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ASSESSING SYSTEM RELIABILITY USING NON-STATIONARY MODELS

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ABSTRACT. Reliability of many stochastic systems depends on several characteristics that are time dependent is considered. Given that the system reliability is determined by just one characteristic, a non-stationary process is used to model the behavior of this characteristic. Having killed this process at system failure time, we discretely sample the process and derive parametric estimates of system reliability. These procedures are shown to extend to situations involving several characteristics that are modulations of a common process. For situations where joint modeling of multiple characteristics is difficult, we have obtained a useful inequality for the system reliability assuming that the characteristics are associated.

1. INTRODUCTION

The reliability, $\bar{F}(t)$, of a system (component) is the probability that the system will preserve its characteristics within specified limits during a specified time interval $[0, t]$. If a system failure is an event in which at least one of the characteristics of the system shift outside certain permissible limits, and if T is the time to failure, then

$$\bar{F}(t) = P(T > t). \tag{1.1}$$

Suppose that the system reliability is determined by a finite number, k, of characteristics. For $i = 1, \ldots, k$, denote the value of the i-th

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characteristic at time t by $X_i(t)$ and assume that it is within permissible limits if $X_i(t) < a_i$, where $a_1, a_2, \ldots, a_k$ are fixed and known values. One may, for example, look upon a_i as the breaking threshold of total damages, $X_i(t)$, by time t. More general ways of defining permissible limits are clearly possible, but will not be pursued in this paper. Obviously, the random time, $T_i(a_i)$, at which that i-th characteristic first crosses its limit is given by

$$T_i(a_i) = \begin{cases} Inf\ \{t \geq 0 : X_i(t) \geq a_i\}, \\ \infty \quad \text{if } X_i(t) < a_i,\ \forall t \geq 0. \end{cases} \tag{1.2}$$

In this setting, the failure time of the system, T, is given by

$$T = \min(T_1(a_1), \ldots, T_k(a_k)). \tag{1.3}$$

In view of (1.1) and (1.3),

$$\bar{F}(t) = P(T_i(a_i) > t, \quad i = 1, \ldots, k). \tag{1.4}$$

Formulation of system reliability by means of equations (1.2) to (1.4) is relevant to engineering disciplines to structural safety, mechanical vibration, etc. For other applications, see Ebrahimi and Ramalingam (1993, 1995).

Certain models based on just one characteristic ($k = 1$) and related computation of $\bar{F}(t)$ have been discussed by Madsen et. al (1986) and the references therein. In the first four sections of this paper, we let $k = 1$ and relabel $X_1(t)$ as $X(t)$. Specifically, we shall consider a non-stationary process given by

$$X(t) = \alpha(t)Y(t), \quad t \geq 0, \tag{1.5}$$

where $\alpha(t)$ is a non-random continuous function such that $\alpha(t) > 0$ for $t \geq 0$ and $Y(t)$ is a process with continuous sample paths and is stationary in the sense of Definition 1.1 on page 443 of Karlin and Taylor (1975). The model in (1.5) is called the uniformly modulated model (UMM) (see Priestly (1988) and $X(t)$ is said to be a modulation of $Y(t)$. As an application of this model, consider a bridge that is subject to seismic disturbances. The failure time of the bridge is higly related to a characteristic $X(t)$ such as displacements during earthquakes. Further illustrations of the model in (1.5) can be found in several works including Fujita and Shibara (1978) and Dargahi-Noubary (1982).

Having let $k = 1$ and relabeled $T_1(a_1)$ as $T(a)$ in (1.4), our main goal will be to develop procedures to assess $\bar{F}(t)$ in the context of the model given by (1.5). Since the permissible limit a is known, it is clear that, for non-zero a, by substituting $X(t)/a(-X(t)/a)$ for $X(t)$ in (1.2), one can reduce $T(a)$ for any $a > 0$ $(a < 0)$ to $T(1)$ $(T(-1))$. Furthermore, the analyses of $T(1)$ and $T(-1)$ are similar. Consequently, we shall discuss only two cases. Case I will deal with $T(0)$, which is relabeled as S, while in Case II, we study $T(1)$, which is termed T.

This paper is organized as follows. In Section 2, we derive the structural properties of processes of the type (1.5) that are stopped at the times S and T. Having derived, in Section 3, the consequences of these results for certain modulations of the Ornstein-Uhlenbeck process, we obtain, in Section 4, the estimation of system reliability under Cases I and II. In Section 5, we discuss a reduction of an important situation in which $k > 1$ to the $k = 1$ problem. For several general characteristics, which are collectively associated (see Definition 5.1) and individually follow a UMM, we derive an inequality for system reliability in terms of the individual characteristics.

2. GENERAL PROPERTIES OF UMM

In many important applications involving catastrophic situations at the time of system failure, it is most prudent that we estimate the reliability of the system prior to system failure by monitoring the characteristic $X(t)$ over time. Thus we shall not necessarily wait till the time to failure, namely S or T as the case may be, in estimating $\bar{F}(t)$. This tantamounts to observing the following killed processes:

$$U(t) = \begin{cases} X(t), & \text{if } t < S \\ 0, & \text{if } t \geq S \end{cases} \tag{2.1}$$

$$V(t) = \begin{cases} X(t), & \text{if } t < T \\ 1, & \text{if } t \geq T \end{cases} \tag{2.2}$$

It should be noted that $U(t)$ and $V(t)$ are absorbed into the states 0 and 1 at the times S and T respectively.

Often, we may know that probabilistic structure of the process $X(t)$ completely, but would like to assess, without necessarily waiting to observe system failure. $\bar{F}(t)$, given the current value of the process at a time $s < t$. The results concerning the laws of the $U(t), V(t)$ processes, presented in this and the next sections, are useful in this regard. However, if the structure of $X(t)$ is specified except for some unknown parameters, then these results can also be used to perform likelihood-based inference for $\bar{F}(t)$. For any two random variables ξ_1 and ξ_2, we denote the conditional probability density function of ξ_1 given ξ_2 by $f_{\xi_1|\xi_2}(\cdot|\cdot)$.

Theorem 2.1. *Let $\{Y(t); t \geq 0\}$ in (1.5) be a strong Markov process. Then $\{U(t); t \geq 0\}$ and $\{V(t); t \geq 0\}$ are time inhomogeneous Markov processes such that*

(i) For $0 \leq t_1 < t_2$,

$$P(U(t_2) = 0|U(t_1) = 0) = 1, \tag{2.3}$$

$$P(V(t_2) = 1|V(t_1) = 1) = 1. \tag{2.4}$$

(ii) For $0 \leq t_1 < t_2$, $u < 0$, $v < 1$,

$$P(U(t_2) = 0|U(t_1) = u) = \int_{t_1}^{t_2} f_{S|Y(0)}(x - t_1|\frac{u}{\alpha(t_1)})dx, \tag{2.5}$$

$$P(V(t_2) = 1|V(t_1) = v) = \int_{t_1}^{t_2} f_{T|Y(t_1)}(x|\frac{v}{\alpha(t_1)})dx, \tag{2.6}$$

(iii) For $0 \leq t_1 < t_2$, $u < 0$, $y < 0$, $v < 1$, $z < 1$,

$$\begin{aligned} P(U(t_2) \leq y|U(t_1) = u) &= P(X(t_2) \leq y|X(t_1) = u) \\ &\quad - \int_{t_1}^{t_2} [P(Y(t_2 - x) \\ &\quad \leq \frac{y}{\alpha(t_2)}|Y(0) = 0)] f_{S|Y(0)}(x - t_1|\frac{u}{\alpha(t_1)})dx, \end{aligned} \tag{2.7}$$

$$
\begin{aligned}
P(V(t_2) &\leq z|V(t_1) = v) = P(X(t_2) \leq z|X(t_1) = v) \\
&- \int_{t_1}^{t_2} [P(Y(t_2 - x) \\
&\leq \frac{z}{\alpha(t_2)}|Y(0) = \frac{1}{\alpha(x)})] f_{T|Y(t_1)}(x|\frac{v}{\alpha(t_1)})dx. \qquad (2.8)
\end{aligned}
$$

Proof. Since the arguments for $U(t)$ and $V(t)$ are similar, we shall prove the results only for $U(t)$. Part (i) is obvious. With regard to (2.5), we have, since $u < 0$,

$$
\begin{aligned}
P(U(t_2) = 0|U(t_1) = u) &= P(S \leq t_2|Y(t_1) \\
&= u/\alpha(t_1)) = \int_{t_1}^{t_2} f_{S|Y(0)}(z - t_1|u/\alpha(t_1))dz.
\end{aligned}
$$

Finally, in order to show (2.7),

$$
\begin{aligned}
&P(U(t_2) \leq y|U(t_1) = u) \\
&= P(X(t_2) \leq y,\ S > t_2|U(t_1) = u) \\
&= P(X(t_2) \leq y|X(t_1) = u) - P(X(t_2) \leq y, S \leq t_2|X(t_1) = u) \\
&= P(X(t_2) \leq y|X(t_1) = u) \\
&- \int_{t_1}^{t_2} [P(Y(t_2 - z) \leq y/\alpha(t_2)|Y(0) = 0) \times f_{S|Y(0)}(z - t_1|u/\alpha(t_1))]dz.
\end{aligned}
$$

Remark 2.1. Suppose that the law of the $X(t)$ process in (1.5) is known except for an unknown vector parameter $\underset{\sim}{\theta}$. Then, the system reliability $\bar{F}(t)$, in Case I, is given by

$$
\bar{F}(t) = \bar{F}(t;\underset{\sim}{\theta}) = P_{\underset{\sim}{\theta}}(U(t) < 0) = 1 - P_{\underset{\sim}{\theta}}(U(t) = 0). \qquad (2.9)
$$

In Case II, we have

$$
\bar{F}(t;\underset{\sim}{\theta}) = P_{\underset{\sim}{\theta}}(V(t) < 1) = 1 - P_{\underset{\sim}{\theta}}(V(t) = 1). \qquad (2.10)
$$

Expressions in terms of $\underset{\sim}{\theta}$ for $\bar{F}(t)$ can therefore be obtained by letting $t_2 = t, t_1 = 0$ in, and unconditioning, equations (2.5) and (2.6) in Cases

I and II respectively. We now turn to the problems of finding the conditional first-passage-time densities $f_{S|Y(0)}(x|y)$ and $f_{T|Y(t_1)}(x|y)$ which are needed in applying Theorem 2.1 to the model in (1.5). These densities will also help us to find the marginal densities of S and T. We start with S. For ease of notation, we shall delete ω from the usual notation $y(\omega;t)$ for a typical sample paths of the process $Y(t)$. Suppose now that, for a particular sample path $y(t)$ of $Y(t)$, $y(0) < 0 < y(t_1)$. Then, during the time interval $(0, t_1)$, the process $Y(t)$ must cross the level zero at least once. Hence, we obtain (see Blake and Lindsey (1973) for details)

$$(2.11) \qquad p(y(t_1)|y(0)) = \int_0^{t_1} p(y(t_1)|y(\theta) = 0) f_{S|Y(0)}(\theta|y(0))d\theta$$

where $p(x|y)$ is the transition density of $Y(t)$ process. Now, $f_{S|Y(0)}(\theta|u)$ can be obtained by solving the integral equation (2.11). By a similar argument, $f_{T|X(t_1)}(\theta|u)$ can be expressed as the solution of the integral equation

$$(2.12) \quad h(x(t_2)|x(t_1)) = \int_{t_1}^{t_2} h(x(t_2)|x(\theta) = 1) f_{T|X(t_1)}(\theta|x(t_1))d\theta,$$

where $x(t_1) < 1 < x(t_2)$ and $h(x|y)$ is the transition density of $X(t)$ process.

3. STRUCTURE OF SOME SPECIAL STOPPED PROCESSES

In this section, we shall derive the probability law of the processes $\{U(t), t \geq 0\}$ and $\{V(t), t \geq 0\}$, under the assumption that the stationary process $\{Y(t), t \geq 0\}$ in (1.5) is a Gaussian Markov process with continuous sample paths. It may be noted that such a process is the celebrated Ornstein-Uhlenbeck process (see Doob (1953)). It is well-known that, for some parameters $\gamma > 0$ and τ^2, and $t \geq 0$,

$$(3.1) \qquad E(Y(t)) = 0$$

$$(3.2) \qquad C(t) = Cov(Y(0), Y(t)) = \tau^2 \exp(-\gamma t).$$

Assuming that the system is working initially $(P(Y(0) < 0) = 1)$ and using the results of Darling and Siegert (1953), we solve the integral equation (2.11) and obtain, for $y < 0$, $x > 0$,

$$f_{S|Y(0)}(x|y) = \frac{2\gamma|y|\exp(-\gamma x)}{\tau\sqrt{2\pi}(1-\exp(-2\gamma x))^{3/2}} \times \exp\left[-\frac{y^2\exp(-2\gamma x)}{2\tau^2(1-\exp(-2\gamma x))}\right]. \tag{3.3}$$

Using (3.3) and Theorem 2.1, and letting $\Phi(\cdot)$ to denote the cumulative distribution of the standard normal random variable, for $0 \le t_1 < t_2$, $u < 0$, $y < 0$, we obtain

$$P(U(t_2) \le y|U(t_1) = u) = \Phi\left[\frac{\frac{y}{\alpha(t_2)} - \frac{u}{\alpha(t_1)}\exp(-(t_2-t_1)\gamma)}{\tau(1-\exp(-2\gamma(t_2-t_1)))^{\frac{1}{2}}}\right]$$

$$-\int_{t_1}^{t_2}\Phi\left[\frac{\frac{y}{\alpha(t_2)}}{\tau\{1-\exp(-2\gamma(t_2-x))\}^{\frac{1}{2}}}\right] f_{S|Y(0)}(x-t_1|\frac{u}{\alpha(t_1)})dx. \tag{3.4}$$

Also, invoking (3.3), we obtain the marginal density of S as

$$f_S(s) = [2\gamma\exp(-\gamma s)]/\pi(1-\exp(-2\gamma s))^{\frac{1}{2}}, \quad s > 0. \tag{3.5}$$

Theorem 3.1. *Letting* $\alpha(t) = \exp(-\gamma t)$, $\gamma > 0$, *and*

$$H(t;y,t_1) = \Phi\left[\frac{(\exp(2\gamma t)) - y\exp(2\gamma t_1)}{\tau((\exp(2\gamma t)) - \exp(2\gamma t_1))^{\frac{1}{2}}}\right] - \Phi\left[\frac{(y-2)\exp(2\gamma t_1) + \exp(2\gamma t)}{\tau((\exp(2\gamma t)) - \exp(2\gamma t_1))^{\frac{1}{2}}}\right] \times \exp\left(-2\frac{(1-y)\exp(2\gamma t_1)}{\tau^2}\right), \tag{3.6}$$

$$h(t;y,t_1) = \frac{2\gamma(1-y)\exp(2\gamma(t+t_1))}{\tau\sqrt{2\pi}(\exp(2\gamma t) - \exp(2\gamma t_1))^{\frac{3}{2}}} \times \exp\left[-\frac{(y\exp(2\gamma t_1) - \exp(2\gamma t))^2}{2\tau^2(\exp(2\gamma t) - \exp(2\gamma t_1))}\right], \tag{3.7}$$

$G(w) = \Phi([z\exp(\gamma t_2) - y\exp(-\gamma(t_2 - 2w))]/\tau(1 - \exp(-2\gamma(t_2 - w)))^{\frac{1}{2}})$,
we obtain

(i) For $0 \leq t_1 < t, y < 1$,

$$P(T > t|X(t_1) = y) = H(t; y, t_1). \tag{3.8}$$

(ii) T *is a defective random variable in the sense that*

$$P(T < \infty|X(t_1) = y) = \exp(-2\tau^{-2}(1 - y)\exp(2\gamma t_1)). \tag{3.9}$$

(iii) For $0 \leq t_1 < t_2, y < 1$,

$$P[V(t_2) = 1|X(t_1) = y] = 1 - H(t_2; y, t_1). \tag{3.10}$$

(iv) For $0 \leq t_1 < t_2, y < 1, z < 1$,

$$P(V(t_2) \leq z|X(t_1) = y) = G(t_1) - \int_{t_1}^{t_2} G(w)h(w; y, t_1)dw. \tag{3.11}$$

Proof. Since it is well-known that $Y(t) = \exp(-\gamma t))B(\exp(-2\gamma t))$, where $B(t)$ is the Wiener process with zero drift and diffusion coefficient τ, we have

$$\begin{aligned} T &= Inf\ \{t \geq 0 : X(t) \geq 1\} \\ &= Inf\ \{t \geq 0 : (\exp(-2\gamma t))B(\exp(2\gamma t)) \geq 1\} \\ &= Inf\ \{\Psi(s), s \geq 1 : B(s) \geq s\}, \end{aligned}$$

where $\Psi(s) = \frac{1}{2\gamma}\log s$. Letting $T^* = Inf\ \{s \geq 1 : B(s) \geq s\}$, it is clear that T has the same distribution as $\Psi(T^*)$. Now, define $T^{**}(y, t_1) = Inf\ \{s \geq 0 : B(s) \geq c + s\}$, where $c = c(t_1, y) = (1 - y)\exp(2\gamma t_1)$. Then, for $t \geq t_1$ and $y < 1$,

$$\begin{aligned} P(T > t|X(t_1) = y) &= P(T^* > \exp(2\gamma t)|B(\exp(2\gamma t_1)) \\ &= y\exp(2\gamma t_1)) \\ &= P((T^{**}(y, t_1) > \exp(2\gamma t) - \exp(2\gamma t_1)). \end{aligned} \tag{3.12}$$

Using the result given in Durbin (1985, equation (20)) and the equation (3.12) we get (3.8), which readily yields (3.9). Parts (iii) follows from (i). Using arguments which are similar to those that lead to (2.7), we obtain (3.11).

Remark 3.1. Since an explicit expression for $f_{T|X(t_1)}(x|y)$ cannot be obtained by solving (2.12) for a general $\alpha(t)$, we have computed $f_{T|X(t_1)}(x|y)$ for the special model $\alpha(t) = \exp(-\gamma t)$. When we let $\alpha(t) = \exp(-\gamma t)$, (1.5) clearly implies that the non-stationarity $X(t)$ levels off when t is large. Such a behavior of $\alpha(t)$ would be of interest in those systems where start-ups or occasional disturbances generally have limited effects in time.

4. RELIABILITY ESTIMATION

In this section, for both Cases I for $T(0)$ and case II for $T(1)$, we derive estimators of $\bar{F}(t)$ based on the following type of data. Observations are obtained independently for each of n identical systems. We let a subscript $i = 1, 2, \ldots, n$ refer to the label of the system and a subscript $j = 1, \ldots, M_i$ denote the label of repeated observations for the i-th system. For the j-th observations on the i-th system, let t_{ij} be the time from the initial observations ($t_{i1} = 0$) and $U_i(t_{ij})(V_i(t_{ij}))$ be the observation on the i-th system at time t_{ij}, $i = 1, \ldots, n$, $j = 1, \ldots, M_i$. Moreover, we assume that, in Case I (II), all the systems are initially working, that is $P(Y(0) < 0) = 1)(P(X(0) < 1) = 1)$. It is important to note that, if $U_i(t_{ij}) < 0(V_i(t_{ij}) < 1)$, the system is operating at time t_{ij}. However, if $U_i(t_{i,j-1}) < 0(V_i(t_{i,j-1}) < 1)$ and $U_i(t_{ij}) = 0(V_i(t_{ij}) = 1)$, then we stop gathering data from the failed i-th system and record the actual time to failure fo the i-th system, $S_i(T_i)$, where $t_{i,j-1} < S_i \leq t_{ij}(t_{i,j-1} < T_i \leq t_{ij})$. Furthermore, we relabel j as m_i and the corresponding proposed observation time, t_{im_i}, is relabeled as $S_i(T_i)$. It should be noted that $U_i(t_{ik}) = 0$ ($V_i(t_{ik}) = 1$), for $k = m_i, \ldots, M_i$. Thus, the observed data sets in Cases I and II are $U_i(t_{ij}), j = 1, \ldots, M_i$, $L_i = Min(S_i, t_{iM_i})$, $i = 1, \ldots, n$ and $V_i(t_{ij}), j = 1, \ldots, M_i$, $D_i = Min(T_i, t_{iM_i}), i = 1, \ldots, n$, respectively.

Remark 4.1. This type of sampling continuous-time parameter processes is not uncommon in reliability and survival analysis. See, for instance, the data set provided by Klein et. al (1984).

4.1. ESTIMATION IN CASE I

At the outset, we assume that in (1.5)

$$\alpha(t) = \exp(-\beta t), \quad \beta > 0, \tag{4.1}$$

and $\{Y(t); t \geq 0\}$ is the Ornstein-Uhlenbeck process with its moments-structure given by (3.1) and (3.2). Since density function of S in (3.5), and hence $\bar{F}(t)$ is free of the parameters β and τ, we shall treat these two parameters as nuisance parameters.

Suppose now that both nuisance parameters are known. Then we can assume, without loss of generality, that $\beta = 1$ and $\tau = 1$. For, if not, we can make the transformation

$$\frac{1}{\tau}(X(t))\exp(-(1-\beta)t) = \left[\frac{Y(t)}{\tau}\right]\exp(-t). \tag{4.2}$$

Thus, in this sub-section, our objective will be to estimate the only unknown parameter γ, and hence $\bar{F}(t)$. We now need a stopped process $Y^*(t)$, given by

$$Y^*(t) = \begin{cases} Y(t), & \text{if } t < S \\ 0, & \text{if } t \geq S \end{cases} = \begin{cases} \exp(\beta t)U(t), & \text{if } t < S \\ 0, & \text{if } t \geq S. \end{cases} \tag{4.3}$$

In view of (4.1) and the simplification that $\beta = 1$, we obtain from (1.5), $Y(t) = (\exp(t))X(t)$. We thus transform our data to $Y_i^*(t_{ij}) = (\exp(t_{ij}))(U_i(t_{ij}))$, $i = 1, \ldots, n$ and $j = 1, \ldots, M_i$. In order to write the log-likelihood function, $l(\gamma)$, of this data set, we denote by $\phi(\cdot)$ the probability density function of the standard normal distribution, and for $0 \leq t_1 < t_2, a \leq 0, b \leq 0$, denote

$$\begin{aligned} J_1(t_1) &= (1-\exp(-2\gamma t_1))^{\frac{1}{2}}, \\ J_2(t_1,t_2) &= [1-\exp(-2\gamma(t_2-t_1))]^{-\frac{1}{2}}, \\ J_3(t_2,a,b) &= \frac{1}{J_1(t_2)}\phi\left[\frac{b-a\exp(-\gamma t_2)}{J_1(t_2)}\right] \end{aligned}$$

$$
\begin{aligned}
& - \int_0^{t_2} (J_2(x,t_2))\phi(bJ_2(x,t_2)) f_{S|Y(0)}(x|a)dx, \\
\lambda(t_1,a;t_2,b) &= J_3(t_2-t_1,a,b), \\
q(t_1,a;t_2) &= f_{S|Y(0)}(t_2-t_1|a).
\end{aligned}
$$

Using (3.3) and (3.4) and the foregoing notations, we introduce a function $g(\cdot)$ by

$$
(4.4) \qquad g(t_1,a;t_2,b) = \begin{cases} \lambda(t_1,a;t_2,b) & \text{if } a<0 \text{ and } b<0 \\ q(t_1,a;t_2,b) & \text{if } a<0 \text{ and } b=0 \\ 1 & \text{if } a=0,\ b=0 \end{cases}
$$

and obtain

$$
(4.5)\ l(\gamma) = \sum_{i=1}^{n}\sum_{j=2}^{M_i} \log[g(t_{i,j-1}, y_i^*(t_{i,j-1}); t_{ij}, y_i^*(t_{ij}))] + \sum_{i=1}^{n} \log\phi(y_i^*(t_{i1})),
$$

Remark 4.2. It is possible that, in some situations, at the time of system failure, we may be able to observe that fact that S_i belongs to one of the intervals $(t_{i,j-1}, t_{ij})j = 2, \ldots, M_i$ and (t_{iM_i}, ∞). In such cases, inference for $\bar{F}(t)$ can be carried out with slight modifications of the likelihood in (4.5). Instead of incorporating the density of the failure time in (4.5), the probabilities of failure during the intervals mentioned above should be included.

In order to maximize the likelihood in (4.5), we consider the equation given by $\dot{l}(\gamma) = 0$, where $\dot{l}(\gamma) = \frac{dl(\gamma)}{d\gamma}$. Since the necessary partial derivatives of $g(\cdot)$ in (4.4) can, in practice, be computed using well-known symbolic software packages such as MACSYMA, in the score function and the second derivative $\ddot{l}(\gamma) = \frac{d^2l(\gamma)}{d\gamma^2}$ can be computed using (4.5). The root of the likelihood equation will be denoted by $\hat{\gamma}$.

A numerical scheme such as the Newton-Raphson may be employed here to arrive at the estimate $\hat{\gamma}$. In order to initialize this iterative scheme, we pretend that our data $Y_i^*(t_{ij}), i = 1, \ldots, n, j = 1, \ldots, M_i$ are observations from the unstopped process $Y(t)$. Letting $\Delta_{ij} = t_{i,j+1} - t_{ij}$

and noting the fact that $\log E[Y_i^*(t_{i,j+1})Y^*(t_{i,j})] = -\gamma\Delta_{ij}$, we take the following as an initial value of γ.

$$\hat{\gamma} = -\sum_{i=1}^{n}\sum_{j=1}^{M_i-1}\{(\log[Y_i^*(t_{i,j+1})Y_i^*(t_{i,j})])\Delta_{ij}\}/\sum_{i=1}^{n}\sum_{j=1}^{M_i-1}\Delta_{ij}^2.$$

We now discuss of the asymptotic properties of $\hat{\gamma}$ and the realted estimate of reliability. For simplicity, we shall assume that $M_1 = M_2 = \cdots = M_n = M > 0$ and the data are equally spaced, that is, $\Delta_{ij} = \Delta > 0$, $i = 1,\ldots,n$, $j = 1,\ldots,M-1$.

Theorem 4.1. *With probability tending to 1 as $n \to \infty$, there exists a solution $\hat{\gamma}$ of the likelihood equation such that (i) $\hat{\gamma}$ converges to γ in probability and (ii) $\sqrt{n}(\hat{\gamma}-\gamma)$ converges weakly to a normal distribution with mean zero and variance σ^2, where*

$$(4.6)\quad \sigma^2 = -[E[\frac{d^2}{d\gamma^2}\sum_{j=2}^{M}\log g((j-1)\Delta, Y^*((j-1)\Delta); j\Delta, Y^*(j\Delta))]]^{-1}.$$

Proof. It is clear that the likelihood in (4.5) is based on n i.i.d. copies, $\underset{\sim}{W}_{-1},\ldots,\underset{\sim}{W}_n$, of $\underset{\sim}{W}$, where $\underset{\sim}{W} = (Y^*(0), Y^*(\Delta),\ldots,Y^*(M\Delta), L)$ and the log-density of W is seen to be

$$\log f(\underset{\sim}{w};\gamma) = \log\phi(y^*(0)) + \sum_{j=2}^{M}\log g((j-1)\Delta, y^*((j-1)\Delta); j\Delta, y^*(j\Delta))$$

Since the regularity conditions of Theorem 4.1 on page 429 of Lehmann (1983) can be easily verified, we obtain the results (i) and (ii).

Corollary 4.1. *As $n \to \infty$ the following hold for the reliability estimate $\bar{F}(t;\hat{\gamma})$, with σ^2 defined in (4.6); (i) $\bar{F}(t;\hat{\gamma})$ converges to $\bar{F}(t;\gamma)$ in probability and (ii) $\sqrt{n}(\bar{F}(t;\hat{\gamma}) - \bar{F}(t;\gamma))$ converges weakly to the normal random variable with mean 0 and variance σ_1^2, where*

$$(4.7)\qquad \sigma_1^2 = 4\sigma^2t^2\exp(-2\gamma t)/[\pi^2(1-\exp(-2\gamma t))].$$

Proof. Part (i) is straightforward. In view of (3.5), $\bar{F}(t) = (2/\pi)$ arcsin$(\exp(-\gamma t))$ and hence we obtain part (ii) by applying the δ-method to Theorem 4.1, (ii).

It may be recalled that the parameters τ and β were presumed to be known when we set out to find an estimate of γ. If, however, the nuisance parameters are not known, then one can estimate τ and β as follows. We define

$$U^*(t) = U(t)|U(t) < 0, Y^{**}(t) = Y(t)|Y(t) < 0 \quad \text{and}$$
$$Z^*(t) = -2\beta t + W_1^*(t), \tag{4.8}$$

where $Z^*(t) = \log[(U^*(t))^2/(U^*(t_{11}))^2]$ and $W_1^*(t) = \log[(Y^{**}(t))^2/(Y^{**}(t_{11}))^2]$. Noting that $E(Y^{**}(t)) = -\frac{2\tau}{\sqrt{2\pi}}$ and $E((Y^{**}(t))^2 = \tau^2$, we get $E(W_1^*(t)) = 0$. The model (4.8) is therefore a special case of a general regression model and we propose the simple least-squares estimate of β, namely $\hat{\beta} = -\{\sum_{i=1}^{n}\sum_{j=1}^{d_i}(t_{ij}Z^*(t_{ij}))\}/2\sum_{i=1}^{n}\sum_{j=1}^{d_i} t_{ij}^2$, where $d_i = \sum_{j=1}^{M_i} I(U_i(t_{ij}) < 0), i = 1, \ldots, n$, and $I(A)$ is the indicator function of A. One can follow the approach of Grenander (1954) and establish the consistency of $\hat{\beta}$ in our set-up. Using $\hat{\beta}$ in (4.3), we can transform the data on $U(t)$ into observations on $Y^{**}(t)$. Since $E(Y^{**}(t)) = -2\tau/\sqrt{2\pi}$, one can use the moment method to estimate τ by $\hat{\tau} = -(\sqrt{\pi/2})\bar{Y}$, where, $\bar{Y} = [\sum_{i=1}^{n}\sum_{j=1}^{d_i} Y^{**}(t_{ij})/\sum_{i=1}^{n} d_i$. Having replaced τ and β by $\hat{\tau}$ respectively in (3.3) and (3.4), we can represent the transition kernel of the process $\{U(t), t \geq 0\}$ in terms of $\hat{\tau}$, $\hat{\beta}$ and the unknown γ. Thus, when the nuisance parameters are unknown, one can write the log-likelihood similar to (4.5) and obtain the pseudo-maximum likelihood estimate of γ.

4.2. ESTIMATION IN CASE II

We specialize the model for $X(t)$ in (1.5) by letting

$$\alpha(t) = \exp(-\gamma t), \quad \gamma > 0, \tag{4.9}$$

and $\{Y(t); t \geq 0\}$ to be the Ornstein-Uhlenbeck process with its moments structure given by (3.1) and (3.2). In this setting, based on the data for Case II that we described at the top this esction, we now give the maximum likelihood estimation of the vector parameter $\underset{\sim}{\theta} = (t, \gamma)$. In order to write the log- likelihood function, $l(\underset{\sim}{\theta})$, of the data set, we use $h(\cdot)$ from (3.7) and for $0 \leq t_1$, $w < t_2$, $a \leq 1$, $b \leq 1$, let

$$\begin{aligned} K(w, a; t_2, b) &= \frac{\exp(\gamma t_2)}{\tau(1-\exp(-2\gamma(t_2-w)))^{1/2}}\phi \\ &\quad \left[\frac{b\exp(\gamma t_2) - a\exp(-\gamma(t_2-2w))}{\tau(1-\exp(-2\gamma(t_2-w)))^{1/2}}\right] \\ \lambda_1(t_1, a; t_2, b) &= K(t_1, a, t_2, b) - \int_{t_1}^{t_2}, K(w, a; t_2, b)h(w; a, t_1)dw, \\ q_1(t_1, a; t_2) &= -h(t_2; a, t_1). \end{aligned}$$

Introducing functions $g_1(\cdot)$ and $g_1^*(\cdot)$ by

$$g_1(t_1, a; t_2, b) = \begin{cases} \lambda_1(t_1, a; t_2, b) & \text{if } a < 1 \text{ and } b < 1 \\ q_1(t_1, a; t_2) & a < 1 \text{ and } b = 1, \\ 1 & \text{if } a = 1,\ b = 1, \end{cases} \tag{4.10}$$

$$g_1^*(x, t) = \frac{1}{t\sqrt{2\pi}\exp(-\gamma t)}\exp\left[-\frac{x^2}{2\tau^2\exp(-2\gamma t)}\right], \tag{4.11}$$

we obtain

$$\begin{aligned} l(\underset{\sim}{\theta}) &= \sum_{i=1}^{n}\sum_{j=2}^{M_i}\log[g_1(t_{i,j-1}, v_i(t_{i,j-1}); t_{ij}, v_i(t_{ij})] \\ &\quad + \sum_{i=1}^{n}\log g_1^*(v_i(t_{i1}), t_{i1}) \end{aligned} \tag{4.12}$$

To maximize this likelihood function, we consider two equations given by $\dot{\underset{\sim}{l}}\ (\underset{\sim}{\theta}) = \underset{\sim}{0}$, where $\dot{\underset{\sim}{l}}\ (\theta) = \dfrac{\partial l(\underset{\sim}{\theta})}{\partial(\underset{\sim}{\theta})}$ is the vector of partial derivates. Since the partial derivatives fo $g_1(t_1, a; t_2, b)$ and $g_1^*(x, t)$ in (4.10) and (4.11) can, in practive, be computed with the help of symbolic software packages such as MACSYMA, the score vector and the 2×2 matrix of second partial derivatives, $\ddot{l}(\underset{\sim}{\theta})$, can be computed using (4.12). The roots of the likelihood equations will be denoted by $\hat{\underset{\sim}{\theta}} = (\hat{\tau}, \hat{\gamma})$. An iterative scheme such as the Newton-Raphson may be used here to obtain $\hat{\underset{\sim}{\theta}}$. The initializing values for τ and γ may be obtained by pretending that the observables $V_i(t_{ij}), j = 1, \ldots, M_i, i = 1, \ldots, n$ are values of the original unstopped process $X(t)$. Define $Z(t) = -2\gamma t + W(t)$, where $Z(t) = \log[X^2(t)/X^2(t_{11})]$ and $W(t) = \log[Y^2(t)/Y^2(t_{11})]$. The simple least-squares estimate of γ is $\tilde{\gamma} = -[\sum_{i=1}^{n}\sum_{j=1}^{M_i} t_{ij} Z(t_{ij})]/2\sum_{i=1}^{n}\sum_{j=1}^{M_i} t_{ij}^2$. Using the transformation $Y(t) = (\exp(\tilde{\gamma} t))X(t)$ and the fact that $E(Y^2(t)) = \tau^2$, a simple estimate of τ^2 is $\tilde{\tau}^2 = [\sum_{i=1}^{n}\sum_{j=1}^{M_i} Y^2(t_{ij})]/\sum_{i=1}^{n} M_i$.

We close this sub-section by stating the asymptotic properties of the maximum likelihood estimate of $\underset{\sim}{\theta}$ and $\bar{F}(t; \underset{\sim}{\theta})$, the proofs of which are similar to those in Theorem 4.1 and Corollary 4.1. For simplicity, we shall assume that $M_1 = M_2 = \cdots = M_n = M$ and the data are equally spaced.

Theorem 4.2. *With probability tending to 1 as $n \to \infty$, there exists a solution $\hat{\underset{\sim}{\theta}}$ of the likelihood equations such that (i) $\hat{\underset{\sim}{\theta}}$ converges in probability to $\underset{\sim}{\theta}$, and (ii) $\sqrt{n}(\hat{\underset{\sim}{\theta}} - \underset{\sim}{\theta})$ converges weakly to a bivariate normal with mean vector zero and covariance matrix $\Gamma(\hat{\underset{\sim}{\theta}})$, where, letting*

$$K_1(\underset{\sim}{\theta}) = \sum_{j=2}^{M}[\log g_1((j-1)\Delta, V((j-1)\Delta); j\Delta, V(j\Delta))] + \log g_1^*(V(0), 0),$$

we obtain

$$(4.13) \quad \Gamma(\underset{\sim}{\theta}) = \begin{bmatrix} -E\left(\frac{\partial^2}{\partial\gamma^2}K_1(\underset{\sim}{\theta})\right) & -E\left(\frac{\partial^2}{\partial\gamma\partial\tau}K_1(\underset{\sim}{\theta})\right) \\ -E\left(\frac{\partial^2}{\partial\gamma\partial\tau}K_1(\underset{\sim}{\theta})\right) & -E\left(\frac{\partial^2}{\partial\tau^2}K_1(\underset{\sim}{\theta})\right) \end{bmatrix}^{-1}.$$

Corollary 4.2. *As $n \to \infty$ the following hold for the reliability estimate*

$\bar{F}(t;\hat{\underset{\sim}{\theta}})$: *(i)* $\bar{F}(t;\hat{\underset{\sim}{\theta}})$ *converges to* $\bar{F}(t;\underset{\sim}{\theta})$ *in probability and (ii)* $\sqrt{n}(\bar{F}(t;\hat{\underset{\sim}{\theta}})-\bar{F}(t;\underset{\sim}{\theta}))$ *converges weakly to the normal random variable with mean 0 and variance* σ_2^2, *where using (3.8) and letting* $f_{Y(0)}(\cdot)$ *be the normal density with mean zero and variance* τ^2,

$$\bar{F}(t;\underset{\sim}{\theta}) = \int_{-\infty}^{1} H(t;y,0) f_{Y(0)}(y)dy, \quad \eta'(\underset{\sim}{\theta}) = \left[\frac{\partial \bar{F}(t;\underset{\sim}{\theta})}{\partial \gamma}, \frac{\partial \bar{F}(t,\underset{\sim}{\theta})}{\partial \tau}\right],$$

we have $\sigma_2^2 = \eta'(\underset{\sim}{\theta})\Gamma(\underset{\sim}{\theta})\eta(\underset{\sim}{\theta})$.

5. ASSESSING $\bar{F}(t)$ UNDER MULTIPLE CHARACTERISTICS

Suppose we have a system whose life time depends on more than one characteristic. Recall that the failure time of the system is given by T in (1.3) through $T_i(a_i), i = 1,\ldots,k$ in (1.2), where we now assume that

$$X_i(t) = \alpha_i(t)Y_i(t), \tag{5.1}$$

$Y_1(t),\cdots,Y_k(t)$ being separable processes, $\alpha_1(t),\alpha_2(t),\ldots,\alpha_k(t)$ being continuous, positive and non- random functions. In this general context, assessing $\bar{F}(t)$ in (1.4) is difficult, especially when the joint dependence structure of the k characteristics is not known. However, if the characteristics are modulation of a common process, $Y(t)$, then using (5.1) we have $T_i(a_i) = Inf\{t; \alpha_i(t)Y(t) \geq a_i\}, i = 1,\cdots,k$, and hence

$$P(T > t) = P(\underset{1\leq i\leq k}{Min}\ T_i(a_i) > t) = P(T^* > t),$$

where $T^* = Inf\{t : Y(t) \geq \underset{1\leq i\leq k}{Min} \frac{a_i}{\alpha_i(t)}\}$. Consequently, in this special multiple-characteristics situation, $\bar{F}(t)$ can be assessed using methods similar to those that we discussed earlier for the single-characteristic problem. Returning to the general model, suppose we can assess $P(T_i(a_i) > t), i = 1,\ldots,k$, using procedures described in previous sections. Then one can use results of this section to derive a bound for $\bar{F}(t)$. We start with a definition.

Definition 5.1. Stochastic processes $\{\eta_1(t); t \geq 0\},\cdots,\{\eta_k(t); t \geq 0\}$ are said to be associated if, for any $0 \leq t_{i1} < t_{i2} < \cdots < t_{im_i} < \infty$ $i =$

$1, \cdots, k$, the random vector $(\eta_1(t_{11}), \cdots, \eta_1(t_{im_1}), \cdots, \eta_k(t_{k1}), \cdots, \eta_k(t_{km_k}))$ is associated in the sense that for any increasing functions f and g,

$$cov(f(\eta_1(t_{11}), \cdots, \eta_1(t_{im_1}), \cdots, \eta_k(t_{k1}), \cdots, \eta_k(t_{km_k})),$$
$$g(\eta_1(t_{11}), \cdots, \eta_1(t_{im_1}), \cdots, \eta_k(t_{k1}), \cdots, \eta_k(t_{km_k}))) \geq 0,$$

provided that the covariance exists.

Theorem 5.1. *Suppose that the processes $\{Y_1(t); t \geq 0\}, \cdots, \{Y_k(t); t \geq 0\}$ given by (1.5) are associated in the sense of Definition 5.1. Then,*

$$\bar{F}(t) = P(T > t) \geq \prod_{i=1}^{k} P(T_1(a_i) > t) \tag{5.2}$$

Proof. We shall prove this result for $k = 2$, the arguments being similar for $k > 2$. Let $t > 0$ be fixed but otherwise arbitrary. Since $X_i(u), i = 1, 2$, are separable processes, one can obviously find sequences of finite sets $\{D_{in}, n \geq 1\}, i = 1, 2$, such that,

$$\begin{aligned} P(T_i(a_i) > t; i = 1, 2) &= P(\sup_{0 \leq u \leq t} X_i(u) < a_i, i = 1, 2) \\ &= \lim_{n \to \infty} P(\sup_{u \in D_{in}} X_i(u) < a_i, i = 1, 2). \end{aligned} \tag{5.3}$$

Since $\{Y_1(s); s \geq 0\}, \{Y_2(s); s \geq 0\}$ are associated processes and $\alpha_1(s) > 0$, $\alpha_2(s) > 0$, for all s, we observe that for any points s_{il} in $[0, t], l = 1, 2, \ldots, n_i$, $i = 1, 2$, the vector $(X_1(s_{11}), X_1(s_{12}), \ldots, X_1(s_{1n_1}), X_2(s_{21}), \ldots, X_2(s_{2n_2}))$ is associated. Thus, using (1.4) and (5.3), we obtain

$$P(T > t) \geq \lim_{n \to \infty} \prod_{i=1}^{2} P \sup_{u \in D_{in}} X_i(u) < a_i) = \prod_{i=1}^{2} P(T_i(a_i) > t).$$

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