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THE HASSE PRINCIPLE FOR A CLASS OF CUBIC SURFACES

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ABSTRACT. We prove the Hasse principle for a class of cubic surfaces. This generalizes on a theorem of Selmer.

1. INTRODUCTION

Selmer [5] established the Hasse principle for cubic surfaces of the form

$$aX^3 + cY^3 = bZ^3 + dW^3 \tag{1}$$

where a, b, c, d are non-zero integers satisfying $c/a = d/b$. On the other hand, Swinnerton-Dyer [6] gave a counter-example to the Hasse principle for cubic surfaces, and Cassels and Guy [3] (see also [1]) gave another counter-example for diagonal cubic surfaces. It is thus interesting to specify classes of cubic surfaces which do satisfy the Hasse principle. In this paper we give such a class. Our Theorem below is in fact a generalization of Selmer's result stated above.

We note that the Hasse principle for cubic forms is discussed extensively in [4]. We would like to thank the referee for his corrections and helpful comments.

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Theorem. *Suppose k is a number field and K is a cubic extension of k. Suppose $a,\ b \in k^*$ and $\theta,\ \phi \in K\backslash k$. The cubic surface*

$$a\ \mathrm{Norm}_{K/k}(X+Y\theta) = b\ \mathrm{Norm}_{K/k}(Z+W\phi) \tag{2}$$

satisfies the Hasse principle; that is, if (2) has k_v-rational points for all valuations v, then it has k-rational points.

We note that for the cubic surface in (1), Selmer's result follows from the above theorem by taking $\theta = \phi = (c/a)^{1/3} = (d/b)^{1/3}$ in the case c/a is not a cube; if c/a is a cube then (1) trivially has rational points.

In the proof of the theorem we will need the following Lemma which is well-known but for which we cannot find any convenient reference; though a similar argument is given in the introduction to [4]. Thus we give a proof.

Lemma. *Suppose $\mathcal{S}$ is a non-singular cubic surface defined over a number field k. Suppose $\mathcal{S}$ has points over $k(\sqrt{d})$ for some $d \in k^*\backslash(k^*)^2$. Then $\mathcal{S}$ has points over k.*

Proof. Suppose $P \in \mathcal{S}(k(\sqrt{d}))$. If P is k-rational then we are finished. Thus suppose that P is not k-rational, and let $Q \in \mathcal{S}(k(\sqrt{d}))$ be its quadratic conjugate. There exists a unique line L passing through P and Q, and this line must be defined over k. There are now two possibilities:
The line L lies on $\mathcal{S}$. Thus $\mathcal{S}$ has infinitely many k-rational points and we are done.
The line L intersects $\mathcal{S}$ in precisely three points. These points will be P, Q and a third point, say, R. Since $\mathcal{S}$ and L are both defined over k, the set $\{P,\ Q,\ R\}$ is rational over k. That is the set is mapped to itself by the action of Galois. Since P, Q are quadratic conjugates it follows that R is k-rational and the proof is finished.

We are now ready to prove the theorem.

Proof of the theorem. There are two possibilities:
The extension K/k is normal, and so, since it is of degree 3, is cyclic. It follows from the hypotheses of the theorem that $a^{-1}b$ is a norm everywhere locally, and thus by a theorem of Hasse ([2] page 251), is a global norm. That is, $a^{-1}b = \mathrm{Norm}_{K/k}(\alpha)$ for some $\alpha \in K$. Note that the elements $1,\ \theta,\ \alpha,\ \alpha\phi$

must be linearly dependent over k. Thus there exists $X, Y, Z, W \in k$, not all zero, such that

$$X + Y\theta = \alpha(Z + W\phi) .$$

Taking norms we deduce the existence of a k-rational point on (2).

The extension K/k is not normal. Let d be the discriminant of K/k, $K' = K(\sqrt{d})$ and $k' = k(\sqrt{d})$. Then K'/k' is a normal cubic extension. Using the same argument as above we see that the surface (2) has k'-rational points. By the Lemma above, the surface has k-rational points, and our theorem is established.

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