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EXISTENCE OF TRAVELLING WAVES FOR A MODEL FROM EPIDEMIOLOGY

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ABSTRACT. This work is concerned with the existence of travelling wave solutions to a system of reaction-diffusion issued from epidemiology. The model considered describes the propagation of an epidemic within a population consisting of infectives and susceptibles. Via a topological argument, we obtain existence of travelling waves for any removal rate and any positive wave speed. The proof involves a new technique for computing the Leray-Schauder topological degree. When the removal rate is zero, we then prove an existence result by use of a shooting method. We consider a general class of interaction term extending the classical Kermack-McKendric model.

1. INTRODUCTION

We consider the propagation of an infectious disease within a population consisting only of two interacting classes, one of infectives and the other of susceptibles. Let $I(x,t)$ and $S(x,t)$ denote respectively the spatial densities of infectives and susceptibles in the position x and at the time t. They are solutions of the following reaction-diffusion system for $(x,t) \in \mathbb{R}\times]0,+\infty[$:

$$(1.1) \qquad \begin{aligned} &\tfrac{\partial I}{\partial t} - a\tfrac{\partial^2 I}{\partial x^2} + \lambda I = ISh(S) \\ &\tfrac{\partial S}{\partial t} - b\tfrac{\partial^2 S}{\partial x^2} = -ISh(S) \end{aligned}$$

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with the initial conditions

$$I(x,0) = I_0(x); \quad S(x,0) = S_0(x). \tag{1.2}$$

The constants a and b refer to the diffusion coefficients (see e.g. [3] and [12] for the biological background).

In this work, we are seeking propagating fronts moving with a speed c, viz. particular solutions of the form

$$I(x,t) = u(x+ct) \text{ and } S(x,t) = v(x+ct). \tag{1.3}$$

Choosing $c > 0$ amounts to assuming a wave moving from right to left. For a survey of more general properties of travelling waves, we refer the reader to Aronson and Weinberger (see [1], [2]). Here, we will assume $a = 0$ and $b > 0$; the other extreme case $b = 0$ and $a > 0$ has been studied by Källen in [9]. Substituting (1.3) into (1.1), we obtain, for the unknowns u and v, the following system of differential equations with respect to the new variable $\xi = x + ct \in \mathbb{R}$:

$$cu' + \lambda u = uvh(v) \tag{1.4a}$$

$$bv'' - cv' = uvh(v) \tag{1.4b}$$

The constant λ stands for the removal rate; it is related to the mortality rate of infectives; the reproduction rate of the infection $\frac{1}{\lambda}$ is the number of secondary infections produced by one primary infective in a susceptible population (see [3], [12]). The particular case $h \equiv h_0 > 0$ corresponds to the Kermack-McKendric model (see [10]); here the transmissibility coefficient h_0 represents the average area of infection swept out by an infective per unit time while $h_0 v$ is the number of new infectives produced per unit time, per infective (see [13]). Sedentary infectives mean that no member can be removed by recovery, death, immunization or by any other means; as an example, consider the spatial spread of an epidemic wave, such as plague or rabies, into a uniform population whose members are sensitive to catch the disease.

Although $\lambda = 0$ is of minor epidemiological interest, for the sake of completeness, we shall study the cases $\lambda > 0$ and $\lambda = 0$ separately.

The case where $h_0 \equiv 1$ and $\lambda > 0$ has been investigated by Hosono and Ilyas in [HI]. For any $b > 0$, and $0 < \lambda < 1$, they showed the existence of a solution $(u, v) \in [C^1(\mathbb{R})]^2$ satisfying the boundary conditions (2.1), (2.3).

One can notice that, from Eq. (1.4a), u may be expressed in terms of v; inserting into (1.4b) leads to a nonlinear differential equation for the unknown v. Studying directly the resulting equation is not easier at all; so, throughout this paper, we confine ourselves to the whole system (1.4).

In this work, we suppose the nonlinear function h: $\mathbb{R} \ni \mathrm{s} \hookrightarrow \mathrm{h(s)} \in \mathbb{R}^+$ to be continuous, locally Lipschitz on $\mathbb{R}$ and strictly positive. Another hypothesis will be assumed to prove existence result for the case $\lambda = 0$. The paper is organized in the following way. After mentioning, in Section 2, some remarks on the boundary conditions, we give, in Section 3, two nonexistence results. In Section 4, using an argument of Leray and Schauder degree type, we prove a general existence theorem when $\lambda > 0$. The case $\lambda = 0$ is discussed in Section 5. A method of shooting type is used to get another existence theorem. For both cases, the wave speed is assumed to be an arbitrary positive constant.

2. REMARKS ON THE B.V.P.

In the sequel, by travelling wave solutions to the evolution problem (1.1)-(1.2) are meant classical solutions to the system of differential equations (1.4); more precisely, we seek a couple of functions (u, v) lying in the positive cone

$$\mathcal{C}^+ = \{(u,v) \in [L^\infty(\mathbb{R}) \times W^{1,\infty}(\mathbb{R})] \cap [C^0(\mathbb{R}) \times C^1(\mathbb{R})];\ u, v \geq 0 \text{ on } \mathbb{R}\}.$$

2.1. The case $\lambda > 0$

Let $(u, v) \in \mathcal{C}^+$ be a solution to system (1.4).

(a) Ahead of the wave, the wave is moving into a state where there are no infectives and after the wave passes, we expect that again there are no infectives; so we should have the boundary conditions:

$$(2.1) \qquad \lim_{x \to -\infty} u(x) = 0; \quad \lim_{x \to +\infty} u(x) = 0.$$

u is then a pulse; rigorously, if we put $\theta(x) := (-cu + bv' - cv)(x)$, then subtracting the two equations in (1.4) and integrating from 0 to x yields

$$\lambda \int_0^x u(t)\,dt = \theta(x) - \theta(0). \tag{2.2}$$

Now, u is a positive function and θ is bounded; hence, the integral in (2.2) converges to a finite limit. From a classical argument, we infer that $\lim_{|x|\to+\infty} u(x)$ exists and is equal to zero. Turning to Eq. (1.4a), we easily conclude that $\lim_{|x|\to+\infty} u'(x) = 0$.

(b) At $+\infty$, it is not possible to know the number of susceptibles, if any, after the passage of the epidemic wave. Therefore, we have to assume $v'(+\infty) = 0$ or

$$\lim_{x\to-\infty} v(x) = \beta; \quad \lim_{x\to+\infty} v(x) = \alpha \tag{2.3}$$

for some positive constants $\alpha < \beta$. In fact, let us write Eq. (1.4b) as

$$(v'(x)e^{-\frac{c}{b}x})' = \frac{uv}{b} h(v) e^{-\frac{c}{b}x}. \tag{2.4}$$

Then, noting that $v \geq 0$ and $v' \in L^\infty(\mathbb{R})$, we integrate Eq. (2.4) from x to $+\infty$ to get $v'(x) \leq 0$, $\forall\, x \in \mathbb{R}$. Moreover, $v'(x) < 0$, $\forall\, x \in \mathbb{R}$; if not, $v'(x_0) = 0$ for some $x_0 \in \mathbb{R}$. Then, integrating (2.4) again from x_0 to $+\infty$ yields $uvh(v)(x) = 0$, for any $x \in [x_0, +\infty[$. Since $h(v)(x) > 0$ and $v'(x) \leq 0$, we then conclude that either $u \equiv 0$ or $v \equiv 0$ on an interval $(x_1, +\infty)$ with some $x_1 \geq x_0$, leading to a contradiction. Then $\lim_{|x|\to+\infty} v(x)$ exists and $\lim_{|x|\to+\infty} v'(x) = \lim_{|x|\to+\infty} v''(x) = 0$ follow from Eq. (1.4b) itself.

2.2. The case $\lambda = 0$

Here, we can check that $u' \geq 0$, $v' \leq 0$ whenever $(u, v) \in \mathcal{C}^+$ is a couple of non-negative solutions. Furthermore, all derivatives of u and v are vanishing at infinity. Therefore, instead of conditions (2.1)-(2.3), we will consider the following ones

$$\lim_{x\to+\infty} u(x) = \beta; \quad \lim_{x\to+\infty} v(x) = 0. \tag{2.5}$$

$$\lim_{x\to-\infty} u(x) = 0; \quad \lim_{x\to-\infty} v(x) = \beta. \tag{2.6}$$

Here β is some positive constant. In fact, subtracting the two equations in system (1.4) and integrating over $\mathbb{R}$ yields $(u+v)(+\infty) = (u+v)(-\infty)$. Since $u(-\infty) = v(+\infty) = 0$ is a consequence of the monotonicity of the solutions, we infer that $u(+\infty) = v(-\infty)$, whence (2.5)-(2.6).

3. NONEXISTENCE RESULTS

Let $0 \leq \alpha < \beta$ be two positive constants. An interesting question is to know, in the case $\lambda > 0$, whether there exists a solution $(u, v) \in \mathcal{C}^+$ to system (1.4) which satisfies the boundary conditions (2.1), (2.3); equivalently, are there any nontrivial heteroclinic orbits $(u, v \geq 0, V \leq 0)$ for the following first order differential system

$$\begin{aligned} u' &= \tfrac{1}{c}(uvh(v) - \lambda u) \\ v' &= \tfrac{1}{b}(V + cv) \\ V' &= uvh(v) \end{aligned}$$

which connect the critical points $(0, \beta, -c\beta)$ and $(0, \alpha, -c\alpha)$ in the $\mathbb{R}^3$ phase space? We aim to prove that, in general, the answer is negative. Setting

$$\underline{\lambda} = \inf_{\alpha \leq s \leq \beta} (sh(s)) \quad \text{and} \quad \overline{\lambda} = \sup_{\alpha \leq s \leq \beta} (sh(s)),$$

we first have

Proposition 3.1. *Let* $c,\ \lambda > 0$ *and* $(u, v) \in \mathcal{C}^+$ *be a solution to problem* (1.4), (2.1), (2.3). *Then*

$$\lambda \in \,]\underline{\lambda}, \overline{\lambda}[. \tag{3.1}$$

Proof. An integration of Eq. (1.4a) over $\mathbb{R}$, using (2.1), yields

$$\int_{-\infty}^{+\infty} u(x)[v(x)h(v(x)) - \lambda]\, dx = 0.$$

Since u is positive, there is some $x_0 \in \mathbb{R}$ such that $v(x_0)h(v(x_0)) = \lambda$; it follows that $\lambda \in [\underline{\lambda}, \bar{\lambda}]$. If $\lambda = \bar{\lambda}$, Eq. (1.4a) implies $u' \geq 0$ on $\mathbb{R}$ which contradicts $u(+\infty) = 0$; the same holds for the case $\lambda = \bar{\lambda}$, which proves the proposition.

The following result is less obvious.

Proposition 3.2. *Problem* (1.4), (2.1), (2.3) *has no solution* (u,v) *in* $\mathcal{C}^+$ *whenever*

$$\sup_{\alpha\le s\le\beta} sh(s) = \alpha h(\alpha). \tag{3.2}$$

Proof. Let $(u,v)\in\mathcal{C}^+$ be a nontrivial solution to problem (1.4), (2.1), (2.3) and set

$$\varepsilon(x) := \frac{\alpha h(\alpha) - v(x)h(v(x))}{c} \quad \text{and} \quad \Lambda := \frac{\alpha h(\alpha)-\lambda}{c}.$$

Assume (3.2); then, in view of (3.1), we have $\Lambda > 0$ and $\varepsilon(x)\ge 0$ for any $x\in\mathbb{R}$.

Moreover, Eq. (1.4a) reads

$$u' = \Lambda u - \varepsilon u \qquad \text{for } x\in\mathbb{R}. \tag{3.3}$$

Since $\lim_{x\to+\infty}\varepsilon(x)=0$, we have

$$\forall\,\varepsilon_0>0,\ \exists\,\overline{x}>0:\quad (x\ge\overline{x}\ \Rightarrow\ 0<\varepsilon(x)\le\Lambda\varepsilon_0). \tag{3.4}$$

It is easily seen that the function $U(x)=e^{\Lambda x}\int_x^{+\infty}u(t)\varepsilon(t)e^{-\Lambda t}\,dt$ also satisfies

$$U' = \Lambda U - \varepsilon u, \quad x\in\mathbb{R}. \tag{3.5}$$

Subtracting (3.5) from (3.3) and integrating over $(0,x)$, one gets

$$u(x) = U(x) + [u(0)-U(0)]e^{\Lambda x}. \tag{3.6}$$

In addition, the following upper bound is straightforward

$$0\le U(x)\le \frac{1}{\Lambda}\sup_{t\ge x}u(t).\sup_{t\ge x}\varepsilon(t); \tag{3.7}$$

hence $U\in L^\infty(\mathbb{R})$. (3.6) then shows that u and U are identical because Λ is positive. Back to (3.7), using (3.4), we deduce finally that

$$0\le\sup_{x\ge\bar{x}}u(x)\le\varepsilon_0\sup_{x\ge\bar{x}}u(x).$$

Choosing $0 < \varepsilon_0 < 1$ leads to a contradiction unless $u \equiv 0$ over $\mathbb{R}$.

Remark 3.3

(a) It is easy to see that if $c = 0$, then $(u, v) = (0, k)$ is the sole solution to system (1.4). In addition, Proposition 3.1 shows that the case $\lambda' \geq \overline{\lambda}$ does not lead to the existence of any travelling wave solution. Indeed, if either the infectives die too rapidly or if the transmissibility factor is too small, the disease can neither spread nor be transmitted. This phenomenon is well known in the biological context (see [3], [12] and [13]).

(b) Proposition 3.2 eliminates the case in which the function $s \hookrightarrow \mathrm{sh(s)}$ decreases. Now, assume for instance that there exists some $\zeta_0 \in \mathbb{R}$ such that this function is only decreasing over $(\zeta_0, +\infty)$ and let $(u, v) \in \mathcal{C}^+$ be a solution to system (1.4), then Proposition 3.2 implies that $\alpha := v(+\infty) < \zeta_0$, yielding an estimate of the final density of susceptibles. The following result shows the existence of a unique maximal density of infectives:

Proposition 3.4. *For any $\lambda > 0$, there is a unique $x_m \in \mathbb{R}$ satisfying $u'(x) \geq 0$, $\forall x \leq x_m$ and $u'(x) \leq 0$, $\forall x \geq x_m$.*

Proof. The function $x \hookrightarrow v(x)h(v(x))$ is decreasing; then, owing to Proposition 3.1, for any $\lambda \in]\underline{\lambda}, \overline{\lambda}[$, there exists some unique $x_m \in \mathbb{R}$ such that $v(x_m)h(v(x_m)) = \lambda$, $v(x)h(v(x)) > \lambda$, $\forall x < x_m$ and $v(x)h(v(x)) < \lambda$, $\forall x > x_m$. From Eq. (1.4a), we infer that

$$cu' = u[vh(v) - \lambda] \begin{cases} > 0, \ \forall x \in]-\infty, x_m[\\ < 0, \ \forall x \in]x_m, +\infty \end{cases}$$

By continuity, $u'(x_m) = 0$ and the proposition follows.

4. EXISTENCE RESULT FOR THE CASE $\lambda > 0$

The main result of this section is

Theorem 4.1. *For any $c, \lambda > 0$, there are some $\beta > \alpha \geq 0$ such that system* (1.4) *has a solution (u, v) satisfying* (2.1), (2.3).

Before starting the proof, it may be useful to outline the method which is involved. Given $a > 0$, we primarily study system (1.4) on the open interval $I_a =]-a, +a[$ under suitable boundary conditions. Using a topological degree argument, we prove the existence of a solution (u_a, v_a) provided a is large enough. To this end, we deal with a family of problems involving some parameter $t \in [0,1]$; afterwards, we define and compute a Leray-Schauder topological degree; this is the crucial point of our analysis. Appropriate apriori estimates upon the solutions will allow us to pass to the limit when $a \to \infty$ to obtain the sought solution. For the general setting, we refer to [4] where a system from combustion theory is studied; as for definitions and general properties of the topological degree, there is an extensive literature (see, for instance, [11]).

4.1. The general framework

Let $c, \gamma, a > 0$ be fixed and take t as a parameter moving in $[0,1]$. Now, consider the Banach space $E = C^0(\overline{I}_a) \times C^1(\overline{I}_a) \times \mathbb{R}$ endowed with the norm

$$\|(u,v,\mu)\|_E = \max\left(\|u\|_{C^0(\bar{I}_a)}, \|v\|_{C^1(\bar{I}_a)}, |\mu|\right).$$

Let us introduce two new positive constants γ and β' satisfying

$$0 < \gamma < \frac{\beta'}{2}; \tag{4.1}$$

then set

$$h^* = \sup_{0 \le s \le \beta'} h(s). \tag{4.2}$$

Finally, define a mapping K_t: $E \longrightarrow E$ which maps each element $(u,v,\mu) \in E$ onto the element $(U, V, u(-a))$ where (U,V) is the unique solution to the following linear problem

$$\begin{aligned} &cU' = tuvh(v) + (1-t)\mathrm{sgn}(u)\mathrm{sgn}(v)f - \lambda u \\ &bV'' - cV' = tuvh(v) + (1-t)\mathrm{sgn}(u)\mathrm{sgn}(v)f. \end{aligned} \tag{4.3}$$

$$\begin{aligned} &bV'(-a) - cV(-a) = c(\mu - \beta') \\ &V(a) = 0. \end{aligned} \tag{4.4}$$

$$U(0) = \gamma. \tag{4.5}$$

Here sgn denotes the sign function

$$\mathrm{sgn} u(x) = \begin{cases} +1, & u(x) > 0 \\ 0, & u(x) = 0 \\ -1, & u(x) < 0 \end{cases}$$

while the function f is defined by

$$f(x) = \begin{cases} \rho e^{\delta x} & \text{if } x \in \mathbb{R}^- \\ \rho e^{-\delta x} & \text{if } x \in \mathbb{R}^+ \end{cases} \tag{4.6}$$

with some constants $\rho, \delta > 0$ to be selected later on. Our aim is to find solutions u, v lying in the strip $\{-a < x < +a,\ 0 < y < \beta'\}$; in particular $0 < \mu < \beta'$. So, in the sequel, the problem of seeking fixed points to the mapping K_t with $0 \le \mu \le \beta'$ will be referred to simply as $(\mathcal{P}_t)$.

Remark 4.2

(a) It is meaningful to note that, with the first order differential equation in u, two conditions have been prescribed for problem $(\mathcal{P}_t)$. The fact that μ is seen as an unknown to be determined as part of the solution justifies this procedure.

(b) The existence and the uniqueness of a triplet (u, v, μ) $(0 < \mu < \beta')$ to the mapping K_0 implies the existence of a non-vanishing topological degree (see [L]); consequently, at least one fixed point for K_1 will be inferred.

(c) The parameterizations introduced to define the mapping K_t together with the auxiliary function f will have to play a key role in the computation of the degree. In particular, the trick which consists in inserting the sign function is aimed mainly to insure the positivity of the sought solutions.

(d) It should be emphasized that problem $(\mathcal{P}_t)$ does not give rise to trivial solutions on $(-a, +a)$. In fact, $(u, v) = \left(0, \beta'\left(1 - e^{c(x-a)}\right)\right)$ is ruled out since $\gamma > 0$. However, $v \equiv 0$ implies $u(x) = \gamma e^{-\frac{\lambda}{c}x}$; now, if a is greater than $\frac{c}{\lambda}\ln(\frac{\beta'}{\gamma})$, condition $\mu = u(-a) = \beta'$ is violated too. Moreover, boundary conditions (4.4) and (4.5) are prescribed so as not to obtain trivial solutions when the limit $a = +\infty$ is achieved.

4.2. Apriori estimates

Lemma 4.3. *Let (u, v, μ) be a solution to problem $(\mathcal{P}_t)$. Then, the following estimates which are independent of t and a hold true on the interval I_a :*

$$\text{(a)} \quad u > 0. \qquad \text{(b)} \quad v > 0 \text{ and } -\tfrac{c\beta'}{b} < v' < 0.$$
$$\text{(c)} \quad 0 < u + v < \beta', \text{ in particular } 0 < u < \beta' \text{ and } \mu \in]0, \beta'[.$$
$$\text{(d)} \quad -\tfrac{\lambda\beta'}{c} < u' < \tfrac{\beta'^2 h^* + \rho}{c} \text{ and } -\tfrac{\beta' c^2}{b^2} < v'' < \tfrac{\beta'^2 h^* + \rho}{b},$$

h^* being defined by (4.2).

Proof. **(a)** Assume $u(x_0) = 0$ for some $x_0 \in I_a$. By virtue of the uniqueness to the Cauchy problem, u is identically zero contradicting (4.5); since $u(0) > 0$, $u > 0$ on I_a.

(b) From the boundary conditions (4.4) both with $0 \leq \mu \leq \beta'$, we have $bv'(-a) \leq cv(-a)$; if v takes some negative value, there would exist $x_0 \in I_a$ such that $v(x_0) < 0$, $v'(x_0) = 0$ and $v''(x_0) \geq 0$. A contradiction is then immediately reached from the equation in v. Hence, $v'(a) < 0$ and integrating the inequality $(v'(x)e^{\frac{c}{b}x})' > 0$ from x to a implies $v'(x) < 0$, for any $x \in I_a$. Likewise, the function $bv' - cv$ is monotone increasing so that $bv' - cv > c(\mu - \beta') > -c\beta'$. The second part of the lemma is proved.

(c) Subtracting equation in v from equation in u and making use of (4.4), we get, after integration, $bv' - c(u + v) > -c\beta'$; whence, with **(b)**, $u + v < \beta'$.

(d) The desirable estimates are read off from problem $(\mathcal{P}_t)$ both with parts **(a)**-**(c)** above.

Corollary 4.4. *For any $c, \lambda > 0$ and any $t \in [0, 1]$, there exists an open set $\Omega \subset E$ such that the Leray-Schauder topological degree $\deg(I - K_t, \Omega, 0)$ is well defined; here I stands for the identity operator on E.*

Proof. Let (u, v, μ) be a solution to problem $(\mathcal{P}_t)$. From the previous lemma together with the continuous embedding $W^{r,\infty}(I_a) \hookrightarrow C^{r-1}(\overline{I}_a)$ $(r = 1, 2)$, there exists some constant $M > 0$ such that

$$\|(u,v)\|_{C^0(\bar{I}_a)\times C^1(\bar{I}_a)} \leq M.$$

Now, consider the open subset

$$\Omega = \left\{(u,v,\mu) \in E; 0 < u < \beta', 0 < v < \beta', -\frac{c\beta'}{b} < v' < 0, 0 < \mu < \beta'\right\}.$$

Then, for any $(c,\lambda,t) \in (\mathbb{R}_*^+)^2 \times [0,1]$, $(u,v,\mu) \in \Omega$. In addition, by the compactness of the embedding $C^r(\bar{I}_a) \hookrightarrow C^{r-1}(\bar{I}_a)$ $(r = 1,2)$, the restriction of the mapping K_t to Ω is compact and uniformly continuous in terms of the parameter t; the proof of Corollary 4.4 follows.

4.3. Computation of the degree

For the particular case $t = 0$, Problem $(\mathcal{P}_0)$ is read:

Find a unique triplet $(u,v,\mu) \in \Omega$ solution to the problem

$$\begin{aligned} cu' &= \mathrm{sgn}(u)\mathrm{sgn}(v)f - \lambda u \\ bv'' - cv' &= \mathrm{sgn}(u)\mathrm{sgn}(v)f. \end{aligned} \tag{4.7}$$

$$u(0) = \gamma. \tag{4.8}$$

$$u(-a) = \mu. \tag{4.9}$$

$$bv'(-a) - cv(-a) = c(\mu - \beta'); \quad v(a) = 0. \tag{4.10}$$

For this purpose, it is equivalent to seek a couple of positive solutions u, v satisfying, for $x \in I_a$,

$$cu' + \lambda u = f \tag{4.11a}$$

$$bv'' - cv' = f \tag{4.11b}$$

with the required boundary conditions (4.8), (4.10). Uniqueness of such a solution is obvious. It remains to choose a convenient function f, that is positive constants ρ, δ in (4.6) ensuring the positivity of u and v; to this end, we have

Proposition 4.5. *There exists* $(\rho,\delta)\in(\mathbb{R}_*^+)^2$ *such that problem* (4.8)-(4.11) *admits a unique solution* $(u,v,\mu)\in E$, *with* u,v *positive.*

Proof. A straightforward computation in (4.11a), using (4.6), yields

$$u(x)=\begin{cases}\frac{\rho}{\lambda+c\delta}e^{\delta x}+(\gamma-\frac{\rho}{\lambda+c\delta})e^{-\frac{\lambda}{c}x} & \text{if } x\in(-a,0)\\ \frac{\rho}{\lambda-c\delta}e^{-\delta x}+(\gamma-\frac{\rho}{\lambda-c\delta})e^{-\frac{\lambda}{c}x} & \text{if } x\in(0,+a)\end{cases}$$

whenever

$$\delta\neq\frac{\lambda}{c}\cdot \tag{4.12}$$

Note that u never traverses the half-positive axis; otherwise, $u'(x_0)<0$ and $u(x_0)=0$ for some $x_0>0$. But, $cu'(x_0)=f(x_0)>0$. Our supposition is thus ruled out. Therefore, taking $\gamma=\frac{\rho}{\lambda+c\delta}$, that is

$$\rho=\gamma(\lambda+c\delta) \tag{4.13}$$

makes the solution u positive on the interval I_a.

To deal with the solution v, let us notice that the latter is the required solution if and only if $v'(a)<0$ (note that, since $v(a)=0$, $v>0$ on $I_a \Leftrightarrow v'(a)<0$). Meanwhile, integrating (4.11b) from $-a$ to a, we get, in the light of (4.6) and (4.10)

$$bv'(a)=c\left(\mu-\beta'\right)+\frac{2\rho}{\delta}\left(1-e^{-a\delta}\right),$$

that is, after computing $\mu=u(-a)$ and using (4.13),

$$bv'(a)=\left(\frac{2\rho}{\delta}-c\beta'\right)-\frac{\rho(c\delta+2\lambda)}{\delta(c\delta+\lambda)}e^{-\delta a}.$$

The needed condition $v'(a)<0$ is thus obtained if

$$\rho<\frac{c\beta'\delta}{2}\cdot \tag{4.14}$$

Given (4.1), the set S of admissible values (δ,ρ), such that both (4.13) and (4.14) are fulfilled, is as depicted in Figure 1. If $\frac{\beta'}{4}<\gamma<\frac{\beta'}{2}$, this set identifies the hatched half-line; when $0 < \gamma \leq \frac{\beta'}{4}$, one must exclude the value $\delta_0 := \frac{\lambda}{c}$ so that (4.12) is, in both cases, accomplished.

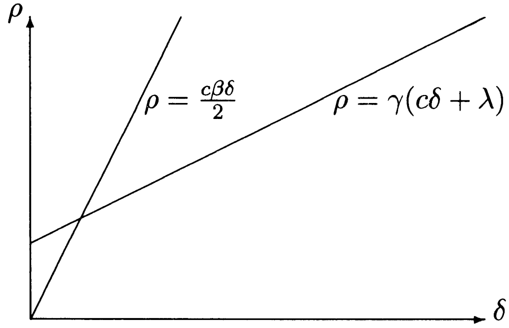

Fig. 1. Admissible set of values (δ, ρ) which guarantee the positivity of u and v.

At last, note that $0 < \mu = u(-a) = \frac{\rho}{\lambda + c\delta} e^{-a\delta} = \gamma e^{-a\delta} < \gamma < \beta'$. The proof of the proposition is by now complete.

4.4 The limit case $a = +\infty$

From Proposition 4.5 both with Remark 4.2, **(b)**, a fixed point (u_a, v_a, μ_a) for the mapping K_1 is obtained. One can extend u_a and v_a to the full real line by setting

$$\tilde{u}_a(x) = \begin{cases} u_a(a), & x \geq a \\ u_a(-a), & x \leq -a \end{cases} \quad \text{and} \quad \tilde{v}_a(x) = \begin{cases} 0, & x \geq a \\ v_a(-a), & x \leq a. \end{cases}$$

Lemma 4.3 provides estimates for $(\tilde{u}_a, \tilde{v}_a)$ which are independent of the parameter $a > 0$ in the space $H^2_{loc}(\mathbb{R}) \times H^1_{loc}(\mathbb{R})$ which is compactly embedded in $X := C^1_{loc}(\mathbb{R}) \times C^0_{loc}(\mathbb{R})$; then, there exists an increasing sequence $\{a_n\}_{n=1}^{\infty}$, such that $(\tilde{u}_{a_n}, \tilde{v}_{a_n})$ converges strongly in X, as $n \to \infty$, to a solution (u, v) to problem (1.4), (2.1); furtheron, (2.3) is satisfied for some α, β with $0 \leq \alpha < \beta \leq \beta'$. This proves the claim of Theorem 4.1.

5. EXISTENCE RESULT FOR THE CASE $\lambda = 0$

5.1. Setting of the problem and outline of the proof

Even if, in the biological context, λ is known to be positive, the case $\lambda = 0$ is, from a mathematical point of view, interesting to be investigated. In this case, system (1.4) reads

$$\begin{aligned} cu' &= uvh(v) \\ bv'' - cv' &= uvh(v) \end{aligned} \tag{5.1}$$

Further to the hypotheses given in Section 1, the nonlinear function h will be assumed of $C^1(\mathbb{R})$ class satisfying

$$\lim_{s \to \pm\infty} |s|h(s) = \ell_{\pm\infty} > 0 \text{ or } +\infty. \tag{5.2}$$

When investigating problem (5.1), (2.5), (2.6), the positivity of the solutions u and v is not a priori known; so the function h need to be defined on the whole real line; assumption (5.2) is then required for technical purpose; recall that the model case corresponds to the case $h(s) \equiv h_0 > 0$. Now, we aim to prove

Theorem 5.1. *There exists some $\beta > 0$, such that for any $c > 0$, problem* (5.1), (2.5), (2.6) *has at least one solution* $(u, v) \in \mathcal{C}^+$.

Let us sketch the proof. We start by dividing the real line into two semi-infinite intervals $\mathbb{R}^+$ and $\mathbb{R}^-$. On $[0, +\infty[$, we define an initial value problem by introducing three shooting parameters as initial conditions. Then, we prove that the solution obtained satisfies the boundary condition (2.6) whenever these parameters are suitably restricted. Subsequently, we show that the solutions can be extended to $x = -\infty$ and verify the required boundary conditions (2.5). This method, known as the shooting method, is by now classical and turns out to be effective for solving a wide class of systems of differential equations especially here because u is increasing and v is decreasing (see [14], [7] and the references therein). The following lemma from classical analysis will be useful for our subsequent reasoning:

Preliminary lemma

(a) *Let $f \in C^1(\mathbb{R}^+)$ be such that $\lim_{x \to +\infty} f(x)$ exists. Then, either f' has no limit as $x \to +\infty$, or $\lim_{x \to +\infty} f'(x) = 0$.*

(b) *Let $f \in C^2(\mathbb{R}^+)$ be such that f, f' and f'' are nonnegative, then $\lim_{x \to +\infty} f(x) = +\infty$.*

5.2. The boundary value problem in $(0, \infty)$

Let γ, η and η' be three positive constants. Consider the solution (u, v) to system (5.1) corresponding to the following initial conditions:

$$u(0) = \gamma,\ v(0) = \eta \ \text{ and } v'(0) = -c\eta'. \tag{5.3}$$

Since h is Lipschitz-continuous, this solution can be continued to the whole real line.

Remark 5.2

(a) The first quadrant is a trap area for the trajectory of u; it means that such a trajectory can never leave this region once it has entered it. Indeed, assume $u(x_0) = 0$ for some $x_0 > 0$. Then, owing to the uniqueness of solutions of initial value problems $u \equiv 0$ on $(0, +\infty)$ which discards the condition $u(0) = \gamma > 0$.

(b) The following conservation statement is straightforwardly derived

$$bv' - c(u + v) = -c(\gamma + \eta + b\eta'), \quad \text{on } \mathbb{R} \tag{5.4}$$

in which $\gamma + \eta + b\eta'$ will play the role of β. Next, let us define the following sets

$$\Gamma^+ = \{(\gamma, \eta, \eta') \in \mathbb{R}^3 :\ \exists\, x_0 > 0 \ \text{ with } \ u(x_0) > \gamma + \eta + b\eta'\}\,.$$
$$\Gamma^- = \{(\gamma, \eta, \eta') \in \mathbb{R}^3 :\ \exists\, \tilde{x}_0 > 0 \ \text{ with } \ u(\tilde{x}_0) < \gamma\}\,.$$

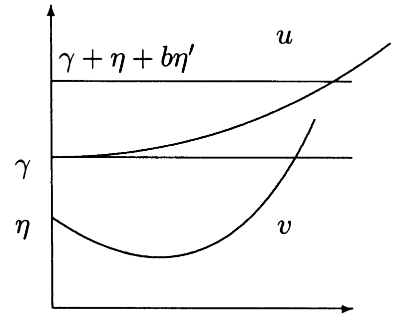

Fig. 2. $(\gamma, \eta, \eta') \in \Gamma^+$.

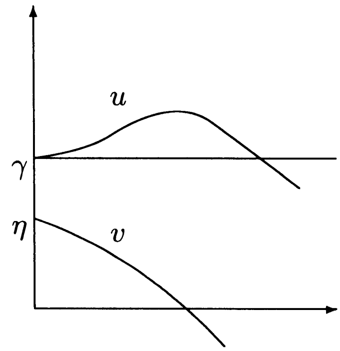

Fig. 3. $(\gamma, \eta, \eta') \in \Gamma^-$.

In what follows, we investigate some properties of Γ^+ and Γ^-. Our goal is to find a triplet of switching parameters (γ, η, η') lying neither in Γ^- nor in Γ^+.

Lemma 5.3. *Γ^+ and Γ^- are disjoint open sets.*

Proof. These sets are open since the solution u of the Cauchy problem depends continuously upon the initial conditions (see [H]); in addition, they are disjoint; otherwise there are some $0 \leq x_1 < x_0 < x_2$ such that one of the following occurs:

• $u'(x_0) = 0$, $u(x) > \gamma + \eta + b\eta'$ for $x \in (x_1, x_2)$ and u is increasing on (x_1, x_0), decreasing on (x_0, x_2).

• $u'(x_0) = 0$, $0 < u(x) < \gamma$ for $x \in (x_1, x_2)$ and u is decreasing on (x_1, x_0), increasing on (x_0, x_2).

Assume the former; therefore $bv'(x) - cv(x) = c(u(x) - (\gamma + \eta + b\eta'))$ is positive on (x_1, x_2). Since $v(x_0) = 0$, we infer that $v'(x_0) > 0$. Now $v \leq 0$ on (x_0, x_2) implies $v'(x_0) \leq 0$ leading to a contradiction. The second case is treated in an analogous manner. In order to describe the behaviour of u near the origin, we consider the representation

$$(5.5) \quad u(x) = \gamma + \frac{\gamma\eta h(\eta)}{c}x + \frac{\gamma}{2}\Big(\frac{\eta^2 h^2(\eta)}{c^2} - \eta'(h(\eta) + \eta h'(\eta))\Big)x^2 + x^2\varepsilon(x),$$

with $\lim_{x\to 0^+} \varepsilon(x) = 0$. The following two lemmas are crucial for the sequel.

Lemma 5.4. Γ^+ *is a nonempty set.*

Proof. For $x > 0$, the relation (5.5) implies

$$\frac{u(x)-\gamma-\eta-b\eta'}{\eta x^2} = \frac{\gamma\eta h^2(\eta)}{2c^2} + \frac{\gamma h(\eta)}{cx} - \frac{1}{x^2} + \frac{\psi(\eta')}{x^2} + \frac{\varepsilon(x)}{\eta} \tag{5.6}$$

where

$$\psi(\eta') := -\frac{\eta'}{\eta}\left(b + \frac{\gamma x^2}{2}\left(h(\eta)+\eta h'(\eta)\right)\right)\cdot$$

Let η, ε_0 and δ_0 be some fixed positive constants. Then, there exists $\underline{x} > 0$ such that $(0 < x \le \underline{x} \;\Rightarrow\; \varepsilon(x) > -\varepsilon_0)$; likewise, there is some $\underline{\eta}' > 0$ such that $(0 < \eta' \le \underline{\eta}' \;\Rightarrow\; \psi(\eta') > -\delta_0)$. Now, consider $0 < x_1 < \underline{x}$, $0 < \eta_1' \le \underline{\eta}'$ and select

$$\gamma > \max\left(\frac{2\varepsilon_0 c^2}{\eta^2 h^2(\eta)}, \frac{c(1+\delta_0)}{x_1 h(\eta)}\right);$$

then (5.6) implies

$$\frac{u(x_1)-\gamma-\eta-b\eta_1'}{\eta {x_1}^2} \ge \left(\frac{\gamma\eta h^2(\eta)}{2c^2} - \frac{\varepsilon_0}{\eta}\right) + \frac{\gamma h(\eta)x_1 - c - c\delta_0}{cx_1^2} > 0$$

which in turn yields some x_0 not necessarily equal to x_1, which can be associated with the triplet (γ, η, η_1') in Γ^+.

Lemma 5.5. Γ^- *is a nonempty set.*

Proof. Following the proof of Lemma 5.4, we start from (5.5) by writing

$$u(x) - \gamma = -\frac{\gamma}{2}\eta' x^2\left(h(\eta)+\eta h'(\eta)\right) + \varphi(\eta) + x^2\varepsilon(x)\cdot \tag{5.7}$$

In order to prove that $(u(x)-\gamma)$ may be negative, we observe that $\lim_{x\to 0^+} \varepsilon(x) = 0$ while $\varphi(\eta) := x\frac{\gamma\eta h(\eta)}{c} + x^2\frac{\gamma\eta^2 h^2(\eta)}{2c^2}$ has limit zero too, as $\eta \to 0^+$, $x > 0$ being fixed. The remaining of the proof mimics that of Lemma 5.4; taking some positive constants ε_0, δ_0, one gets two new positive constants

$\underline{x}$, $\underline{\eta}$ such that $(0 < x \leq \underline{x} \Rightarrow \varepsilon(x) < \varepsilon_0)$ and $(0 < \eta \leq \underline{\eta} \Rightarrow \varphi(\eta) < \delta_0)$. Considering some constants $0 < x_1 \leq \underline{x}$ and $0 < \eta_1 \leq \underline{\eta}$, we infer, by (5.7), the following upper bound

$$u(x_1) - \gamma \leq x_1^2 \left(\varepsilon_0 - \frac{\gamma}{2}\eta' h(\eta_1)\right) + \left(\delta_0 - \frac{\gamma\eta' x_1^2 \eta_1 h'(\eta_1)}{2}\right).$$

Now, we distinguish between two cases:

(i) $h'(\eta_1) \neq 0$: Choosing $\eta' > \max\left(\frac{2\varepsilon_0}{\gamma h(\eta_1)}, \frac{2\delta_0}{x_1^2 \gamma \eta_1 h'(\eta_1)}\right)$ makes $(u(x_1) - \gamma)$ negative.

(ii) $h'(\eta_1) = 0$: Is is sufficient to select $\eta' > \frac{2}{\gamma x_1^2 h(\eta_1)}(\delta_0 + x_1^2 \varepsilon_0)$. Therefore, $(\gamma, \eta_1, \eta') \in \Gamma^-$, ending the proof of the lemma. We are now ready to prove

Theorem 5.6. *There exists some* $\beta > 0$ *such that problem* (5.1), (2.6) *has at least a solution.*

Proof. It will be carried out in three steps.

Step 1. Actually, there is some $(\gamma_0, \eta_0, \eta_0') \notin \Gamma^+ \cup \Gamma^-$. On the contrary, $C_{\mathbb{R}^3}(\Gamma^+ \cup \Gamma^-) = \emptyset$, that is $C_{\mathbb{R}^3}(\Gamma^+) \cap C_{\mathbb{R}^3}(\Gamma^-) = \emptyset$ whence $C_{\mathbb{R}^3}(\Gamma^+) \subset \Gamma^-$. However, in view of Lemma 5.3, $\Gamma^+ \subset C_{\mathbb{R}^3}(\Gamma^-)$ therefore $C_{\mathbb{R}^3}(\Gamma^+) = \Gamma^-$ and $\mathbb{R}^3 = \Gamma^+ \cup \Gamma^-$; making use of Lemmas 5.3-5.5, we get a violation of the connexity of the $\mathbb{R}^3$ space. From definitions of Γ^+ and Γ^-, there corresponds, to the triplet $(\gamma_0, \eta_0, \eta_0')$, a solution u which remains in the strip $\{x > 0;\ \gamma_0 < u(x) < \gamma_0 + \eta_0 + b\eta_0'\}$.

Step 2. We claim that $v > 0$ on $(0, +\infty)$; otherwise, there are some $0 < x_1 < x_2$ satisfying $v(x_1) = 0$, $v'(x_1) < 0$ and $v(x) < 0$ on (x_1, x_2). Eq. (5.1b) then shows that $bv' - cv$ is monotone decreasing over (x_1, x_2); hence $bv'(x) - cv(x) < bv'(x_1) < 0$ for any $x \in (x_1, x_2)$ which in turn implies that $v' < 0$ on this interval. Turning back to Eq. (5.1b), we can see that v'' is negative too. In addition, the fourth quadrant is now a trap area for v; on the contrary, there is some $x_3 > x_2$ such that $v(x_3) < 0$, $v'(x_3) = 0$ and $v''(x_3) \geq 0$; a contradiction is then achieved from Eq. (5.1b) itself. We have just shown that v as well as its derivatives still remain negative for any $x \geq x_1$. In view of part (b) of the preliminary lemma, we conclude that

$\lim_{x \to +\infty} v(x) = -\infty$. Therefore, u is monotone decreasing for $x \geq x_1$ so that there exists $\ell = \lim_{x \to +\infty} u(x)$; from Eq. (5.1a) both with (5.2), we infer that $\lim_{x \to +\infty} u'(x) < 0$ contradicting part (a) of the preliminary lemma.

Step 3. To check that $v' < 0$ on $(0, +\infty)$, we argue by contradiction and assume that $v'(x_0) = 0$, $v(x_0) > 0$ for some $x_0 > 0$ while necessarily $v'(x) > 0$ for any $x > x_0$; therefore $v'' > 0$ for $x > x_0$ and $\lim_{x \to +\infty} v(x) = +\infty$ is a consequence of part (b) of the preliminary lemma. Then u is bounded, positive and increasing on $(x_0, +\infty)$; hence $\lim_{x \to +\infty} u(x)$ exists and is strictly positive; turning back to Eq. (5.1a) together with (5.2) and part (a) of the preliminary lemma, we find that $\lim_{x \to +\infty} u'(x) = 0$ and we reach a contradiction. Then, u is increasing to some limit l while v decreases to another limit l'. From system (5.1), we infer that u', v' and v'' vanish at $+\infty$ so that $l' = 0$ and $l > 0$. Conservation statement (5.4) yields $l = \gamma_0 + \eta_0 + b\eta_0'$ which plays the role of β, ending the proof of Theorem 5.6.

5.3. The extension to $\mathbb{R}^-$

In extending the solution (u, v) obtained in the previous section to $(-\infty, 0)$, we note that if u remains positive, so must do v and $-v'$; otherwise $v'(x_0) = 0$ and $v(x_0) > 0$ for some $x_0 < 0$ with $0 \geq bv''(x_0) = u(x_0)v(x_0)h(v(x_0)) > 0$ leading to a contradiction. In other words, if u and v become negative, u does so first. Nevertheless, u can never cross the x axis; on the contrary, u and u' must change sign at the same point which is impossible. Therefore, u and u' are positive so that there exists $\ell = \lim_{x \to -\infty} u(x)$. Relation (5.4) then shows that v and $-v'$ are bounded; hence, we have got some $\ell' = \lim_{x \to -\infty} v(x)$; finally, $\ell = 0$ and again $\ell' = \gamma_0 + \eta_0 + b\eta_0'$ follows from (5.4).

Theorem 5.1 is now proved with $\beta = \gamma_0 + \eta_0 + b\eta_0'$. The graphs of u and v are as depicted in Figure 4.

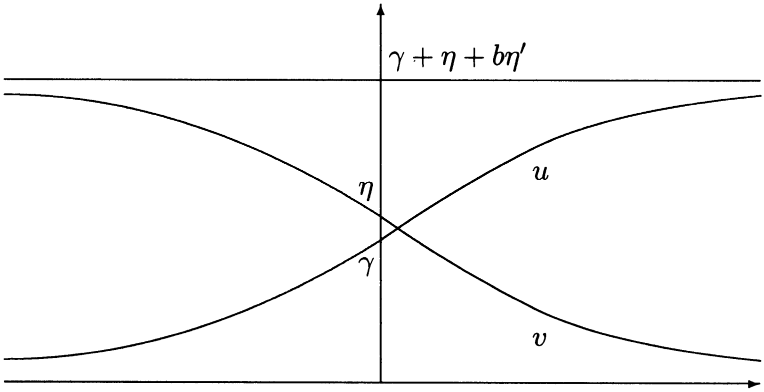

Fig. 4. Schematic sketch of u and v.

5.4. A lower bound of the wave speed

In the sequel, (u, v) stands for the solution to problem (5.1), (2.5), (2.6), (5.3). We have

Proposition 5.7. *The following estimate of the parameter c holds true*

$$c > \frac{1}{\eta + b\eta'}\sqrt{2b\gamma \int_0^\eta sh(s)\,ds}\cdot \tag{5.8}$$

Proof. Multiply Eq. (5.1b) respectively by v and v' and then integrate over $(0, +\infty)$. We get two identities

$$bc\eta\eta' + \frac{c}{2}\eta^2 - b\int_0^\infty |v'(x)|^2 dx = \int_0^\infty uvh(v)(x)\,dx. \tag{5.9}$$

$$\frac{b}{2}c^2\eta'^2 + c\int_0^\infty |v'(x)|^2 dx = \int_0^\infty uv(-v')h(v)(x)\,dx. \tag{5.10}$$

Combining (5.9) and (5.10) and noting that $u \geq \gamma$ on $(0, +\infty)$, we infer

$$b\gamma \int_0^\eta sh(s)\,ds < b\int_0^\infty uv(-v')h(v)(x)\,dx < \frac{c^2}{2}(\eta + b\eta')^2,$$

whence the required inequality (5.8).

Remark 5.8. In fact, Proposition 5.7 provides a global estimate of the shooting parameters γ, η, η' for which a solution to system (5.1) has been obtained. For the particular case $h \equiv 1$, (5.8) reads $0 < \frac{\eta\sqrt{b\gamma}}{\eta+b\eta'} < c$ which also tells us that $\lim_{x\to+\infty} u(x) = \lim_{x\to-\infty} v(x) > \gamma + \frac{\eta}{c}\sqrt{b\gamma}$.

5.5. Asymptotic behaviour of the solutions

Now set

$$\beta\colon = \gamma + \eta + b\eta', \ \underline{h} = \inf_{s\in\mathbb{R}} h(s), \ \overline{h} = \sup_{s\in\mathbb{R}} h(s)$$

and denote

$$r_1 = \frac{c - \sqrt{c^2 + 4b\beta\overline{h}}}{2b}, \ s_1 = \frac{c - \sqrt{c^2 + 4b\gamma\underline{h}}}{2}, \ t_2 = \frac{c + \sqrt{c^2 + 4b\gamma\overline{h}}}{2b}.$$

From simple comparison principles, the asymptotic behaviour at infinity of u and v are readily checked. More precisely, we can prove

Proposition 5.9. *We have*

$$\begin{aligned}
&\textbf{(a)} \quad \gamma e^{\frac{\overline{h}\beta}{c}x} \le u(x) \le \gamma e^{\frac{\underline{h}\eta}{c}x}, \quad \forall x \le 0.\\
&\textbf{(b)} \quad \eta e^{t_2 x} \le v(x) \le \beta + (\eta - \beta)e^{\frac{c}{b}x}, \quad \forall x \le 0.\\
&\textbf{(c)} \quad \eta e^{r_1 x} \le v(x) \le \eta e^{s_1 x}, \quad \forall x \ge 0.
\end{aligned}$$

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