

It is thus clear that the definitions (1.2) and (1.3) will not in general coincide. An example, where they differ, is a vector field theory with

$$\mathcal{L} = \frac{1}{2} g_{\mu v} A^\mu A^v + \ldots \tag{1.4}$$

This term contributes $-\frac{1}{2} g^{\mu v} A^\sigma A_\sigma$ to $T^{\mu v}$ of (1.2), whereas it contributes $A^\mu A^v - \frac{1}{2} g^{\mu v} A^\sigma A_\sigma$ to $T^{\mu v}$ of (1.3).

One concludes that expressions for $T^{\mu v}$ which are taken, in curved space-time , from their lorentz-invariant form will not coincide with $T^{\mu v}$ obtained from (1.3) and need not satisfy the covariant conservation condition (1.1), since they are not in general required to satisfy the Lorentzian conservation condition.

The field equations of GR are obtained from an action principle for the complete system of matter and gravity, using the definition (1.3). Since (1.1) already holds by construction, one is either excluding those systems for which the flat-space restriction of $T^{\mu v}$ is not conserved, or is forcing them to satisfy (1.1) regardless of their flat-space state. This is remedied by the modified version of the field equations which we propose.

Finally, we mention that support for the relaxation of the conservation condition also comes from the horizon problem of the standard model. It has been observed [2] that resolution of the horizon problem requires that one either relaxes the conservation condition or admit negative pressure for temperatures in excess of about one Tev. Relaxation of the conservation condition would imply the creation of entropy by irreversible processes that might dominate the high temperature regime. Since negative pressure would not solve the entropy problem, it appears that the only viable resolution to both the entropy and horizon problems, within classical GR, is relaxation of the conservation condition.

2. GENERAL RELATIVITY WITHOUT THE CONSERVATION CONDITION

We derive the modified field equations using the conventional approach - see ref. [3] - except that we do not impose the conservation condition. The field equations are

$$G_{\mu v} = -8\pi G T^{\mu v} \tag{2.1}$$

where $G_{\mu v}$ is the gravitational tensor to be determined. The requirement that $G_{\mu v}$ contains only terms that are linear in the second, or quadratic in the first, derivatives of the metric tensor restricts $G_{\mu v}$ to the form

$$G_{\mu v} = \alpha R_{\mu v} + \beta R g_{\mu v} \tag{2.2}$$

where α and β are constants. Then one requires that the non- relativistic Newtonian equation

$$\nabla^2 g_{00} = -8\pi T_{00} \tag{2.3}$$

be obtained in the limit of the weak stationary field. This imposes the condition

$$\beta = \frac{\alpha(\alpha-2)}{2(3-2\alpha)} \quad \alpha \neq 0, 3/2. \tag{2.4}$$

When this is substituted into (2.2) and (2.1), one obtains the modified field equations:

$$R_{\mu v} - \frac{1}{2}\gamma R g_{\mu v} = -kT_{\mu v} \tag{2.5}$$

where

$$\gamma = \frac{2-\alpha}{3-2\alpha}, \quad k = \frac{8\pi G}{\alpha}. \tag{2.6}$$

In these equations the constant α is an arbitrary parameter. Standard GR has $\alpha = 1$.

Since the conservation condition (1.1) has not been explicitly imposed, it will obviously not be automatically satisfied. In fact one finds

$$T^{\mu}_{v;\mu} = \frac{\gamma - 1}{2k} R_{,v} \tag{2.7}$$

or, using $R = k(2\gamma - 1)^{-1}T$,

$$T^{\mu}_{v;\mu} = \frac{-1}{2}(1-\alpha)T_{,v} \tag{2.8}$$

where $T = g_{\mu v}T^{\mu v}$. It is thus possible to accommodate systems for which, in flat space, $T^{\mu}_{v,\mu} \neq 0$. For conserved systems, the condition $T^{\mu}_{v,\mu} = 0$ may be transformed into $T^{\mu}_{v;\mu} = 0$. The field equations would then require these to satisfy T =constant. If one imposes $T^{\mu v} \to 0$ as spatial infinity is approached, one finds

$$T^{\mu v}_{;\mu} = 0 \Rightarrow T = 0, \tag{2.9}$$

i.e. covaniantly conserved systems must have a traceless energy - momentum tensor. This conclusion, which requires $\alpha \neq 1$, is a remarkable consequence of MGR. There is in flat space no connection between the conservation and tracelessness conditions.

The important case of the pure electromagnetic field is an example of this result. In this case

$$T^{\mu v} = \frac{1}{4} g^{\mu v} F_{\alpha\beta} F^{\alpha\beta} - g^{\mu\alpha} F^{v\beta} F_{\alpha\beta}, \tag{2.10}$$

and $T^{\mu v}_{,\mu} = 0$ The generalization of this to $T^{\mu v}_{;\mu} = 0$ requires $g_{\mu v}T^{\mu v} = 0$ in curved space-time. Since this is already satisfied in flat space, both conditions are consistently transformed from flat to curved space-time without imposing any additional constraint on $T^{\mu v}$.

Conversely, one deduces from equ. (2.8) that MGR coincides with standard GR for all systems with a traceless energy momentum tensor. For such systems $R = 0$ so that

$$R_{\mu v} = -kT_{\mu v}. \tag{2.11}$$

The only difference between this and the corresponding equation of GR is the replacement of G by G/α on the right hand side. Any observable consequences of this change should lead to a determination of α. That the change is possible requires $\alpha > 0$. It is also possible to write the field equations (2.5) in a form that maintains the conventional statements:

(a) $R^{\mu v}_{\text{gravity}} = R^{\mu v} - \frac{1}{2} R g^{\mu v}$, and

(b) The field equations are $T^{\mu v}_{\text{gravity}} = T^{\mu v}_{\text{matter}}$.

This form is

$$R_{\mu v} - \frac{1}{2} R g_{\mu v} = -k\theta_{\mu v}, \tag{2.12}$$

where

$$\theta_{\mu v} = T_{\mu v} + \frac{1}{2}(1-\alpha) T g_{\mu v}. \tag{2.13}$$

On notes that $\theta_{\mu v}$ is completely determined by $T_{\mu v}$ and vanishes in the absence of matter.

At this stage one considers MGR to be a theory with two constants: k and α, or equivalently G and α. The Newtonian limit shows that αk, or G, is the universal gravitational constant. It may well be that α is also a universal constant. Equation (2.8) would then indicate a universal rate of energy dissipation, or energy generation, for matter systems in interaction with the gravitational field. The universal constant α may then be determined by studying (2.8) in a simple system for which this equation is non-trivial. It may also be that α is system-dependent. It would then indicate the dissipation or generation rate for that system and appear in the additional term in equation (2.13) which modifies the energy-momentum tensor of the system for use in the field equations (2.12). We keep an open mind about these two possibilities and proceed to consider further consequences of the modified theory.

3. CONSEQUENCES OF THE FIELD EQUATIONS

Empty space: In empty space the field equations reduce to those of standard GR

$$R_{\mu v} = 0. \tag{3.1}$$

This is extremely significant on two accounts. The first is that one is allowed to keep, in MGR, the extensive literature [4] that exists on the classes of exact solutions of Einstein's vacuum field equations of various types and symmetries. The second is that some of these solutions, such as the Schwarzchild and Kerr solutions, have been widely used to explore the most important consequences of GR. In particular, one notes that the crucial tests of the perihelion of Mercury, the deflection of light, the gravitational red shift and the delay in radar echoes are all based on the Schwarzchild solution. Similarly the existence of singularities and black holes is based on vacuum solutions. Thus all these features, which are characteristic of standard GR, are maintained in MGR.

B. Radiation: As previoulsy remarked, for systems with traceless energy-momentum tensor, the form of the field equations of GR and MGR are the same; equ. (2.11). These include Einstein Maxwell fields for which many exact solutions exist [4]. They also include general radiation and highly relativistic thermodynamic systems. Thus the modified and the standard field equations are equivalent in their description of the pure radiation era in cosmology. It follows that, even with the proposed modification, entropy could not have been generated during the pure radiation era. One must therefore associate the generation of entropy with the advent of massive particles, i.e., with the phase transitions that create mass. Thus the successes of the standard model of early cosmology are maintained while deviations are provided for, where the standard model is expected to be inadequate.

C. Homogeneous perfect fluid: The energy momentum tensor of a perfect fluid in a homogeneous Robertson-Walker (RW) metric is

$$T_{\mu v} = (\rho + p)u_\mu u_v + p g_{\mu v}, \tag{3.2}$$

where $u^2 = -1, \rho = \rho(t),\ p = p(t)$. In this case $T = 3p - \rho$. Equations (2.12) and (2.13) then show that the field equations in MGR for any homogeneous perfect fluid with energy-momentum $T_{\mu v}$ (not necessarily conserved) are equivalent to those of a perfect fluid with conserved energy-momentum tensor $\theta_{\mu v}$. Similarly a given equation of state $p = p(\rho)$, will transform into an equation of state $P_\theta = P_\theta(\rho_\theta)$ where

$$\theta_{\mu v} = (\rho_\theta + p_\theta)u_\mu u_v + p_\theta g_{\mu v}. \tag{3.3}$$

Conversely to any conserved perfect fluid with tensor $\theta_{\mu v}$ and equation of state $P_\theta = P_\theta(\rho_\theta)$ there is a corresponding perfect fluid with tensor $T_{\mu v}$ and equation of state $p = p(\rho)$, where

$$T_{\mu v} = \theta_{\mu v} - \frac{1}{2}\frac{1-\alpha}{2\alpha - 1}\theta g_{\mu v}, \tag{3.4}$$

$\theta = \theta^\mu_\mu$.

This observation enables one to use known exact solutions for conserved systems in standard GR to immediately construct exact solutions for the corresponding perfect fluids in MGR. Thus for these systems relaxation of the conservation condition does not introduce any analytic complexity.

As an example of this procedure, consider a dust-like, or matter-dominated, universe with a flat RW metric. In standard GR the scale factor $a_s(t)$ is given by

$$a_s(t) = (\sqrt{m}t)^{2/3}. \tag{3.5}$$

To obtain the solution in MGR, we find from $T_{\mu v} = \rho u_\mu u_v$ that $\theta_{\mu v} = \rho u_\mu u_v - \frac{1}{2}(1-\alpha)\rho g_{\mu v}$. Thus one has a perfect fluid with

$$P_\theta = -\frac{1-\alpha}{3-\alpha}\rho_\theta. \tag{3.6}$$

For this system, the solution to (2.12) for a flat RW metric gives the scale factor $a(t)$ where

$$a(t) = \left(\frac{2\sqrt{m}}{3-\alpha}t\right)^{1-\frac{\alpha}{3}} \tag{3.7}$$

This changes the relation between the present age of the universe, t_p, and the present value of Hubble's constant, H_p, from $t_p = \frac{2}{3}H_p^{-1}$ in standard GR to $t_p = \left(1 - \frac{\alpha}{3}\right) H_p^{-1}$ in MGR. For a given observational value of H_p, this increases the estimate of the age of the universe by a factor of $\frac{3-\alpha}{2}$ where $0 < \alpha < 1$.

In this example one also has

$$\rho = Ca^{-\frac{6}{3-\alpha}} \tag{3.8}$$

which indicates that the energy E_m of non-relativistic matter is not conserved. For, equ. (3.8) gives $\rho a^3 = Ca^{3\frac{1-\alpha}{3-\alpha}}$ so that E_m increases as the universe expands. This is associated with generation of entropy given by

$$Tds \sim (1-\alpha) a^{\frac{-2\alpha}{3-\alpha}} da. \tag{3.9}$$

Our interpretation is that entropy is generated during the process of the creation of massive non-relativistic matter, i.e. the transformation of gravitational energy into dust. These remarks pave the way to our discussion of the cosmological model in MGR in the next section.

Built-in cosmological constant: It is well known that the addition of a term $\Lambda g_{\mu v}$, Λ a constant, to the left hand side of Einstein's field equations is compatible with the conservation constraint (1.1). It does, however, violate the condition on the Newtonian limit and leads to residual vacuum curvature. In a homogeneous RW universe, generation of entropy requires that Λ be time-dependent [5] so that (1.1) is violated. A number of variable- Λ cosmological models exist in the literature [5,6]. The field equations of MGR may, in fact, be written in a form that exhibits a time-dependent term and presents the theory as a variable-Λ model:

$$R_{\mu v} - \frac{1}{2}Rg_{\mu v} + \Lambda g_{\mu v} = -kT_{\mu v}, \tag{3.10}$$

$$\Lambda = \frac{1}{2}\frac{1-\alpha}{3-2\alpha}R. \tag{3.11}$$

In this respect one observes that

(a) $\Lambda(t) \to 0$ as $t \to \infty$ in a perfect- fluid RW universe, i.e. one has a decaying- Λ cosmology.

(b) $\lambda \equiv 0$ in vacuum, so that no additional Λ-force occurs in the vacuum solutions.

(c) $\Lambda \equiv 0$ for traceless $T_{\mu v}$, so that exponential de Sitter or inflationary expansion cannot arise during the pure-radiation era. Thus, if desired, exponential inflation can only occur in a pre-radiation period. This is consistent with the assumption that the initial period of the cosmic expansion was controlled by the interaction of a scalar field with gravitation [7].

4. THE COSMOLOGICAL MODEL

At any time during the evolution of the universe one may write the total energy density in the form

$$\rho = \rho_r + \rho_{rm} + \rho_m \tag{4.1}$$

where ρ_r is the contribution of pure radiation (i.e. of the different types of massless particles), ρ_{rm} the contribution of massive matter in equilibrium with (electromagnetic) radiation and ρ_m the contribution of decoupled massive matter.

In a classical cosmological model, one may distinguish three stages of evolution. Stage I is an era of pure radiation with

$$\rho = \rho_r, \quad p = \frac{1}{3}\rho_r, \rho_r = bT^4 \tag{4.2}$$

where b is a known constant. Stage II is a transitional era of radiation and relativistic (elementary-particle) matter in thermal equilibrium, with

$$\rho = \rho_r + \rho_{rm}, \quad p_r = \frac{1}{3}\rho_r = b'T^4. \tag{4.3}$$

The expressions for $\rho_{rm}(T)$ and $P_{rm}(T)$ are model-dependent. One usually assumes ideal gas distributions and applies the formulae of nuclear statistical equilibrium to study nucleosynthesis during this period. It would be useful to reconsider this standard analysis in MGR.

State III is the matter-dominated era in which non-relativistic (atomic then galactic) matter is decoupled from radiation. In this era,

$$\rho = \rho_r + \rho_m, \;\; \rho_r = cT_r^4, \; p_r = \frac{1}{3}\rho_r, \; p_m = 0. \tag{4.4}$$

The matter component may be thermodynamically modelled by an ideal gas with $p_m \neq 0$ and a temperature T_m related to ρ_m and p_m by the equation of state of a classical non-relativistic ideal gas. It is actually a classical thermodynamic system that is gradually transformed from an ideal gas of hydrogen into an ideal gas of galaxies. To circumvent this, one simply studies the matter component as a mechanical system that evolves under the field equations.

For a homogeneous universe with RW metric and $T_{\mu v}$ given by (3.2), the field equations (2.5) give

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{k}{3}[\rho - \beta(3p - \rho)] - \frac{\varepsilon}{a^2}, \tag{4.5}$$

$$\frac{d}{da}[a^3\{\rho - \beta(3p - \rho)\}] + 3a^2[p + \beta(3p - \rho)] = 0, \tag{4.6}$$

where $\beta = \frac{1}{2}(1 - \alpha)$ and ε is the curvature constant.

As remarked previously, these equations coincide with those of the standard cosmological model for the stage of pure radiation, except that one replaces G by G/α:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{3}k\rho_r - \frac{\varepsilon}{a^2},$$

$$\frac{d}{da}(a^3\rho_r) + a^2\rho_r = 0. \tag{4.7}$$

Thus for small t, irrespective of the value of ε, $a \sim t^{1/2}$, $\rho_r \sim a^{-4}$, $T \sim a^{-1}$, S is constant and the model is indistinguishable from the standard model during this stage. An earlier field- theoretic era may be constructed to remove the initial singularity. This may for example, be an extension of the string-motivated model of ref [8].

During the transitional era of stage II, the field equations give

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{k}{3}[\rho_r + (1+\beta)\rho_{rm} - 3\beta P_{rm}] - \frac{\varepsilon}{a^2}$$

$$\frac{d}{da} \quad [a^3\{\rho_r + (1+\beta)\rho_{rm} - 3\beta P_{rm}\}] + 3a^2\left[\frac{1}{3}\rho_r - \beta\rho_{rm} + (1+3\beta)P_{rm}\right] = 0. \tag{4.8}$$

One may apply the usual assumptions of thermal and chemical equilibrium for the various species of radiation, elementary particles or nuclei that populate the universe in several successive ranges of temperature. The results will not be much affected by the presence of β, since for the most part the conditions are relativistic. For example, one would still find that

$$\frac{T_\gamma}{T_\upsilon} \approx \left(\frac{11}{4}\right)^{1/3}, \tag{4.9}$$

since this is based on $\rho_{e\pm} \sim T^4$, so that entropy is approximately constant in MGR, which is required for (4.9). A notable exception is the generation of entropy during this stage in MGR whereas entropy is strictly constant in the standard model. Equation (4.8) gives

$$Tds = d(\rho_{rm}V) + P_{rm}dV = \beta V d(3P_{rm} - \rho_{rm}). \tag{4.10}$$

In the ideal gas model, for large T, $d(3P_{rm} - \rho_{rm}) > 0$. Thus $\beta \geq 0$ i.e. $0 < \alpha \leq 1$.

The generation of entropy during this stage is an important physical feature, since this period includes the spontaneous symmetry breaking that generated quark and lepton mass, as well as the phase transition that created hadrons.

During the matter-dominated era, the field equations give

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{k}{3}(1+\beta)\rho_m - \frac{\varepsilon}{a^2}, \tag{4.11}$$

$$(1+\beta)\frac{d}{da}(a^3\rho_m) - 3\beta a^2\rho_m = 0, \tag{4.12}$$

assuming that ρ_m, $3\beta\rho_m >> \rho_r$. For the radiation component, the entropy is (assuming massless neutrinos)

$$S_r = S_\gamma + S_v \propto a^3 T_\gamma^3 \tag{4.13}$$

which is not constant, since equ. (4.7) is no longer valid. Equation (4.12) yields

$$\rho_m = \rho_{mp}\left(\frac{a_p}{a}\right)^{\frac{3}{1+\beta}}, \tag{4.14}$$

where the constant of integration is evaluated at the present time $t = t_p$, writing ρ_{mp} and a_p for the present values of ρ_m and a. Thus in contrast to standard cosmology, the energy content of non-relativistic matter is not constant. It increases with the expansion according to

$$\rho_m a^3 \sim a^{\frac{3\beta}{1+\beta}}. \tag{4.15}$$

The time-dependence of the scale factor, $a(t)$, is obtained from equations (4.11) and (4.14). From equ. (4.11) one gets

$$\varepsilon = a_p^2 H_p^2 \left[\frac{1+\beta}{1-2\beta}\Omega_p - 1\right] \tag{4.16}$$

where

$$\Omega = \frac{8\pi G\rho_{pm}}{3H^2}. \tag{4.17}$$

Thus the universe is critical for

$$\Omega_p = \eta, \quad \eta = \frac{1-2\beta}{1+\beta}. \tag{4.18}$$

Since $\eta < 1$, this is a considerable departure from the standard model, which is critical for $\Omega_p = 1$. If for example, $\beta = \frac{1}{4}$, the universe is closed when $\Omega_p > 0.4$.

The observational limits on Ω_p are

$$0.1 < \Omega_p < 4. \tag{4.19}$$

It thus appears that, with any appreciable value of β, a closed universe is more likely. We shall, first consider the case $\varepsilon = 1$, so that

$$\dot{a}^2 = \left(\frac{a_\circ}{a}\right)^\eta \left[1 - \left(\frac{a}{a_\circ}\right)^\eta\right], \tag{4.20}$$

where

$$a_\circ = a_p \left[\frac{\Omega_p}{\Omega_p - \eta}\right]^{\frac{1}{\eta}}. \tag{4.21}$$

Equ. (4.20) should be integrated subject to the boundary condition $a = a_{dec}$ at $t = t_{dec}$, where '*dec*' denotes coupling of matter and radiation. We shall, however, replace this condition by $a = 0$ at $t = 0$. Then equ. (4.20) gives

$$t = \frac{a_\circ}{\eta} \int_0^{\left(\frac{a}{a_\circ}\right)^\eta} u^{\frac{1}{\eta} - \frac{1}{2}} (1-u)^{-\frac{1}{2}} du,$$

i.e.,

$$t = \frac{2a_\circ}{2+\eta} \left(\frac{a}{a_\circ}\right)^{1+\eta/2} F\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{\eta}; \frac{3}{2} + \frac{1}{\eta}; \left(\frac{a}{a_\circ}\right)^\eta\right), \tag{4.22}$$

where F is the hypergeometric function. This equation determines $a = a(t)$ implicitly. The corresponding equ. for the standard model ($\eta = 1$) is

$$t = a_\circ \left[\sin^{-1}\left(\frac{a}{a_\circ}\right)^{\frac{1}{2}} - \left(\frac{a}{a_\circ}\right)^{\frac{1}{2}} \left(1 - \frac{a}{a_\circ}\right)^{\frac{1}{2}}\right]. \tag{4.23}$$

Equations (4.20), (4.21) and (4.22) may be used to express the age of the universe t_p in terms of H_p, Ω_p and η. For, equations (4.20) and (4.21) give

$$a_p = \frac{1}{H_p} \left(\frac{\eta}{\Omega_p - \eta}\right)^{\frac{1}{2}}, \tag{4.24}$$

so that equation (4.22) yields

$$t_p = N^{(+)}(\Omega_p, \eta) H_p^{-1} \tag{4.25}$$

where

$$N^{(+)}(\Omega_p, \eta) = \frac{2}{2+\eta} \left(\frac{\eta}{\Omega_p}\right)^{\frac{1}{2}} F\left(\frac{1}{2}, \frac{1}{2} + \frac{1}{\eta}; \frac{3}{2} + \frac{1}{\eta}; 1 - \frac{\eta}{\Omega_p}\right). \tag{4.26}$$

For the standard model (with $\varepsilon = 1$) one has

$$(4.27)\quad N^{(+)}(\Omega_p, \eta) = \frac{1}{\Omega_p - 1}\left[\frac{\Omega_p}{(\Omega_p - 1)^{\frac{1}{2}}}\sin^{-1}\left(\frac{\Omega_p - 1}{\Omega_p}\right)^{\frac{1}{2}} - 1\right].$$

We shall consider Ω_p, H_p and t_p as given observables. Equation (4.27) is then a rigid test of the (closed) standard model. From work on the age of the globular star clusters in the halo of the Milky Way [9] one deduces that

$$(4.28)\qquad t_p \geq (16 \pm 2) \times 10^9 y.$$

Taking for the Hubble parameter, h_p , in the relation

$$(4.29)\qquad H_p^{-1} = 0.98 h_p^{-1} \times 10^{10} y,$$

the observable range $0.5 \leq h_p \leq 0.85$, we get

$$(4.30)\qquad 11.53 \times 10^9 y \leq H_p^{-1} \leq 19.6 \times 10^9 y.$$

From these one secures the lower bound

$$(4.31)\qquad t_p H_p \geq 0.71,$$

and, if equality is assumed in (4.28), also the upper bound $t_p H_p \leq 1.56$. The bound (4.31) appears to exclude the closed standard model, since equ (4.27) gives $t_p H_p = (0.67, 0.57, 0.51, 0.47)$ for $\Omega_p = (1, 2, 3, 4)$, which covers the whole of the allowed range, (4.19), for the closed model.

For $\Omega_p < 1$, the standard model is open and equation (4.27) is replaced by

$$(4.32)\, N^{(-)}(\Omega_p, 1) = \frac{1}{1 - \Omega_p}\left[1 - \frac{\Omega_p}{(1 - \Omega_p)^{1/2}}\sin h^{-1}\left(\frac{1 - \Omega_p}{\Omega_p}\right)^{\frac{1}{2}}\right].$$

Using this equation one finds that the lower bound (4.31) is satisfied only if $\Omega_p \leq 0.7$. It thus appears that the presently available observational data would exclude the matter dominated standard model for all $\Omega_p > 0.7$. It is only tenable in the range $0.1 \le \Omega_p \le 0.7$.

Equation (4.26) shows that the situation is quite different in the matter-dominated model of MGR, due to the presence of the parameter η.

We first note that equ. (4.26) also holds for the case $\varepsilon = -1$, i.e., for the open model with $\Omega_p < \eta$, so that

$$N^{(-)}(\Omega_p, \eta) = N^{(+)}(\Omega_p, \eta). \tag{4.33}$$

This follows from the fact that, in this case, equ. (4.22) is replaced by

$$t = \frac{2a_1}{2+\eta}\left(\frac{a}{a_1}\right)^{1+\eta/2} F\left(\frac{1}{2}, \frac{1}{2}+\frac{1}{\eta}; \frac{3}{2}+\frac{1}{\eta}; -\left(\frac{a}{a_1}\right)^{\eta}\right), \tag{4.34}$$

where

$$a_1 = a_p\left(\frac{\Omega_p}{\eta - \Omega_p}\right)^{\frac{1}{\eta}} = H_p^{-1}\left(\frac{\eta}{\Omega_p}\right)^{\frac{1}{2}}\left(\frac{\Omega_p}{\eta-\Omega_p}\right)^{\frac{1}{\eta}+\frac{1}{2}} \tag{4.35}$$

Thus equ. (4.26) may be used for all cases and we shall denote $t_p H_p$ by $N(\Omega_p, \eta)$. One notes the special cases:

$$N(\Omega_p, 1) = \frac{2}{3}\Omega_p^{-\frac{1}{2}} F\left(\frac{1}{2}, \frac{3}{2}; \frac{5}{2}; 1 - \Omega_p^{-1}\right), \tag{4.36}$$

which may be written in the forms (4.27) or (4.32) for $\Omega_p \gtrless 1$;

$$N(\Omega_p, \Omega_p) = \frac{2}{2+\Omega_p}, \quad \Omega_p \le 1, \tag{4.37}$$

$$N(\Omega_p, 0) = \sqrt{\pi}\,\Omega_p^{-\frac{1}{2}} e^{1/\Omega_p}\left[1 - \Phi(\Omega_p^{-\frac{1}{2}})\right], \tag{4.38}$$

where $\Phi(x)$ is the error function.

Using these equations one finds (see Fig. (1)) that the observational data restrict the values of N to the range

$$0.71 \le N \le 0.96, \tag{4.39}$$

where the upper limit corresponds to $\Omega_p = 0.1, \eta = 0$. For $0.1 \leq \Omega_p \leq 0.7$, it is possible to admit values of η in the whole range [0,1], where the unique fixed value is determined by the values of Ω_p and N. For $0.7 < \Omega_p \leq 1.4$ we must have $0 \leq \eta < 1$ and the standard model is excluded. Thus the admissible values of Ω_p lie in the range

$$0.1 \leq \Omega_p \leq 1.4. \tag{4.40}$$

Should observations show N or Ω_p to exceed 0.96 or 1.4, respectively, one must discard the matter-dominated, zero-pressure, cosmological model, even when energy conservation is relaxed.

If, on the other hand, observations fix definite values for Ω_p and N that lie within the admissible region of the MGR matter-dominated model, then a unique value of η is determined. For example, the values $\Omega_p = 1$, $N = 0.73$ yield $\eta = 0.25$. One is then able to measure the energy non-conservation parameter at a value that definitely excludes the standard model. Fig. (1) shows the standard model to be extremely restrictive since N is uniquely determined by Ω_p when $\eta = 1$. For example $\Omega_p = 0.5$ yields $N = 0.75$ (when $\eta = 1$), so that $t_p = 15 \times 10^9 y$ implies $H_p^{-1} = 2 \times 10^{10} y$ i.e. $h = 0.49$, which is at the limit of the observational range. In the MGR model, $\Omega_p = 0.5$ is consistent with $0.75 \leq N \leq 0.85$, so that $t_p = 15 \times 10^9 y$ yields $0.49 \leq h \leq 0.56$ which provides some margin for agreement with observation.

We finally note that equ. (3.11) enables one to obtain an expression of the effective decaying cosmological "constant" of the matter-dominated model based on MGR:

$$\Lambda = -\frac{1-\alpha}{\alpha} 4\pi G \rho_m \sim a^{-\frac{3}{1+\beta}}. \tag{4.41}$$

If α is small, as seems to be the indication of fig. (1), then $\beta \approx \frac{1}{2}$ and the rate of decay in (4.41) will be close to that of the critical-density model of ref. [5] in which $\Lambda \sim a^{-2}$. From (4.41) one obtains for the present value

$$|\Lambda_p| = \frac{3}{2}\frac{1-a}{\alpha}\Omega_p H_p, \tag{4.42}$$

of the order of $10^{-20} y^{-2}$.

In Section V we give a summary of our results and some concluding remarks.

5. SUMMARY AND CONCLUSIONS

(1) Motivated by our desire to solve the entropy problem of the standard cosmological model, we have proposed that the energy-conservation condition be relaxed. The result is a modified version of classical general relativity with a single free parameter, embodied in the field equations (2.5).

(2) A number of immediate consequences follow from the modified field equations:

a) Covariantly conserved systems must possess a traceless energy momentum tensor.

b) In empty space the modified field equations coincide exactly with those of standard general relativity; equations (3.1).

c) It follows from (b) that the crucial test of general relativity, the existence of singularities, black holes and all features connected with the exact vacuum solutions remain intact.

d) For systems with a traceless energy-momentum tensor the modified field equations coincide with the original ones, equ (2.11), except that G is replaced by G/α where $\alpha \neq 1$.

e) It follows from (d) that the treatment of Einstein-Maxwell fields, general fields of pure radiation, and highly relativistic thermodynamics systems remains unchanged. In particular, the modified and original field equations are equivalent in their description of the radiation era in cosmology.

f) The modified field equations may be written in a form that exhibits a cosmological term - equ. (3.10) - with a definite dependence on the scalar curvature, equ (3.11). In a perfect-fluid RW universe, this provides an effective decaying cosmological "constant", that vanishes identically in vacuum and in pure- radiation systems.

(3) In the matter-dominated era of the homogeneous RW universe based on the modified field equations one finds that the total energy of non-relativistic matter increases with the expansion, equ. (4.15). This might be interpreted as the continuous transformation of gravitational energy into massive matter, as the curvature of the universe unfolds and it evolves towards flatness.

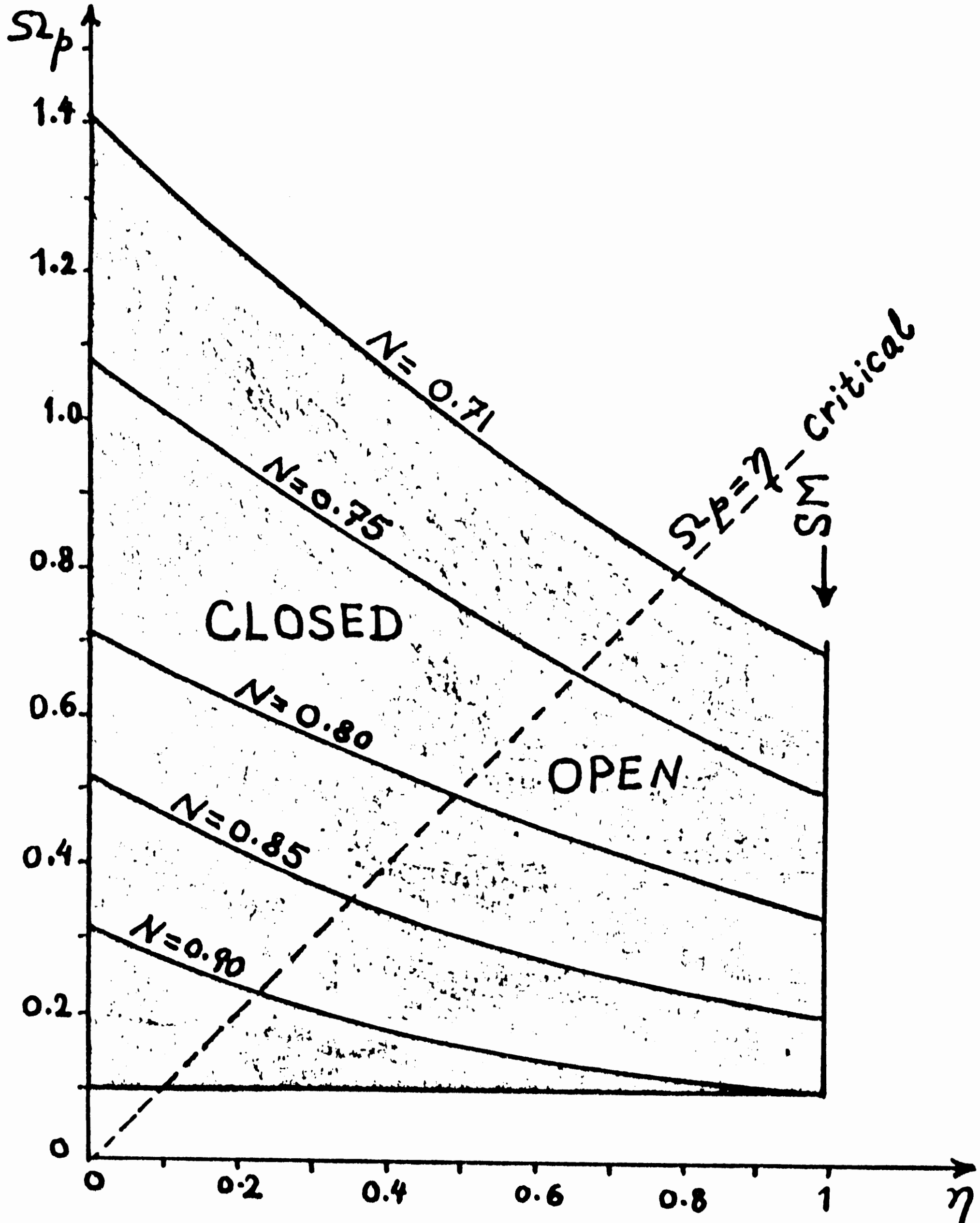
(4) The present value of the product $t_p H_p$ is given, for the modified matter-dominated model by equ. (4.26) where $0 \leq \eta \leq 1$ and $\eta = 1$ is the un-modified model. The universe is closed for $\Omega_p > \eta$. One finds that the observational lower bound in equ. (4.31), which seems to exclude the closed unmodified model, is consistent with open and closed possibilities in the modified version.

(5) The observational lower bound in equ. (4.31) restricts the original matter-dominated model to be open and admits values of Ω_p that are limited to $\Omega_p \leq 0.7$. For the modified version both open and closed possibilities are consistent with the data and the upper bound on Ω_p is 1.4. Exact knowledge of Ω_p and $t_p H_p$ would either uniquely determine the free parameter η in the range $0 \leq \eta \leq 1$ or show that the matter-dominated cosmological model is inconsistent with the data, even when energy conservation is relaxed.

(6) Should the need arise for generalizing the matter-dominated zero-pressure cosmological model, the obvious extension is to include electro-magnetic radiation and relativistic massive neutrinos, where the latter are the dominant component. Exploration of the wider margin for consistency with the observational data, that is provided by the modified model, should however precede such generalizations.

Figure Caption

Region consistent with the modified matter-dominated model is shaded. The standard model is the edge $\eta = 1$ of this region.



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