Arab J. Math. Sc.
Volume 2, Number 1, June 1996
pp. 87-98

ON NEW RENEWAL BETTER THAN USED CLASSES OF AGEING

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ABSTRACT. Suppose that a device is subject to shocks occurring randomly in time according to the counting process $N = \{N(t), t \geq 0\}$. Let $\bar{P}_k$, be the probability that the device survives the first k shocks, $k = 0, 1, 2, \ldots$, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \ldots$. The probability that the device survives beyond t, $\bar{H}(t) = \sum_{k=0}^{\infty} P\{N(t) = k\}\bar{P}_k$, is proved to have the new renewal better (worse) than used NRBU (NRWU) property under some conditions on $\{\bar{P}_k\}_{k=0}^{\infty}$ when N is homogeneous and nonhomogeneous Poisson processes. Laplace transform and generating function characterizations for these NRBU (NRWU) properties are given.

1. INTRODUCTION

Suppose that a device is subject to shocks occuring randomly in time according to the counting process $N = \{N(t), t \geq 0\}$. Let the device have the probability $\bar{P}_k$ of surviving k shocks $k = 0, 1, 2, \ldots$, where $1 = \bar{P}_0 \geq \bar{P}_1 \geq \ldots$ The probability $\bar{H}(t)$ that the device survives beyond t is given by

$$\bar{H}(t) = \sum_{k=0}^{\infty} P\{N(t) = k\}\bar{P}_k. \tag{1.1}$$

Such shock models have been studied by Esary et al. (1973) when N is a homogeneous Poisson process and by A-Hameed and Proschan (1973, 1975) when N is a nonhomogeneous Poisson process. In all these cases the authors prove that $\bar{H}(t)$ is IFR, IFRA, DMRL, NBU, or NBUE under suitable conditions on N if $\{\bar{P}_k\}_{k=0}^{\infty}$ has the corresponding discrete property. Klefsjo (1981) has considered (1.1) for the HNBUE class. Abouammoh et al. (1988) have studied some shock models for NBUFR and NBAFR classes. Abouammoh and Hendi (1991) considered shock models for the new better than used renewal failure rate (NBURFR) and the new better than average renewal failure rate (NBARFR). Recently Abouammoh and Ahmed (1993) studied NRBU (NRWU) closure properties under some reliability operations such as convolutions, mixtures and coherent systems. They examine also, the relationships between these classes and other classes.

The main theme of this paper is to establish different results of shock models for the class of new renewal better than used (NRBU) distributions and its dual class of new renewal worse than used (NRWU) distributions. In section 3 the survival function $\bar{H}(t)$ is studied when N is a homogeneous and nonhomogeneous Poisson processes. The Laplace transforms for these classes and characterization of generating functions of the renewal failure rate classes are established in section 4.

2. THE NEW RENEWAL BETTER THAN USED PROPERTIES

Let H be a life distribution, that is $H(0-) = 0$ and $\bar{H}(t) = 1 - H(t)$, $\forall t \geq 0$ be its corresponding survival function and $\mu_H = \int_0^\infty \bar{H}(u)du$. The new renewal better than used (NRBU) and the new renewal worse than used (NRWU) properties are given below.

Definition 2.1 : A life distribution H with $H(0-) = 0$ and finite mean $\mu_H = \int_0^\infty \bar{H}(u)du$ is said to have the NRBU(NRWU) property if

$$\mu_H \bar{H}(t+x) \leq (\geq) \bar{H}(t) \int_x^\infty \bar{H}(u)du, \qquad \forall\, t, x \geq 0 \tag{2.1}$$

Classes of life distributions with these properties are introduced by Abouammoh and Ahmed (1993). Their behaviour under some reliability operations such as convolution, mixing and formation of coherent systemes has been also studied. Moreover, they investigated the relationships between these classes with other classes.

Now consider a device with life distribution H which is replaced instantaneously upon failure by a sequence of mutually independent devices. These devices are independent of the first device and each has the same life distribution H. In the long run the residual life of a device under operation is given by

$$W_H(t) = \mu_H^{-1} \int_0^t \bar{H}(u)du, \qquad t \geq 0 \tag{2.2}$$

the stationary renewal distribution of H. Using (2.2) relation (2.1) can be expressed as

$$\bar{H}(x|t) \leq \bar{W}_H(x); \qquad t, x \geq 0 \tag{2.3}$$

where $\bar{W}_H(x) = 1 - W_H(x)$. The last inequality (2.3) account for the name NRBU.

Now we give the following definition of the discrete NRBU and NRWU properties.

Definition 2.2 : A life distribution or its survival $\bar{P}_k = 1 - P_k$, $k = 0, 1, 2, \ldots$ is called NRBU (NRWU) if

$$\mu \bar{P}_{l+k} \leq (\geq) \bar{P}_l \sum_{j=k}^{\infty} \bar{P}_j \ ; \qquad k \geq 0, 1, \ldots \tag{2.4}$$

where $\mu = \sum_{k=0}^{\infty} \bar{P}_k$.

3. A POISSON SHOCK MODEL LEADING TO NRBU

In this section we consider the shock model given by (1.1) such that the arriving shocks to the device occur according to a counting process N which is a homogeneous and nonhomogeneous Poisson processes.

Assume first that a device is subjected to shocks occuring randomly in time according to a homogeneous Poisson process with constant intensity λ. Thus the shock model (1.1) is reduced to the form

$$\bar{H}(t) = \sum_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \bar{P}_k. \tag{3.1}$$

Now we prove that the discrete NRBU property of $\bar{P}_k$, $k = 0, 1, 2, \ldots$ is preserved for $\bar{H}(t)$ under the model (3.1).

Theorem 3.1. *The survival $\bar{H}(t)$ in model (3.1) is NRBU if $\{\bar{P}_k\}_{k=0}^{\infty}$ is discrete NRBU.*

Proof. By (3.1)

$$\mu_H = \int_0^\infty \bar{H}(u)du = \int_0^\infty \sum_{k=0}^{\infty} e^{-\lambda u} \frac{(\lambda u)^k}{k!} \bar{P}_k du$$

$$= \frac{\mu}{\lambda}. \tag{3.2}$$

Note that $\bar{H} \in$ NRBU if

$$\mu_H \bar{H}(t+x) \leq \bar{H}(t) \int_x^\infty \bar{H}(u)du \tag{3.3}$$

L. H. S. of (3.3) is

$$\begin{aligned}
\mu_H \bar{H}(t+x) &= \frac{\mu}{\lambda} \sum_{k=0}^{\infty} e^{-\lambda(t+x)} \frac{[\lambda(t+x)]^k}{k!} \bar{P}_k \\
&= \frac{\mu}{\lambda} e^{-\lambda(t+x)} \sum_{k=0}^{\infty} \bar{P}_k \sum_{j=0}^{k} \frac{\binom{k}{j} (\lambda t)^j (\lambda x)^{k-j}}{k!} \\
&= \frac{\mu}{\lambda} e^{-\lambda(t+x)} \sum_{k=0}^{\infty} \bar{P}_k \sum_{j=0}^{k} \frac{(\lambda t)^j}{j!} \frac{(\lambda x)^{k-j}}{(k-j)!} \\
&= \frac{\mu}{\lambda} e^{-\lambda(t+x)} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} \sum_{k=j}^{\infty} \frac{(\lambda x)^{k-j}}{(k-j)!} \bar{P}_k, \\
&= \frac{\mu}{\lambda} e^{-\lambda(t+x)} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} \sum_{l=0}^{\infty} \frac{(\lambda x)^l}{l!} \bar{P}_{j+l}.
\end{aligned}$$

Since $\bar{P}_k$ is NRBU, i.e. $\mu \bar{P}_{j+l} \leq \bar{P}_j \sum_{i=l}^{\infty} \bar{P}_i$, then

$$\mu_H \bar{H}(t+x) \leq \frac{1}{\lambda} e^{-\lambda(t+x)} \sum_{j=0}^{\infty} \frac{(\lambda t)^j}{j!} \sum_{l=0}^{\infty} \frac{(\lambda x)^l}{l!} \bar{P}_j \sum_{i=l}^{\infty} \bar{P}_i$$

$$\begin{aligned}
&= \frac{1}{\lambda}e^{-\lambda(t+x)}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\bar{P}_j\sum_{l=0}^{\infty}\frac{(\lambda x)^l}{l!}\sum_{i=l}^{\infty}\bar{P}_i \\
&= \frac{1}{\lambda}e^{-\lambda(t+x)}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\bar{P}_j\sum_{i=0}^{\infty}\bar{P}_i\sum_{l=0}^{i}\frac{(\lambda x)^l}{l!} \\
&= e^{-\lambda t}\sum_{j=0}^{\infty}\frac{(\lambda t)^j}{j!}\bar{P}_j\sum_{i=0}^{\infty}\bar{P}_i(\frac{1}{\lambda}\sum_{l=0}^{i}\frac{(\lambda x)^l}{l!}e^{-\lambda x}) \\
&= \sum_{j=0}^{\infty}e^{-\lambda t}\frac{(\lambda t)^j}{j!}\bar{P}_j\int_x^{\infty}\sum_{i=0}^{\infty}\frac{(\lambda u)^i}{i!}e^{-\lambda u}\bar{P}_i du.\square
\end{aligned}$$

By reversing all inequalities we get a dual theorem in the NRWU case.

Next, let the shocks in (1.1) occur according to non-homogeneous Poisson process, with mean value function $\Lambda(t)$ and event rate $\Lambda'(t)$. Let $\bar{P}_k, k = 0, 1, \ldots$, be the probability that the device survives the first k shocks, in this case model (1.1) is reduced to

$$\bar{H}(t) = \sum_{k=0}^{\infty} e^{-\Lambda(t)}\frac{[\Lambda(t)]^k}{k!}\bar{P}_k, \quad t \geq 0. \tag{3.4}$$

Now we investigate the preservation of discrete NRBU (NRWU) property of $\bar{P}_k, k = 0, 1, 2, \ldots$ for $\bar{H}(t)$ under the model (3.4).

Theorem 3.2. *$\bar{P}_k, k = 0, 1, 2, \ldots$ are preserved for $\bar{H}(t)$ under model (3.4) if $\{\bar{P}_k\}_{k=0}^{\infty}$ is discrete NRBU, $\Lambda'(0) \neq 0$ and $\Lambda'(\infty) = \infty$ if*

$$\Lambda'(0)\bar{H}_k(t)\{\sum_{k=0}^{\infty}\bar{H}_k(x)\} \leq \Lambda'(x)\bar{H}(t+x)(\sum_{j=k}^{\infty}\bar{P}_j) \tag{3.5}$$

Proof. The survival function $\bar{P}_k$ has the discrete NRBU properties if $\mu\bar{P}_{l+k} \leq \bar{P}_l\sum_{j=k}^{\infty}\bar{P}_j, l, k = 0, 1, 2, \ldots$ i.e.

$$\bar{P}_l \geq \mu\bar{P}_{l+k}/(\sum_{j=k}^{\infty}\bar{P}_j), \qquad l, k = 0, 1, 2, \ldots$$

Multiplying both sides of the above inequality by kernal $e^{-\Lambda(t)}\frac{\Lambda^l(t)}{l!}$ and taking summation over $l = 0, 1, \ldots$ we get

$$\bar{H}(t) \geq \mu \bar{H}_k(t)/(\sum_{j=k}^{\infty} \bar{P}_j) \quad k > 0, \tag{3.6}$$

where

$$\bar{H}_k(t) = \sum_{l=0}^{\infty} e^{-\Lambda(t)} \frac{\Lambda^l(t)}{l!} \bar{P}_{k+l}, \quad k = 1, 2, \ldots \ .$$

The survival function $\bar{H}(t)$ has continuous NRBU properties if

$$\bar{H}(t) \geq \mu_H \frac{\bar{H}(t+x)}{\int_x^{\infty} \bar{H}(u)du} \ . \tag{3.7}$$

Note that

$$\begin{aligned} \mu_H &= \int_0^{\infty} \bar{H}(u)du \\ &= \sum_{k=0}^{\infty} \bar{P}_k \int_0^{\infty} e^{-\Lambda(u)} \frac{\Lambda^k(u)}{k!} \frac{d\Lambda(u)}{\Lambda'(u)}, \end{aligned}$$

where $du = \dfrac{d\Lambda(u)}{\Lambda'(u)}$.

Applying 2^{nd} mean value theorem for the above equality and taking $\Lambda'(0) \neq 0$ and $\Lambda'(\infty) = \infty$, we get $\mu_H = \frac{\mu}{\Lambda'(0)}$ and (3.7) may be written as follows:

$$\bar{H}(t) \geq \frac{\mu \Lambda'(x) \bar{H}(t+x)}{\Lambda'(0)\{\sum_{k=0}^{\infty} \bar{H}_k(x)\}}; \qquad t, x \geq 0. \tag{3.8}$$

From (3.6) and (3.8) we conclude that $\bar{H}(t)$ satisfies the assumption of NRBU property if

$$\mu \bar{H}_k(t)/(\sum_{j=k}^{\infty} \bar{P}_j) \leq \frac{\mu \Lambda'(x) \bar{H}(t+x)}{\Lambda'(0)\{\sum_{k=0}^{\infty} \bar{H}_k(x)\}}.$$

The proof is complete.

The proof for the NRWU case can be obtained by reversing all the inequalities in the above proof.

4. LAPLACE TRANSFORMS AND GENERATING FUNCTIONS FOR NRBU

Here we establish the necessary and sufficient conditions for the life distribution to have the NRBU properties by using the laplace transforms. These conditions can be used to investigate the closure under convolution.

Now let F be a distribution function such that $F(0-) = 0$ and let $\phi(s) = \int_0^\infty e^{-su} dF(u)$; $s \geq 0$ be the Laplace transform of $F(x)$. Define

$$(4.1) \qquad a_n(s) = \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \left(\frac{1-\phi(s)}{s}\right); \qquad n \geq 0 \ , \ s \geq 0$$

Let $b_{n+1}(s) = s^{n+1} a_n(s)$ for $n \geq 0$ and $b_0(0) = 1$ for $s \geq 0$. The transforms $a_n(s)$ and $b_{n+1}(s)$ have the forms

$$(4.2) \qquad a_n(s) = \frac{1}{n!} \int_0^\infty u^n e^{-su} \bar{F}(u) du$$

$$(4.3) \qquad b_{n+1}(s) = \frac{1}{n!} \int_0^\infty s^{n+1} u^n e^{-su} \bar{F}(u) du, \qquad n \geq 0, \ s \geq 0$$

Venogradov (1973) characterized the IFR property in terms of $b_n(s)$. Block and Savits (1980) obtained similar characterization for the IFRA, DMRL, NBU and NBUE properties. Abouammoh et al. (1988) have characterized the NBAFR property by $b_n(s)$. Abouammoh and Hendi (1990) characterized the NBURFR and NBARFR properties by $b_n(s)$. In the following we establish the corresponding characterization for the new renewal better than used (NRBU).

Theorem 4.1. *Let F be a life distribution with $F(0-) = 0$, then F has the NRBU property if and only if (iff)*

$$(4.4) \qquad \bar{F}(t) \sum_{j=n+1}^{\infty} b_{j+1}(s) \geq \mu c_{n+1}(s,t) \qquad \text{for } n > 0, s > 0$$

where

$$c_{n+1}(s,t) = s \int_0^\infty e^{-su} \frac{(su)^n}{n!} \bar{F}(t+u) du, \qquad t > 0.$$

Proof. Assume that F is NRBU, then by using the form (4.3) for $n > 0$, $s > 0$,

$$\begin{aligned}\sum_{j=n+1}^{\infty} b_{j+1}(s) &= \sum_{j=n+1}^{\infty} s^{j+1} \int_0^{\infty} e^{-su} \frac{(u)^j}{j!} \bar{F}(u) du \\ &= s \int_0^{\infty} \bar{F}(u) \Big(\sum_{j=n+1}^{\infty} \frac{(su)^j}{j!} e^{-su} \Big) du \\ &= s \int_0^{\infty} \bar{F}(u) \Big(\int_0^{u} e^{-sv} \frac{(sv)^n}{n!} dv \Big) du,\end{aligned}$$

i.e.,

$$\sum_{j=n+1}^{\infty} b_{j+1}(s) = s \int_0^{\infty} e^{-sv} \frac{(sv)^n}{n!} \Big(\int_v^{\infty} \bar{F}(u) du \Big) dv.$$

Since F is NRBU i.e. $\mu \bar{F}(t+u) \leq \bar{F}(t) \int_u^{\infty} \bar{F}(v) dv$, then

$$\begin{aligned}\sum_{j=n+1}^{\infty} b_{j+1}(s) &\geq s \int_0^{\infty} e^{-su} \frac{(su)^n}{n!} \mu \frac{\bar{F}(t+u)}{\bar{F}(t)} du \\ &= \frac{\mu s}{\bar{F}(t)} \int_0^{\infty} e^{-su} \frac{(su)^n}{n!} \bar{F}(t+u) du,\end{aligned}$$

i.e.

$$\sum_{j=n+1}^{\infty} b_{j+1}(s) \geq \frac{\mu}{\bar{F}(t)} c_{n+1}(s,t).$$

This completes the proof of necessary part.

To prove that the condition (4.4) is sufficient, note that it may be written in the form

$$\int_0^{\infty} G_n(u) \bar{F}(u) du \geq \frac{\mu}{\bar{F}(t)} c_{n+1}(s,t), \tag{4.5}$$

where

$$G_n(u) = s \sum_{j=n+1}^{\infty} \frac{(su)^j}{j!} e^{-su}.$$

It is obvious that

$$G_n(u) = \int_0^{u} s \frac{(sv)^n}{\Gamma(n+1)} e^{-sv} dv = P\{\sum_{i=1}^{n+1} Y_i \leq u\},$$

where $Y_1, Y_2, \ldots, Y_{n+1}$ are mutually independent and exponential with rate s. Hence $G_n(u)$ represent a gamma distribution function with parameters $(n+1, s)$ and its characteristic function is given by $\phi_{n+1}(w) = (1 - \frac{iw}{s})^{-(n+1)}$.

Letting $\frac{(n+1)}{s} \longrightarrow x$, it can be shown that

$$\lim_{n\to\infty} \phi_{n+1}(w) = \exp(iwx),$$

that is

$$G_n(u) \longrightarrow I_x(u) = \begin{cases} 0 & \text{for } u < x, \\ 1 & \text{for } u \geq x. \end{cases} \tag{4.6}$$

On the other hand, by Lemma (2.3) due to Block and Savits (1980),

$$\lim_{n\to\infty} c_{n+1}(s,t) = \bar{F}(t+x). \tag{4.7}$$

The condition (4.5) implies that

$$\lim_{n\to\infty} \int_0^\infty G_n(u)\bar{F}(u)du \geq \lim_{n\to\infty} \frac{\mu}{\bar{F}(t)} c_{n+1}(s,t).$$

From (4.6) and (4.7), it follows that

$$\int_x^\infty \bar{F}(u)du \geq \frac{\mu}{\bar{F}(t)} \bar{F}(t+x).$$

This complete the proof. □

The corresponding Laplace transforms of the dual NRWU properties are satisfied with inequalities sign reversed.

In the following we translate NRBU (NRWU) properties in terms of generating functions.

Let $p_0 = 1 - \bar{P}_0$ and $p_i = \bar{P}_{i-1} - \bar{P}_i, i = 1, 2, \ldots$ be the probability mass function (pmf) of a nonnegative random variable X. The probability generating function of X is

$$\psi(\theta) = E[\theta^X] = 1 - \sum_{j=0}^{\infty}(1-\theta)\theta^j \bar{P}_j. \tag{4.8}$$

Relation (4.8) may be written in the form

$$\psi(\theta) = 1 - \sum_{j=0}^{\infty} P(Y=j)\bar{P}_j \tag{4.9}$$

where Y has the geometric distribution

$$g_k = P(Y=k) = (1-\theta)\theta^k, \quad k = 0, 1, 2, \ldots \tag{4.10}$$

Let $Y_i, i = 1, 2, \ldots, n$ be iid random variables with common pmf given by (4.10). The variable $V = \sum_{i=1}^{n} Y_i$ has the negative binomial distribution,

$$P(V = n+j) = \binom{n+j-1}{j} \theta^j (1-\theta)^n. \qquad j = 0, 1, \ldots \tag{4.11}$$

Next define

$$B_n(\theta) = \begin{cases} \sum_{j=0}^{\infty} \binom{n+j-1}{j} \theta^j (1-\theta)^n \bar{P}_j, & \text{for } n = 1, 2, \ldots \\ 1 & \text{for } n = 0. \end{cases} \tag{4.12}$$

The form (4.12) has the following interesting physical meaning. Suppose that a device is subjected to two different types of shocks W_1 and W_2 say. At every time unit a shock of type W_1 occurs with probability $1-\theta$. If Y_i denote the number of W_1 shocks between the $(i-1)^{\underline{th}}$ and $i\underline{th}$, $i \in N$ of W_2 shocks, then Y_i has geometric distribution with pmf given by (4.10) and V has a negative binomial distribution given by (4.11). Hence $B_n(\theta), n \in N$, represents the probabilities that the device survives n shocks of type W_1 type, where $\bar{P}_j$ represents the probability that the device survives the first j shocks of type W_2.

Using the form (4.9) Abouammoh and Hendi (1990, 1991) found conditions for discrete life distributions, namely IFR, IFRA, NBU, NBUFR, NBAFR, NBURFR and NBARFR in terms of $B_n(\theta)$. Next, we translate the NRBU (NRWU) properties in terms of $B_n(\theta)$. The proof of the following Theorem is direct and therefore omitted.

Theorem 4.2. *Let $B_n(\theta)$ be given by (4.12), and*

$$C_{n,m}(\theta) = \sum_{l=0}^{\infty} \binom{n+l-1}{l} \theta^l (1-\theta)^n \bar{P}_{l+m}, \qquad l, m = 1, 2, \ldots, .$$

Then $\{\bar{P}_j\}_{j=0}^{\infty}$ is NRBU (NRWU) iff

$$B_n(\theta) \geq (\leq) C_{n,m}(\theta).$$

ACKNOWLEDGEMENT

The authors are grateful to Professor A. M. Abouammoh, King Saud Universtiy, Riyadh, Saudi Arabia for valuable comments and stimulating discussion when preparing this paper. Also the authors are grateful to the referee for his constructive comments which has helped in improving the presentation of the paper.

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Date received April 12, 1994.