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ON CERTAIN INTEGRAL OPERATORS DEFINED ON SOME CLASSES OF UNIVALENT FUNCTIONS

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ABSTRACT. Let A denote the class of analytic functions f defined in the unit disc satisfying the conditions $f(0) = 0 = f'(0) - 1$. Let b be a nonzero complex number and let $S_n(b), K_n(b)$ and $C_n^*(b)$ be the classes defined by virtue of the Ruscheweyh derivative. In this paper we study some properties of $S_n(b)$ and $K_n(b)$. Integral operators $I_\lambda(f)$ are also discussed for these classes.

1. INTRODUCTION

Let S denote the subclass of A consisting of univalent functions in the unit disc U and for a nonzero complex number b, let $C(b), S^*(b)$, and $K(b)$ denote the subclasses of S consisting of convex, starlike and close-to-convex functions of complex order b, respectively [3], [2].

The Hadamard product of two functions $f, g \in A$ is denoted by $f * g$. For $n \in N_0 = \{0, 1, 2, 3, ...\}$, let $D^n f = \frac{z}{(1-z)^{n+1}} * f$, so that

$$D^n f = z(z^{n-1} f)^n / n!$$

For $n \in N_0$, a function $f \in A$ is said to belong to the class S_n, if and only if for $z \in U$

$$\text{Re}\frac{z(D^n f(z))'}{D^n f(z)} > 0.$$

The functions in S_n are starlike and hence univalent [12].

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Definition 1.1. For $n \in N_0$, a function $f \in S$ is said to belong to the class $K_n(b)$ if and only if

$$\operatorname{Re}\left[1+\frac{1}{b}\left\{\frac{(D^n f(z))'}{D^n g(z)}-1\right\}\right]>0$$

for some $g \in S_n$ [2]. Here $K_0(b)=K(b)$.

Definition 1.2. A function $f \in S$ is in $C_n^*(b)$, where $n \in N_0$, if and only if

$$\operatorname{Re}\left[1+\frac{1}{b}\left\{\frac{[z(D^n f(z))']'}{(D^n g(z))'}-1\right\}\right]>0$$

for all $z \in U$ and for some function $g, zg' \in S_n$. From the definition it follows that $f \in C_n^*(b)$ if and only if $zf' \in K_n(b)$ [8].

We now introduce the following class $S_n(b)$ as follows:

For $n \in N_0$, a function $f \in S$ is in $S_n(b)$ if and only if

$$\operatorname{Re}\left[1+\frac{1}{b}\left\{\frac{z(D^n f(z))'}{D^n f(z)}-1\right\}\right]>0, \quad z \in U.$$

We note that $S_0(b)=S(b)$ and $S_1(b)=C(b)$.

The class $S_n=S_n(1)$ was considered earlier by Singh and Singh [12].

Let the operator $I_\lambda : A \to A$ be defined by $f=I_\lambda(F)$, as

$$f(z)=\frac{1}{\lambda}z^{1-\frac{1}{\lambda}}\int_0^z t^{\frac{1}{\lambda}-2}F(t)dt, \tag{1.1}$$

where $0<\lambda \leq 1$.

2. MAIN RESULTS

Theorem 2.1. *Let f be defined as in (1.1), and let $F \in S_n(b)$. Then $f \in S_n(\gamma)$ where γ satisfies the following*

$$|\gamma|=\frac{\lambda+2\lambda|b|+\sqrt{\lambda^2+4\lambda^2|b|^2+4\lambda^2|b|-8\lambda}}{4\lambda}$$

(where $0 < Arg\, b = Arg\, \gamma \leq \pi$*).*

Proof. Let

$$\frac{z(D^n f(z))'}{D^n f(z)} = \gamma p(z) + (1-\gamma) \tag{2.1}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$. We will prove that Re $p(z) > 0$.

From (1.1) it follows that

$$D^n F(z) = (1-\lambda) D^n f(z) + \lambda z (D^n f(z))',$$

hence

$$\frac{z(D^n F(z))'}{D^n F(z)} = \frac{\dfrac{(1-\lambda) z (D^n f(z))'}{D^n f(z)} + \dfrac{\lambda z (z (D^n f(z))')'}{D^n f(z)}}{1 - \lambda + \dfrac{\lambda z (D^n f(z))'}{D^n f(z)}} \tag{2.2}$$

From (2.1) we have

$$\frac{z(z(D^n f(z))')'}{D^n f(z)} = \frac{z(D^n f(z))'}{D^n f(z)} [\gamma p(z) + 1 - \gamma] + \gamma z p'(z).$$

Thus

$$\frac{z(D^n F(z))'}{D^n F(z)} = \gamma p(z) + (1-\gamma) + \frac{\lambda \gamma z p'(z)}{1 - \lambda\gamma + \lambda\gamma p(z)}.$$

or

$$1 + \frac{1}{b}\left(\frac{z(D^n F(z))'}{D^n F(z)} - 1\right) = 1 + \frac{1}{b} \left[\gamma p(z) - \gamma + \frac{\lambda\gamma z p'(z)}{1 - \lambda\gamma + \lambda\gamma p(z)}\right]. \tag{2.3}$$

Now we can form the function $\psi(u,v)$ by taking $u = p(z), v = zp'(z)$, in (2.3) as

$$\psi(u,v) = \frac{b-\gamma}{b} + \frac{\gamma}{b} u + \frac{\lambda\gamma v}{b(1 - \gamma\lambda + \lambda\gamma u)}. \tag{2.4}$$

Taking $D = \mathbb{C}/\{\frac{\gamma\lambda-1}{\lambda\gamma}\} \times \mathbb{C}$, we have

$$\begin{aligned}\operatorname{Re}\psi(iu_2, v_1) &= \frac{r_1 - r_2}{r_1} \\ &+ \frac{r_2}{r_1}\frac{\lambda v_1(1 - \lambda r_2\cos\theta - \lambda r_2 \sin\theta u_2)}{(1 + \lambda^2 r_2^2 + \lambda^2 r_2^2 u_2^2 - 2\lambda r_2\cos\theta - 2\lambda r_2\sin\theta u_2)}\end{aligned}$$

where $r_1 = |b|$ and $r_2 = |\gamma|$.

i.e.

$$\begin{aligned}(2.5)\qquad \operatorname{Re}\psi(iu_2, v_1) &= [(r_1 - r_2)(1 + \lambda^2 r_2^2 - 2\lambda r_2\cos\theta) \\ &+(r_1 - r_2)\lambda^2 r_2^2 u_2^2 - 2\lambda r_2 \sin\theta u_2(r_1 - r_2) \\ &+\lambda r_2 v_1(1 - \lambda r_2\cos\theta) \\ &+\lambda^2 r_2^2 \sin\theta u_2 v_1]/r_1(1 + \lambda^2 r_2^2 + \lambda^2 r_2^2 u_2^2 - 2\lambda r_2\cos\theta - 2\lambda r_2 \sin\theta u_2).\end{aligned}$$

Since $0 < \operatorname{Arg} b \leq \pi$ (2.5) becomes

$$\operatorname{Re}\psi(iu_2, v_1) \leq \frac{A + Bu_2^2}{r_1(1 + \lambda^2 r_2^2 + \lambda^2 r_2^2 u_2^2 - 2\lambda r_2\cos\theta - 2\lambda r_2\sin\theta u_2)}$$

where

$$\begin{aligned} A &= (r_1 - r_2)(1 - \lambda r_2)^2 - \frac{1}{2}\lambda r_2(1 - \lambda r_2) \\ B &= (r_1 - r_2)\lambda^2 r_2^2 - \frac{1}{2}\lambda r_2(1 - \lambda r_2)\end{aligned}$$

and

$$r_1(1 + \lambda^2 r_2^2 + \lambda^2 r_2^2 u_2^2 - 2\lambda r_2\cos\theta - 2\lambda r_2\sin\theta u_2) > 0.$$

Now, $\operatorname{Re}\psi(iu_2, v_1) \leq 0$ if and only if $A \leq 0$ and $B \leq 0$.

From $A \leq 0$ we obtain $r_2 \geq r_2'$, where

$$r_2' = \frac{\lambda + 2 + 2\lambda r_1 - \sqrt{\lambda^2 + 4\lambda^2 r_1 + 4\lambda^2 r_1^2 + 4\lambda - 8\lambda r_1 + 4}}{4\lambda}.$$

From $B \leq 0$ we obtain $r_2 \geq r_2''$

$$r_2 \geq r_2'' = \frac{\lambda + 2\lambda r_1 + \sqrt{\lambda^2 + 4\lambda^2 r_1^2 + 4\lambda^2 r_1 - 8\lambda}}{4\lambda}$$

It can be verified that if $0 < \lambda \le 1$, then

$$r_2'' \ge r_2',$$

By [7] we have Re $p(z) > 0$ which implies that $f \in S_n(\gamma)$.

Theorem 2.2. *Let* $0 < \lambda \le 1$. *Let* f *be given by (1.1), where* $F \in K_n(b)$. *Then* $f \in K_n(\gamma)$ *where* $|b| < |\gamma|$ *and arg* $b = arg\gamma$.

Proof. Let $G \in S_n$ and $g(z)$ be given by

$$g(z) = \frac{1}{\lambda} z^{1-\frac{1}{\lambda}} \int_0^z t^{\frac{1}{\lambda}-2} G(t)dt.$$

We set

$$\frac{z(D^n f(z))'}{D^n g(z)} = \gamma p(z) + (1-\gamma) \tag{2.6}$$

where $p(z) = 1 + \sum_{n=1}^{\infty} C_n z^n$. We need to show that Re $p(z) > 0$ for $z \in U$.

$$\frac{z(D^n F(z))'}{D^n G(z)} = \frac{\dfrac{(1-\lambda)z(D^n f(z))'}{D^n g(z)} + \dfrac{\lambda z[z(D^n f(z))']'}{D^n g(z)}}{1 - \lambda + \dfrac{\lambda z(D^n g(z))'}{D^n g(z)}} \tag{2.7}$$

Since $g \in S_n$ then

$$\frac{z(D^n g(z))'}{D^n g(z)} = H(z) \quad \text{where Re } H(z) > 0. \tag{2.8}$$

From (2.6) we have

$$z(D^n f(z))' = D^n g(z)\gamma p(z) + D^n g(z)(1-\gamma)$$

and

$$\begin{aligned} \frac{z[z(D^n f(z))']'}{D^n g(z)} &= \frac{z(D^n g(z))'\gamma p(z)}{D^n g(z)} \\ &\quad + \gamma z p'(z) + (1-\gamma) z \frac{(D^n g(z))'}{D^n g(z)} \\ &= \frac{z(D^n g(z))'}{D^n g(z)} [\gamma p(z) + (1-\gamma)] + \gamma z p'(z) \\ &= H(z)[\gamma p(z) + 1 - \gamma] + \gamma z p'(z) \end{aligned} \tag{2.9}$$

Using (2.6), (2.8) and (2.9) in (2.7), it follows that

$$(2.10)\qquad 1+\frac{1}{b}\left(\frac{z(D^nF(z))'}{D^nG(z)}-1\right) = \frac{b-\gamma}{b}+\frac{\gamma u}{b} +\frac{\lambda\gamma\nu}{b(\lambda H+1-\lambda)}=\psi(u,\nu)$$

where $u=p(z),\ \nu=zp'(z)$ in (2.10)

$$\operatorname{Re}\psi(iu_2,v_1)=\frac{r_1-r_2}{r_1}+\frac{r_2}{r_1}\frac{\lambda v_1[\lambda h_1+1-\lambda]}{(\lambda h_1+1-\lambda)^2+\lambda^2h_2^2}$$

where $H(z)=h_1(x,y)+ih_2(x,y),\ h_1>0$, and $\gamma=r_2e^{i\theta},\ b=r_1e^{i\theta}$.

By taking $v_1\le -\frac{1}{2}(1+u_2^2)$, we obtain

$$\begin{aligned}\operatorname{Re}\psi(iu_2,v_1) &= \frac{r_1-r_2}{r_1}-\frac{r_2}{r_1}\frac{\lambda(1+u_2^2)[\lambda h_1+1-\lambda]}{2[(\lambda h_1+1-\lambda)^2+\lambda^2h_2^2]}\\ &= \frac{A+Bu_2^2}{2C}\end{aligned}$$

where

$$\begin{aligned}C &= r_1\{[\lambda h_1+1-\lambda]^2+\lambda^2h_2^2\}>0\\ A &= 2(r_1-r_2)[(\lambda h_1+1-\lambda)^2+\lambda^2h_2^2]-r_2\lambda(\lambda h_1+1-\lambda)\\ B &= -\lambda r_2(\lambda h_1+1-\lambda)\end{aligned}$$

We notice that $\operatorname{Re}\psi(iu_2,v_1)\le 0$ if and only if $A\le 0$ and $B\le 0$. From $A\le 0$ we obtain $r_2\ge r_2'$ where

$$r_2'=\frac{2r_1\{(\lambda h_1+1-\lambda)^2+\lambda^2h_2^2\}}{2[(\lambda h_1+1-\lambda)^2+\lambda^2h_2^2]+\lambda(\lambda h_1+1-\lambda)}.$$

Thus $r_1\ge r_2'$ and by [7], we have $\operatorname{Re}p(z)>0$ which implies that $f\in K_n(\gamma)$.

Corollary 2.1. *Let* $0<\lambda\le 1$. *Let* f *be defined by (1.1), where* $F\in C_n^*(b)$. *Then* $f\in C_n^*(\gamma)$ *where* γ *is defined in Theorem (2.2).*

Theorem 2.3. *Let $F \in S_n(b)$ and let*

$$f(z) = (n+1)z^{-n} \int_0^z t^{n-1} F(t)dt \tag{2.11}$$

Then $f \in S_{n+1}(b)$.

Proof. From (2.11), we obtain

$$z(D^{n+1}f(z))' + nD^{n+1}f(z) = (n+1)D^{n+1}F(z) \tag{2.12}$$

and

$$z(D^n f(z))' + nD^n f(z) = (n+1)D^n F(z). \tag{2.13}$$

Using the identity

$$z(D^n F(z))' = (n+1)D^{n+1}F(z) - nD^n F(z) \tag{2.14}$$

in (2.12) and (2.13), we obtain

$$(n+1)D^{n+1}F(z) = (n+2)D^{n+2}f(z) - D^{n+1}f(z) \tag{2.15}$$

and

$$D^n F(z) = D^{n+1}f(z). \tag{2.16}$$

In view of the identity (2.14) and the relations (2.15) and (2.16), $F \in S_n(b)$ yields

$$\mathrm{Re}\left[1 + \frac{1}{b}\left\{\frac{(n+2)D^{n+2}f(z) - (n+1)D^{n+1}f(z)}{D^{n+1}f(z)} - 1\right\}\right] > 0$$

which implies that

$$\mathrm{Re}\left[1 + \frac{1}{b}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}f(z)} - 1\right\}\right] > 0, \quad z \in U.$$

This proves that $f \in S_{n+1}(b)$.

Theorem 2.4. *Let $F \in K_n(b)$ and let f be given by (2.11). Then $f \in K_{n+1}(b)$.*

Proof. Let $g(z) = (n+1)z^{-n}\int_0^z t^{n-1}G(t)dt$ where $G \in S_n$. Then $g \in S_{n+1}$. Since $F \in K_n(b)$, it follows that for $z \in U$,

$$\operatorname{Re}\left[1+\frac{1}{b}\left\{\frac{z(D^nF(z))'}{D^nG(z)}-1\right\}\right] > 0, \quad G \in S_n$$

and thus using (2.16), we conclude that for $z \in U$,

$$\operatorname{Re}\left[1+\frac{1}{b}\left\{\frac{(z(D^{n+1}f(z))'}{D^{n+1}g(z)}-1\right\}\right] > 0.$$

Theorem 2.5. *Let $f \in K_n(b)$ and $\varphi \in C$, then $(\varphi * f) \in K_n(b)$.*

Proof. If $g \in S_n$, then $(\varphi * g) \in S_n$ [9].

$$\begin{aligned}\operatorname{Re}\left[\frac{z[D^n(\varphi * f)(z)]'}{D^n(\varphi * g)(z)}\right] &= \operatorname{Re}\left[\frac{z[\varphi * D^nf]'}{\varphi(z) * D^ng(z)}\right]\\ &= \operatorname{Re}\left[\frac{\varphi(z) * z\dfrac{(D^nf(z))'}{D^ng(z)}\cdot D^ng(z)}{\varphi(z) * D^ng(z)}\right]\end{aligned}$$

By [6] with $F(z) = z\dfrac{(D^nf(z))'}{D^ng(z)}$, $D^ng(z) \in S^*$, we obtain that $D_n(\varphi * f)(z) \in K(b)$ which completes the proof of the theorem.

Theorem 2.6. $K_{n+1}(b) \subset K_n(b)$, *for each $n \in N_0$.*

Proof. Let $f \in K_{n+1}(b)$, then for $z \in U$

$$\operatorname{Re}\left[1+\frac{1}{b}\left\{\frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)}-1\right\}\right] > 0$$

for some $g \in S_{n+1}$.

Define $\omega(z)$ in U such that

$$\frac{z(D^nf(z))'}{D^ng(z)} = \frac{1+(1-2b)\omega(z)}{1+\omega(z)} \tag{2.17}$$

where $\omega(0) = 0$ and $\omega(z) \neq -1$. We show that $|\omega(z)| < 1$.

Since

$$z(D^n f(z))' = (n+1)D^{n+1}f(z) - nD^n f(z) \tag{2.18}$$

we obtain from (2.17)

$$\begin{aligned} z(D^{n+1}f(z))' &= \frac{1}{n+1}\left[z(D^n g(z))' \cdot \frac{1+(1-2b)\omega(z)}{(1+\omega(z))}\right. \\ &\left. + D^n g(z)\left\{\frac{-2bz\omega'}{(1+\omega)^2} + n\frac{1+(1-2b)\omega}{1+\omega}\right\}\right] \end{aligned} \tag{2.19}$$

Now apply (2.18) for the function g and use (2.19) to obtain

$$\begin{aligned} \frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} &= \frac{1+(1-2b)\omega}{1+\omega} \\ &+ \frac{1}{n+1}\frac{D^n g(z)}{D^{n+1}g(z)} \cdot \left[\frac{-2b \cdot z\omega'}{(1+\omega(z))^2}\right] \end{aligned} \tag{2.20}$$

Since $S_{n+1} \subseteq S_n$, this implies that $g \in S_n$ and hence there exists an analytic function $\omega_1(z)$ with $\omega_1(0) = 0$ and $|\omega_1(z)| < 1$, such that

$$\frac{D^{n+1}g(z)}{D^n g(z)} = \frac{1-\omega_1(z)}{1+\omega_1(z)} \tag{2.21}$$

Thus using (2.21) in (2.20), we have

$$\begin{aligned} \frac{z(D^{n+1}f(z))'}{D^{n+1}g(z)} &= \frac{1+(1-2b)\omega(z)}{1+\omega(z)} \\ &+ \frac{1}{n+1} \cdot \left(\frac{1+\omega_1(z)}{1-\omega_1(z)}\right) \cdot \left(\frac{-2bz\omega'(z)}{(1+\omega(z))^2}\right) \end{aligned} \tag{2.22}$$

Suppose now that for $z \in U$

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1, \quad (\omega(z_0) \neq -1).$$

Then it follows from [6], that

$$z_0\omega'(z_0) = K\omega(z_0) \quad \text{where } K \geq 1.$$

Setting $\omega(z_0) = e^{i\theta}$ and $\omega_1(z) = re^{i\varphi}$ in (2.22), we get

$$\mathrm{Re}\left[1 + \frac{1}{b}\left\{\frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}g(z_0)} - 1\right\}\right] = \frac{-4K(1+\cos)(1-r_0^2)}{(n+1)|(1+e^{i\theta})^2|^2|1-r_0e^{i\varphi}|^2}.$$

Hence

$$\mathrm{Re}\left[1 + \frac{1}{b}\left\{\frac{z_0(D^{n+1}f(z_0))'}{D^{n+1}g(z_0)} - 1\right\}\right] < 0$$

where $g \in S_{n+1}$ and $K \geq 1$. This contradicts our hypothesis that $f \in K_{n+1}(b)$. Thus $|\omega(z)| < 1$ and so $f \in K_n(b)$.

Corollary 2.2. *For each* $n \in N_0$, $C^*_{n+1}(b) \subseteq C^*_n(b)$.

Proof.

$$\begin{aligned} f \in C^*_{n+1}(b) &\Leftrightarrow D^{n+1}f \in C^*(b) \\ &\Leftrightarrow z(D^{n+1}f)' \in K(b) \\ &\Leftrightarrow D^{n+1}(zf') \in K(b) \\ &\Leftrightarrow zf' \in K_{n+1}(b) \\ &\Rightarrow zf' \in K_n(b) \\ &\Leftrightarrow D^n(zf') \in K(b) \\ &\Leftrightarrow z(D^nf)' \in K(b) \\ &\Leftrightarrow f \in C^*_n(b) \end{aligned}$$

and this proves the theorem.

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