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PARETO OPTIMUM AND EQUILIBRIUM POINTS OF PRIVATE OWNERSHIP ECONOMIES - A SIMPLER APPROACH WITHOUT FIXED POINT THEOREMS

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ABSTRACT. Simple proofs of two fundamental theorems of mathematical economics have been obtained without fixed point theorems.

0. INTRODUCTION

In articulating his long chapter on economic equilibrium, Pareto [4, p.108], remarked that economic equilibrium can be defined in different ways which come to the same thing in the end. Subsequent researches in the field have shown how accurate he was a century ago. Pareto's special contribution is his description of optimal ophelimity (commonly known as Pareto optimum). Pareto in his book, Manuel d'économie politique ([4], [1909] and [1971, p.261]), describes the economic equilibrium and remarks that

.., the members of a collectivity enjoy maximum ophelimity in a certain position when it is impossible to find a way of moving from that position very slightly in such a manner that the ophelimity enjoyed by each of the individuals of that collectivity increases or decreases. That is to say, any small displacement in departing from that position necessarily has the effect of increasing the ophelimity which certain individuals enjoy, and decreasing that which others enjoy, of being agreeable to some and disagreeable to others.

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An allocation is specified by the consumption levels of each consumer and the input and output levels of each producer. An allocation is said to be Pareto optimal (efficient) if it is not possible to organize the production and distribution so as to increase the utility of one or more individuals without causing loss to the utility of others. Edgeworth [3] and Pareto [4, p.534] considered the relation between competitive equilibrium and optimal allocations by starting from the latter. The concept of Pareto optimum is used nowadays in almost all branches of economics, investments theory, traffic problems, problems of human migration and so on.

Debreu [2] in his book has proved the existence of an equilibrium $((\bar{x}_i), (\bar{y}_j), \bar{p})$ of a private ownership economy (commonly known as Arrow-Debreu model of economy) and has shown that given an equilibrium $((\bar{x}_i), (\bar{y}_j), \bar{p})$ of the economy $\mathcal{E}$, $((\bar{x}_i), (\bar{y}_j))$ is a Pareto optimum if $\bar{x}_i$ is not a satiation point and given a Pareto optimum $((\bar{x}_i), (\bar{y}_j))$ of $\mathcal{E}$, there is a price system $\bar{p}$ such that $((\bar{x}_i), (\bar{y}_j), \bar{p})$ is an equilibrium point of $\mathcal{E}$ provided $\bar{x}_i$ is not a satiation point, where $((\bar{x}_i), (\bar{y}_j))$ is an allocation and $\bar{p}$ is a price system. These two theorems are known as the two fundamental theorems of mathematical economics. Arrow's and Debreu's works on economic equilibria are and will be considered as landmarks in the theory of economic equilibria. In this paper we have obtained the same two results. Our approach is different from Debreu's. We have not used any fixed point theorem while Debreu used Kakutani's fixed point theorem. We have first proved the existence of Pareto optimum while Debreu first proved the equilibrium point. However we should point out that we have used some background results from Debreu. With upper semicontinuous utility and concave utility functions, similar results have been obtained in [5] and [6] in a quite different method.

1. ECONOMY AND RELATED TERMS

An economy $\mathcal{E}$ is defined by: m consumers indexed by $i = 1, 2, \ldots, m$; n producers indexed by $j = 1, 2, \ldots, n$; for each $i = 1, 2, \ldots, m$, a consumption set $(X_i, \prec_i)$ where X_i is a nonempty subset of $\mathbb{R}^l$ and $\preceq_i$ is a preordering, i.e. a reflexive and transitive relation on X_i; for each $j = 1, 2, \ldots, n$, a nonempty set Y_j of $\mathbb{R}^l$, the production set for the producer j, and a *priori* given vector $w \in \mathbb{R}^l$, called the total resources of $\mathcal{E}$.

A state of the economy $\mathcal{E}$ is an $(m+n)$-tubles of points of $\mathbb{R}^l$, which can be represented by a point of $\mathbb{R}^{(m+n)l}$.

Given a state $(x, y) = ((x_i), (y_j))$ of $\mathcal{E}$, the point $x - y = \sum_{i=1}^{m} x_i - \sum_{j=1}^{n} y_j$ is called the net demand and the point $z = \sum_{j=1}^{n} x_i - \sum_{j=1}^{n} y_j - w$ is called the excess demand. Thus every point of the set $Z = X - Y - \{w\}$ represents an excess demand corresponding to a state, where $X = \sum_{i=1}^{m} X_i$ and $Y = \sum_{j=1}^{n} Y_j$. A state $(x, y) = ((x_i), (y_j))$ of $\mathcal{E}$ is called a market equilibrium if $x - y = w$, i.e. if excess demand is 0. A state $(x, y) = ((x_i), (y_j))$ of $\mathcal{E}$ is said to be attainable if $x_i \in X_i$ for each $i = 1, 2, \ldots, m$, $y_j \in Y_j$ for each $j = 1, 2, \ldots, n$ and $x - y = w$. The set of all attainable states of $\mathcal{E}$ is denoted by A.

Lemma 1.1 (A priori bound). Let $\mathcal{E}$ be an economy such that $X = \sum_{i=1}^{m} X_i$ has a lower bound for $\preceq$ (i.e. there is a point $a_i \in \mathbb{R}^l$ such that $a_i \preceq x_i$ for all $x_i \in X_i$ (co-ordinate wise) for each i so that $a = \sum_{i=1}^{m} a_i$ is a lower bound of X for $\preceq$), Y is closed, convex and $Y \cap \Omega = \{0\}$, where Ω is the nonnegative orthant of $\mathbb{R}^l$.

If $Y \cap (-Y) \subset \{0\}$, then A is bounded.

For proof see Debreu [2, p.77].

If $(X_i, \preceq_i)$ is the commodity space and $\preceq_i$ the preference preordering of the i-th consumer where X_i is a subset of $\mathbb{R}^l$, an increasing function $u_i : X_i \to \mathbb{R}$ (i.e. $x_i, x_i' \in X_i$ with $x_i \preceq_i x_i' \Rightarrow u_i(x_i) \leq u_i(x_i')$) is called a utility function for the consumer i. Without any loss of generality we may assume u_i to be nonnegative.

Let us consider the following conditions on $(X_i, \preceq_i)$:

(a) for each $x_i' \in X_i$, the sets $\{x_i \in X_i : x_i \preceq_i x_i')$ and $\{x_i \in X_i : x_i \succeq_i x_i'\}$ are closed subsets of X_i.

If $(X_i, \preceq_i)$ is such that X_i is connected and $\preceq_i$ satisfies (a), then there is a continuous utility function $u_i : X_i \to \mathbb{R}$. For proof of this statement see (Debreu [2, p. 56]).

(b) $(X_i, \preceq_i)$ is said to satisfy the convexity condition if for any x_i, x_i' with $x_i \succ_i x_i', tx_i + (1-t)x_i' \succ_i x_i'$ for all $t \in (0,1]$ where $x_i \succ_i x_i'$ means $x_i \succeq_i x_i'$ and not $x_i \neq x_i'$.

This is equivalent to

(b′) For every $x_i' \in X_i$, the set $\{x_i \in X_i : x_i \succ_i x_i'\}$ is convex.

A real valued function f defined on a convex set Y is said to be quasi-concave if for each real number t, the set $\{y \in Y : f(y) > t\}$ is either empty or convex.

Now if we have a continuous quasi-concave function $u_i : X \to \mathbb{R}$, then we can define a preordering $\preceq_{u_i}$ by $x_i, x_i' \in X_i$, $x_i \preceq_{u_i} x_i'$ if and only if $u_i(x_i) \leq u_i(x_i')$. It is easy to see that u_i is increasing with respect to $\preceq_{u_i}$ and more over $\preceq_{u_i}$ satisfies the conditions (a) and (b′). We can also easily verify that given $(X_i, \preceq_i)$, if $u_i : X_i \to \mathbb{R}$ is a nonnegative increasing function with respect to $\preceq_i$, then $\preceq_i$ is equivalent to $\preceq_{u_i}$.

Also it is clear that the utility function u_i represented by the preference ordering $\preceq_i$ satisfying (a) and (b′) is continuous and quasi-concave.

Remark 1.1. In the light of the above discussion it follows that if the economy $((X_i, \preceq_i), (Y_j))$ is given by preferences, we can pass into the economy $((X_i \preceq_{u_i}), (Y_j))$ given by utilities u_i and vice-versa.

Throughout the rest of the paper we will assume that the economy $\mathcal{E}$ is given by $\mathcal{E} = ((X_i, \preceq_i), (Y_j))$ where $\preceq_i$ $(=\preceq_{u_i})$ is the preference preordering and u_i is the utility function of the i-th consumer.

That is, we will assume each preference can be represented by a utility function. The greatest element of X_i respect to $\preceq_i$ $(=\preceq_{u_i})$ is called a satiation assumption.

(c) Insatiability condition: No satiation consumption exists for the i-th consumer.

PRIVATE OWNERSHIP ECONOMIES (THE ARROW-DEBREU MODEL)

A private ownership economy is defined by $\mathcal{E} = ((X_i, \preceq_i), (Y_j), w)$ where for each i, there is a point w_i (resources of the i-th consumer) of $\mathbb{R}^l$ such that $\sum_{i=1}^{m} w_i = w$ and for each pair (i, j), there is a nonnegative number $\Theta_{i,j}$ such that $\sum_{i=1}^{m} \Theta_{i,j} = 1$ for each j. [For economic interpretation of $\Theta_{i,j}$ see Debreu [2, p.79]).

An equilibrium point of the private ownership economy $\mathcal{E}$ is an $(m + n + 1)$-tuble $((\bar{x}_i), (\bar{y}_j), \bar{p})$ of points of $\mathbb{R}^l$ such that:

(i) for each $i = 1, 2, \ldots, m, \bar{x}_i$ is a greatest element of $\{x_i \in X_i : \bar{p} \cdot x_i \preceq \bar{p}w_i + \sum_{j=1}^{n} \Theta_{i,j}(\bar{p} \cdot \bar{y}_j)\}$ with respect to $\preceq_i$;

(ii) for each $j = 1, 2, \ldots, n$, $\bar{y}_j$ maximizes the profit relative to $\bar{p}$ on Y_j; and

(iii) $\bar{x} - \bar{y} = w$ where $\bar{x} = (\bar{x}_i)$ and $\bar{y} = (\bar{y}_j)$ and $w = \sum_{i=1}^{m} w_i$. (For economic interpretation see Debreu [2, p.79]).

PARETO OPTIMUM

Let us consider an economy $\mathcal{E} = ((X_i, \preceq_i), (Y_j), w)$. We now define a preordering $\preceq_a$ on A, the set of all attainable states of $\mathcal{E}$ as follows: given two attainable states or allocations $((x_i), (y_j))$ and $((x'_i), (y'_j))$, $((x_i), (y_j)) \preceq_a ((x'_i), (y'_j))$ if and only if $x_i \preceq_i x'_i$ for each i, i.e. each consumer i desires his consumption x'_i at least as much as his consumption x_i. The preordering $\preceq_a$ may not be complete, i.e. two attainable states may be incomparable. Given two comparable attainable states $((x_i), (y_j))$ and $((x'_i), (y'_j))$, the second one is said to be better than the first one, if $x_i \preceq_i x'_i$ for each i and $x_i \prec x'_i$ for at least one i. In this case we write $((x_i), (y_j)) \prec_a ((x'_i), (y'_j))$.

A Pareto optimum of the economy $\mathcal{E}$ is an attainable state $((x_i), (y_j))$

for which there is no attainable state better than $((x_i),(y_j))$, i.e. there is no attainable state $((x'_i),(y'_j))$ such that $((x_i),(y_j)) \prec_a ((x'_i),(y'_j))$.

PURE EXCHANGE ECONOMY

For simplicity we now consider the pure exchange economy, i.e., the economy where the production sector is inactive, i.e., $Y = \sum_{j=1}^{n} Y_j = \{0\}$.

The model of pure exchange economy $\mathcal{E}$ is described by m consumers indexed by $i = 1, 2, \ldots, m$; the commodity space $\Omega = \{(x^1, x^2, \cdots, x^l) \in \mathbb{R}^l\}$ which is a positive cone of $\mathbb{R}^l$; the space of all state of economy $\Omega^m = \{x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^{lm} : x_i \in \Omega, i = 1, 2, \cdots, m$ and $\sum_{i=1}^{m} x_i = w$ where $w \in \mathbb{R}^l\}$ is the total resources of the economy and for each $i = 1, 2, \cdots, m$, $u_i : \Omega \to \mathbb{R}_+$ is a continuous function called the utility for the i-th consumer. Each element $(x_1, x_2, \cdots, x_m)$ is called an attainable state. As above, given two attainable states (x_i) and (x'_i), $(x_i) \preceq_a (x'_i)$ if and only if $x_i \preceq_i x'_i$, i.e., $u_i(x_i) \leq u_i(x'_i)$ for each $i = 1, 2, \cdots, m$. Pareto optimum of the economy is a maximal element of A with respect to $\preceq_a$. Now let us define the function $f : A = \Omega^m \to \mathbb{R}_+$ by

$$f(x) = f(x_1, x_2, \cdots, x_m) = \sum_{i=1}^{m} u_i(x_i)$$

for each $x = (x_1, \cdots, x_m) \in A$. Thus $f(x)$ measures the total utility or satifaction for the consumers on the state $x \in A$. Clearly f is a continuous function defined on the compact set A and hence $M = \sup_{x \in A} f(x)$ is achieved. Now it is easy to see that each point $\hat{x} \in f^{-1}(M)$ is a Pareto optimum of the pure exchange economy $\mathcal{E}$. Indeed if $\hat{x} \in A$ is not a Pareto optimum, i.e., not maximal element of A with respect to $\preceq_a$, then there must be an element $\bar{x} = (\bar{x}_i) \in A$ such that $\hat{x} \prec_a \bar{x}$, i.e., $u_i(\hat{x}_i) \leq u_i(\bar{x}_i)$ for all i and $u_i(\hat{x}_i) < u_i(\bar{x}_i)$ for at least one i. This means that $M = f(\hat{x}) = \sum_{i=1}^{m} u_i(\hat{x}_i) < \sum_{i=1}^{m} u_i(\bar{x}_i) = f(\bar{x})$ which is a contradiction. Hence $\hat{x}$ is a Pareto optimum.

It is also interesting to note that the total utility or the total satisfaction of all the consumers is the same at each Pareto optimum. The essence of our approach in this paper lies in this simple fact.

We now state the following theorem concerning the existence of Pareto optimum of a private ownership economy.

Theorem 1.1. Let $\mathcal{E} = ((X_i, \preceq_i), (Y_j), (w_i), (\Theta_{i,j}))$ be a private ownership economy such that

(a) for each $i = 1, 2, \cdots, m$

(i) X_i is closed, convex and has a lower bound for $\preceq$ (see Lemma 1.1 for definition);

(ii) for every $x_i' \in X_i$ the sets $\{x_i \in X_i, x_i \preceq_i x_i'\}$ and $\{x_i \in X_i : x_i \preceq_i x_i'\}$ are closed in X_i, i.e., u_i is continuous;

(iii) for each $x_i' \in X_i$, the set $\{x_i \in X_i : x_i \prec_i x_i'\}$ is convex, i.e., u_i is quasi-concave;

(iv) there exists $x_i^0 \in X_i$ such that $x_i^0 \ll w_i$ (i.e. each co-ordinate of x_i^0 is strictly less than the corresponding co-ordinate of w_i);

(b) for each $j = 1, 2, \cdots, n, \quad 0 \in Y_j$;

(c) Y is closed and convex;

(d) $Y \cap (-Y) \subset \{O\}$:

(e) $Y \supset (-\Omega)$, where Ω is the nonnegative orthant of $\mathbb{R}^l$.

Then there exists a Pareto optimum of $\mathcal{E}$.

Proof. Note that the specification of $(\Theta_{i,j})$ is not needed for the concept of Pareto optimality. For a while let us consider the economy $\bar{\mathcal{E}} = ((X_i, \preceq_i), (\overline{coY_j}), (w_i))$, where $\overline{coY_j}$ is the closed convex hull of Y_j. We note that $Y = \sum_{j=1}^n Y_j = \sum_{j=1}^n (\overline{coY_j})$ (see Debreu [2, p.84]). By (d) and (e) we have $Y \cap \Omega = \{O\}$, thus all the conditions of lemma 1.1 are satisfied for $\bar{\mathcal{E}}$. Hence the set $\bar{A}$ of all attainable sets of $\bar{\mathcal{E}}$ is bounded and hence in $\bar{\mathcal{E}}$ the attainable consumption set of each consumer and the attainable production set of each producer are bounded. (Given an economy $\bar{\mathcal{E}}$, a consumption for the i-th consumer (resp. a production for j-th producer) is attainable if it is i-th (resp. j-th) component of some attainable state of $\bar{\mathcal{E}}$, see Debreu [2, 5.3, p.76]). Now we take a sufficiently large closed bounded cube K with centre O having in its interior these m attainable consumption sets and n attainable production sets of the economy $\bar{\mathcal{E}}$.

For each $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ we set $\hat{X}_i = X_i \cap K$ and $\hat{Y}_j = \overline{coY_j} \cap K$. Then we note that for each $i, \hat{X}_i$ is compact and convex and satisfies (a) (ii) and (iii). By virtue of a (iv), (b) and (e), the set $\bar{A}$ of attainable states of $\bar{\mathcal{E}}$ is nonempty and x_i^0 is an attainable consumption for i-th consumer. Also $0 \in \hat{Y}_j$ which is compact and convex. Let us now consider the economy $\hat{\mathcal{E}} = ((\hat{X}_i, \preceq_i), (\hat{Y}_j), (w_i))$. First we note that if $((\bar{x}_i), (\bar{y}_j)) \in \bar{A}$, then $\bar{x}_i \in \hat{X}_i$ and $\bar{y}_j \in \hat{Y}_j$. Hence $((\bar{x}_i), (\bar{y}_j)) \in \hat{A}$, the set of attainable states of the economy $\hat{\mathcal{E}}$. At the end of this proof, we will show the existence of a Pareto optimum of the economy $\hat{\mathcal{E}}$. In here we prove that if $((\bar{x}_i), (\bar{y}_j))$ is a Pareto optimum of $\mathcal{E}$, then for each $j = 1, 2, \cdots, n$, there is a production $\hat{y}_j$ such that $\sum_{j=1}^n \bar{y}_j = \sum_{j=1}^n \hat{y}_j$ and $((\bar{x}_i), (\hat{y}_j))$ is a Pareto optimum of the economy $\mathcal{E}$.

Since $((\bar{x}_i), (\bar{y}_j)) \in \hat{A}$, $\bar{x}_i \in \hat{X}_i \subset X_i \cap K$ for each i and $\bar{y}_j \in \hat{Y}_j \subset \overline{coY_j} \cap K$ for each j. Let $y \in \sum_{j=1}^n \overline{coY_j} = Y = \sum_{j=1}^n Y_j$ (by an earlier note). Hence for each j, there is a production $\hat{y}_j$ such that $y = \sum_{j=1}^n \bar{y}_j = \sum_{j=1}^n \hat{y}_j$. Thus $((\bar{x}_i), (\hat{y}_j)) \in A$. If possible, let $((\bar{x}_i), (\hat{y}_j))$ be not a Pareto optimum of $\mathcal{E}$. Then there is a state $((x_i'), (y_j')) \in A$ which is better than $((\bar{x}_i), (\hat{y}_j))$ i.e. $x_i' \succ_i \bar{x}_i$ for some $i = 1, 2, \cdots, m$. Let $x_i(t) = (1-t)\bar{x}_i + tx_i'$ and $y_j(t) = (1-t)\hat{y}_j + ty_j'$ for each $t \in (0, 1)$. Since A is convex, it is easy to verify that $((x_i(t)), (y_j(t)) \in A$ for each $t \in (0, 1)$. By (a) (iii) $x_i(t) \succ_i \bar{x}_i$. Now as t is close enough to 0, $x_i(t)$ would be in the interior of K because $\bar{x}_i$ is in the interior of K. Hence $((x_i(t)), (y_j(t)))$ would belong to $\hat{A}$ and $x_i(t) \succ_i \bar{x}_i$ which would contradict the maximality of $((\bar{x}_i), (\bar{y}_j))$. Thus we have proved that $((\bar{x}_i), (\hat{y}_j))$ is a Pareto optimum of the economy $\mathcal{E}$.

Hence it remains to prove that the economy $\hat{\mathcal{E}}$ has a Pareto optimum. To this end we define a function $f : \hat{A} \to \mathbb{R}$ by

$$f(x, y) = \sum_{i=1}^m u_i(x_i)$$

for $(x,y) = ((x_i),(y_i)) \in \hat{A}$. Since for each $i = 1,2,\cdots,n$, u_i is continuous, f is a continuous function in the compact set $\hat{A}$. Hence $M = \sup_{(x,y)\in\hat{A}} f(x,y)$ is achieved. We claim that each point in $f^{-1}(M)$ is a Pareto optimum of $\hat{\mathcal{E}}$, i.e., is a maximal element with respect to $\preceq_a$ in $\hat{A}$. Let $(\hat{x},\hat{y}) \in f^{-1}(M)$, i.e., $f(\hat{x},\hat{y}) = \sum_{i=1}^{m} u_i(\hat{x}_i)$, where $(\hat{x}) = (\hat{x}_i)$, $\hat{y} = (\hat{y}_j)$. If $(\hat{x},\hat{y})$ is not a maximal element in $\hat{A}$, then there is an element $(x^0,y^0) = ((x_i^0),(y_i^0))$ such that $(\hat{x},\hat{y}) \prec_a (x^0,y^0)$, i.e., $u_i(\hat{x}_i) \preceq u(x_i^0)$ for all i and $u_i(\hat{x}_i) \prec u_i(x_i^0)$ for some i. This will mean $M = f(\hat{x},\hat{y}) < f(x^0,y^0)$ which is impossible. Thus $(\hat{x},\hat{y})$ is a Pareto optimum. This completes the proof of the theorem. □

Remark 1.2. For economic interpretations of the condition (d) and (e), we refer to Debreu [2].

Theorem 1.2. *Let $\mathcal{E} = ((X_i,\preceq_i),(Y_j),(W_i),(\Theta_{i,j}))$ be a private ownership economy satisfying the conditions of Theorem 1.1. If there is no satiation consumption in x_i, then there exists an equilibrium of the economy.*

(a) for each $i = 1,2,\cdots,m$

(i) X_i is closed convex and has a lower bound for $\preceq$ (see Lemma 1.1.);

(ii) there is no satiation consumption in X_i;

(iii) for each x_i' in X_i, the set $\{x_i \in X_i, x_i \succeq_i x_i'\}$ and $\{x_i \in X_i : x_i \preceq_i x_i'\}$ are closed, i.e., u_i is continuous;

(iv) for each $x_i' \in X_i$, the set $\{x_i \in X_i : x_i \prec_i x_i\}$ is convex, i.e., u_i is quasi-concave;

(v) there exists $x_i^0 \in X_i$ such that $x_i^0 \ll w_i$, i.e., each coordinate of x_i^0 is strictly less than the corresponding co-ordinate of w_i; and

(b) for each $j = 1,2,\cdots,n$, $0 \in Y_j$;

(c) Y is closed and convex;

(d) $Y \cap (-Y) \subset \{O\}$;

(e) $Y \supset (-\Omega)$, where Ω is the nonnegative orthant of $\mathbb{R}^l$. Then there exists an equilibrium of the economy.

Proof. By Theorem 1.1, there is a Pareto optimum $((x_i^*), (y_j^*))$; since by hypothesis, there is no satiation consumption in x_i, x_i^* is not a satiation consumption. Hence by Theorem 6.4 of Debreu [2, p.59] (the proof of which does not involve any fixed point theorem), there is a price system $p \neq 0$ such that $((x_i^*), (y_j^*), p)$ is an equilibrium point of the economy.

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