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ON THE AUTOMORPHISM GROUP OF SOME SPECIAL $S^1.O(n, R)$-INVARIANT DOMAINS IN C^n

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ABSTRACT. In this note, we develop a new method to determine the automorphism group of some special $S^1.O(n, \mathbb{R})$-invariant domains in $\mathbb{C}^n$ with singular boundary, that can be interesting for more general convex domains in $\mathbb{C}^n$. In particular, it can be applied in the case of the minimal ball introduced in [4] as a solution of the problem of the minimal norm in $\mathbb{C}^n$.

1. INTRODUCTION AND RESULTS

The Riemann conformal mapping theorem asserts that a simply connected domain in $\mathbb{C}$, different from $\mathbb{C}$, is biholomorphically equivalent to the open unit disc. The situation is quite different in several complex variables : a small perturbation of the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ can be nonequivalent to $\mathbb{B}^n$, even if it is simply connected. This shows that a domain in not completely described by its topological proprieties. The structure of the automorphism group depends in an essential way of the structure of the boundary of the domain. A fundamental result showing this fact is due to Wong-Rosay ([11], [13]), stating that a strongly pseudoconvex bounded domain in $\mathbb{C}^n$ with a noncompact automorphism group is biholomorphic to the unit ball. However, the automorphism group of a strongly pseudoconvex domain may have various structure. For example, Bedford-Dadok [2] proved that every compact Lie group can be the automorphism group of a strongly pseudoconvex domain.

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For a survey of results and a rich bibliography on the subject we refer the reader to [6].

Let N_p be the function defined by :

$$N_p(z) = \left\{ (\frac{|z|^2 - |z^2|}{2})^{\frac{p}{2}} + (\frac{|z|^2 + |z^2|}{2})^{\frac{p}{2}} \right\}^{\frac{1}{p}} \quad \text{for } 1 \leq p < \infty$$

and

$$N_\infty(z) = \left(\frac{|z|^2 + |z^2|}{2} \right)^{\frac{1}{2}}$$

where $z^2 = \sum_{1 \leq j \leq n} z_j^2$. We consider the domain

$$B_p = \{z \in \mathbb{C}^n : \ N_p(z) < 1\}$$

and we denote by $Aut(B_p)$ its automorphism group. According to M. Baran [1] the function N_p is a norm in $\mathbb{C}^n$ and its dual norm defined by $N_p^*(z) = \sup\{|z.w|, \ N_p(w) \leq 1\}$ is equal to N_q with $\dfrac{1}{p} + \dfrac{1}{q} = 1$. This family contains the minimal ball introduced by M.K. Hahn and P. Pflug [4] for $p = \infty$, the Lie ball for $p = 1$ and the Euclidean ball for $p = 2$. The automorphism groups of these domains have been studied by several authors. K.T. Kim [8] proved that $Aut(B_\infty) = S^1.O(n, \mathbb{R})$ with scaling method. W. Zwolnek [14] gives an alternative proof of this result by using some geodesics in B_∞ and M. Baran [1] determined the automorphism groups of B_p for $p > 2$. See also [10] for related result (where the authors use a different approach to study the automorphism of invariant bounded domains by a compact linear group). Note that $Aut(B_p) \subset S^1.O(n, \mathbb{R})$.

Our aim in this note is to present a new method of proof to study the automorphism group of B_p that can be useful for more general convex domains. In particular, we determine the automorphism group of B_p for all $p > 1$. The main result is the following

Theorem. *For $1 < p \leq \infty$ and $p \neq 2$, $Aut(B_p) = S^1.O(n, \mathbb{R})$.*

For the case of the Lie ball and the Euclidean ball, their automorphism groups are well known, see for example [5] for the Lie ball and [12] for the Euclidean ball.

The proof is based on two results. The first one is the Kaup-Upmeir theorem [7] giving the structure of the orbit :

In every complex Banach space E with open unit ball $D \subset E$ there is a closed $\mathbb{C}$-linear subspace $V \subset E$ such that $V \cap D$ is the orbit of the origin $0 \in E$ under the action of the automorphism group of D.

The second result is the local version of Wong-Rosay's theorem proved by S. Pinchuk [9] :

The unit ball is a model for the class of C^2-strongly pseudoconvex domains at an accumulation point.

2. STRUCTURE OF B_p

In this section we present some properties of the boundary of B_p. Set

$$\mathbb{H} = \{z \in \mathbb{C}^n : z^2 = 0\} \text{ and } \mathbb{K} = \{z \in \mathbb{C}^n : |z|^2 = |z^2|\} = S^1.\mathbb{R}^n.$$

Proposition 2.1. *(a) For $1 \leq p < \infty$ the function $N_p \in C^\infty(\mathbb{C}^n \backslash \mathbb{K})$ and $N_\infty \in C^\infty(\mathbb{C}^n \backslash \mathbb{H})$.*
(b) For $1 < p < \infty$, $N_p \in C^1(\mathbb{C}^n \backslash \{0\})$.
(c) For $2 \leq p < \infty$, $N_p \in C^2(\mathbb{C}^n \backslash \{0\})$.

Proof. **(a)** Let $1 \leq p < \infty$. For $z \in \mathbb{C}^n \backslash \mathbb{K}$, the function N_p can be written as :

$$N_p(z) = 2^{\frac{1-q}{p}} |z| \left(1 + \sum\nolimits_{k \geq 1} \frac{q(q-1)...(q-2k+1)}{(2k)!} (\frac{|z^2|}{|z|^2})^{2k}\right)^{\frac{1}{p}}$$

with $q = \dfrac{p}{2}$. Hence, clearly N_p belongs to $C^\infty(\mathbb{C}^n \backslash \mathbb{K})$.

Moreover, it is easy to verify that for $z \in \mathbb{C}^n \backslash \{\mathbb{H} \cup \mathbb{K}\}$:

$$\frac{\frac{\partial N_p}{\partial z_j}(z)}{[N_p(z)]^{1-p}} = \frac{1}{4}\left\{ (\frac{|z|^2 - |z^2|}{2})^{\frac{p}{2}-1}(\bar{z}_j - \frac{\bar{z}^2 z_j}{|z^2|}) + (\frac{|z|^2 + |z^2|}{2})^{\frac{p}{2}-1}(\bar{z}_j + \frac{\bar{z}^2 z_j}{|z^2|})\right\}.$$

and for $z \in \mathbb{H}\backslash\{0\}$:

$$\frac{\partial N_p}{\partial z_j}(z) = 2^{\frac{1}{p}-\frac{3}{2}}\frac{\bar{z}_j}{|z|}.$$

For the case $p = \infty$ the computation is similar.

(b) Let $1 < p < \infty$. In view of (a), it suffices to prove that $N_p \in C^1(\mathbb{K}\backslash\{0\})$.

We have $\frac{\partial N_p}{\partial z_j}(w) - \frac{1}{4|z|}(\bar{z}_j + \frac{\bar{z}^2 z_j}{|z^2|}) \to 0$ as $w \to z \in \mathbb{K}\backslash\{0\}$, then $N_p \in C^1(\mathbb{C}^n\backslash\{0\})$ and $\frac{\partial N_p}{\partial z_j}(z) = \frac{1}{4|z|}(\bar{z}_j + \frac{\bar{z}^2 z_j}{|z^2|})$ for all $z \in \mathbb{K}\backslash\{0\}$.

(c) Let now $2 \leq p < \infty$. We denote by $u(z) = |z|^2 + |z^2|$, $v(z) = |z|^2 - |z^2|$ and $q = \frac{p}{2}$. Thus, we have $N_p = (\frac{u^q + v^q}{2^q})^{\frac{1}{p}}$.

Since the function $u^q \in C^\infty(\mathbb{C}^n\backslash\{0\})$, it suffices to verify that $v^q \in C^2(\mathbb{C}^n\backslash\{0\})$. A simple computation gives :

$$\frac{\partial^2 v^q}{\partial z_k \partial z_j}(z) = q(q-1)(|z|^2 - |z^2|)^{q-2}(\bar{z}_j - \frac{\bar{z}^2 z_j}{|z^2|})(\bar{z}_k - \frac{\bar{z}^2 z_k}{|z^2|})$$

$$+q(|z|^2 - |z^2|)^{q-1}\left(\frac{z_j.z_k(\bar{z}^2)^2}{|z^2|^3} - \delta_j^k \frac{\bar{z}^2}{|z^2|}\right)$$

$$\frac{\partial^2 v^q}{\partial \bar{z}_k \partial z_j}(z) = q(q-1)(|z|^2 - |z^2|)^{q-2}(\bar{z}_j - \frac{\bar{z}^2 z_j}{|z^2|})(z_k - \frac{z^2 \bar{z}_k}{|z^2|})$$

$$+q(|z|^2 - |z^2|)^{q-1}(\delta_j^k - \frac{z_j.\bar{z}_k}{|z^2|}).$$

Since

$$\sum_{j,k}(\bar{z}_j - \frac{\bar{z}^2 z_j}{|z^2|})(z_k - \frac{z^2 \bar{z}_k}{|z^2|}) = \left|\sum_{j=1}^{n}(\bar{z}_j - \frac{\bar{z}^2 z_j}{|z^2|})\right|^2 \leq 2(|z|^2 - |z^2|)$$

and $$\left|\sum_{j,k}(\bar{z}_j - \frac{\bar{z}^2 z_j}{|z^2|})(\bar{z}_k - \frac{\bar{z}^2 z_k}{|z^2|})\right| = \left|\left(\sum_{j=1}^{n} \bar{z}_j - \frac{\bar{z}^2 z_j}{|z^2|}\right)^2\right| \leq$$

$$\left|\sum_{j=1}^{n}(\bar{z}_j - \frac{\bar{z}^2 z_j}{|z^2|})\right|^2 \leq 2(|z|^2 - |z^2|),$$

it follows that N_p is C^2-smooth for $2 \leq p < \infty$. □

Remark. Note that if $p \in 2\mathbb{N}$, the function N_p is smooth every where except the origin.

For $1 \leq p \leq \infty$, we denote by $\partial\mathbb{H}_p = \mathbb{H} \cap \partial B_p$ and $\partial\mathbb{K}_p = \partial\mathbb{K} = \mathbb{K} \cap \partial B_p$. Next, we give a description of the boundary of B_p.

Proposition 2.2. *(a) The set of strongly pseudoconvex boundary points of B_p is equal to $\partial B_p \backslash \partial\mathbb{K}$ for $1 < p < 2$ and is equal to $\partial B_\infty \backslash \partial\mathbb{H}_\infty$ for $p = \infty$.*
(b) For $2 \leq p < \infty$, B_p is a strongly pseudoconvex domain.
(c) For $p = 1$, there is no strongly pseudoconvex boundary point.

Proof. **(a)** Let $u(z) = |z|^2 + |z^2|$, $v(z) = |z|^2 - |z^2|$, $q = \frac{p}{2}$, $k = \frac{z^2}{|z^2|}$, $\lambda = \frac{u}{v}$ and $\rho_q(z) = u(z)^q + v(z)^q$. Then we have $N_p = (\frac{\rho_q}{2^q})^{\frac{1}{p}}$.

Let $z \in \partial B_p \backslash \{\partial\mathbb{H}_p \cup \partial\mathbb{K}\}$. The complex Hessian of ρ_q at (z, ξ) is given by the formula :

$$L(\rho_q)(z,\xi) = q(q-1)\left\{[v(z)]^{q-2}|\sum_{j=1}^{n}\frac{\partial v(z)}{\partial z_j}\xi_j|^2 + [u(z)]^{q-2}|\sum_{j=1}^{n}\frac{\partial u(z)}{\partial z_j}\xi_j|^2\right\}+$$

$$+q\left\{[v(z)]^{q-1} + |u(z)|^{q-1}\right\}|\xi|^2 + q|z^2|^{-1}\left\{[u(z)]^{q-1} - [v(z)]^{q-1}\right\}| < \xi|\bar{z} > |^2$$

where $< \xi/\bar{z} >= \sum_j \xi_j z_j$.

The complex tangent space to ∂B_p at z is defined by :

$$T_z^c(\partial B_p) = \left\{\xi \in \mathbb{C}^n :< \xi|u(z)^{q-1}(z+k\bar{z}) + v(z)^{q-1}(z-k\bar{z}) >= 0\right\}.$$

An element ξ belongs to $T_z^c(\partial B_p)$ if and only if it satisfies the equation:

$$< \xi|z >= \bar{k}\frac{1-\lambda^{q-1}}{1+\lambda^{q-1}} < \xi|\bar{z} > .$$

On the complex tangent space, we have :

$$\sum_j \frac{\partial u}{\partial z_j}(z)\xi_j = \frac{2\bar{k}}{1+\lambda^{q-1}} < \xi|\bar{z} >$$

$$\sum_j \frac{\partial v}{\partial z_j}(z)\xi_j = \frac{-2\bar{k}\lambda^{q-1}}{1+\lambda^{q-1}} < \xi|\bar{z} > .$$

After simplification, the Levi-form of ρ_q at (z,ξ) is defined by the formula :

$$L(\rho_q)(z,\xi) = qv^{q-1}(1+\lambda^{q-1})|\xi|^2 +$$

$$+2qv^{q-2}\left[\frac{\lambda^{q-1}-1}{\lambda-1} + 2(q-1)\frac{\lambda^{2q-2}+\lambda^{q-2}}{(1+\lambda^{q-1})^2}\right]| < \xi|\bar{z} > |^2.$$

As $\sigma = \dfrac{\lambda^{q-1}-1}{\lambda-1} + 2(q-1)\dfrac{\lambda^{2q-2}+\lambda^{q-2}}{(1+\lambda^{q-1})^2}$ is negative, we have :

$$\inf_{(\xi\in T_z^c(\partial B_p),|\xi|=1)} L(\rho_q)(z,\xi) = qv^{q-1}\left[(1+\lambda^{q-1}) + \frac{2\sigma}{v}\max_{\xi\in T_z^c(\partial B_p)} \frac{|<\xi|\bar{z}>|^2}{|\xi|^2}\right].$$

We compute $\max_{\xi\in T_z^c(\partial B_p)} \frac{|<\xi|\bar{z}>|^2}{|\xi|^2}$. This quantity is the square of the norm of the projection of $\bar{z}$ on $T_z^c(\partial B_p)$. Since $T_z^c(\partial B_p)$ is the orthogonal of the vector $(\lambda^{q-1}+1)z + k(\lambda^{q-1}-1)\bar{z}$, a simple computation gives that

$$\max_{\xi\in T_z^c(\partial B_p)} \frac{|<\xi|\bar{z}>|^2}{|\xi|^2} = -\lambda\frac{(\lambda^{q-1}+1)^2}{2(\lambda^{2q-1}+1)}v.$$

It follows that :

$\inf_{(\xi\in T_z^c(\partial B_p),|\xi|=1)} L(\rho_q)(z,\xi)$

$$= qv^{q-1}(1+\lambda^{q-1})\left[1+\frac{\lambda(\lambda^{q-1}+1)}{\lambda^{2q-1}+1}\left(\frac{\lambda^{q-1}-1}{\lambda-1}+2(q-1)\frac{\lambda^{2q-2}+\lambda^{q-2}}{(1+\lambda^{q-1})^2}\right)\right]$$

$$= qv^{q-1}\frac{(\lambda^q+1)(\lambda^{q-1}-1)}{\lambda^{2q-1}+1}\left[\frac{(2q-1)\lambda^{q-1}}{1+\lambda^{q-1}}+\frac{\lambda^{2q-1}-1}{(1+\lambda^{q-1})(\lambda-1)}\right].$$

Thus, $L(\rho_q)(z,\xi)$ is bounded below by a strictly positive expression for any $q>\frac{1}{2}$. For $q=\frac{1}{2}$ this expression is zero, then B_1 has no strict pseudoconvexity point (in (c) we will give another proof).

Let now z be a point of $\partial\mathbb{H}_p$. The complex tangent space at z is defined by :

$$T_z^c(\partial B_p) = \left\{\xi\in\mathbb{C}^n : \sum_{j=1}^n z_j\bar{\xi_j} = 0\right\}$$

and the Levi-form of ρ_q at (z,ξ) is given by:

$$L(\rho_q)(z,\xi) = \frac{p}{2^{\ p/2}}|z|^{p-4}\left\{(p-1)|z\circ\xi|^2 + |z|^2|\xi|^2 - |z\circ\xi|^2\right\}.$$

By using the Cauchy-Schwarz inequality, we conclude that the points of $\partial\mathbb{H}_p$ are strongly pseudoconvex for $1 < p < \infty$.

For $p = \infty$, a simple computation of the Levi-form shows that $\partial B_\infty \backslash \partial\mathbb{H}_\infty$ is strongly pseudoconvex.

(b) In view of (a), to show that the boundary of B_p is strongly pseudoconvex for $2 \leq p < \infty$, it suffices to prove it for points in $\partial\mathbb{K}$. The complex tangent space at a point z of $\partial\mathbb{K}$ is given by :

$$T_z^c(\partial B_p) = \{\xi \in \mathbb{C}^n :< \xi | z + k\bar{z} >= 0\}.$$

Then ξ belongs to $T_z^c(\partial B_p)$ if and only if $< \xi|z >= -\bar{k} < \xi|\bar{z} >$. After simplification the Levi-form of ρ_q at a point (z, ξ) of $\partial\mathbb{K} \times T_z^c(\partial B_p)$ is given by the formula :

$$L(\rho_q)(z,\xi) = q|z|^{p-2}(|\xi|^2 + \frac{|< \xi|\bar{z} >|^2}{|z^2|})$$

which is strictly positive for $\xi \neq 0$. This finishes the proof of (b).

(c) The Lie ball is a homogeneous domain [5] which is not biholomorphic to Euclidean ball of $\mathbb{C}^n$. Then in view of the Wong-Rosay theorem it follows that its boundary contains no strict pseudoconvexity points. □

By using the structure of the boundary of B_p, we deduce the following :

Proposition 2.3. *For $p \neq q$, the domains B_p and B_q are biholomorphically inequivalent. In particular B_p is not biholomorphic to the Euclidean ball of $\mathbb{C}^n$ for $p \neq 2$.*

Since $\mathbb{K} = S^1.\mathbb{R}^n$, obviously we have the following

Lemma 2.1. *If $V \subset \mathbb{K}$ is a real-subspace of dimension m, then there exists a real number α such that $e^{-i\alpha}V$ is a subspace of $\mathbb{R}^m$.*

Proof of proposition 2.3. Assume that B_p and B_q are biholomorphic. Since B_p and B_q are bounded, circular, convex domains, according to [3], there exists a linear automorphism L transforming B_p into B_q. We consider several cases:

• Assume that $1 \leq p \leq \infty$ and $q = 2$. As in [1], we consider the points

$$z_0 = 2^{-\frac{1}{p}}(e_1 + ie_2), \quad z_1 = 2^{-\frac{1}{p}}(e_1 - ie_2)$$

where $e_1 = (1, 0..., 0)$ and $e_2 = (0, 1, 0.., 0)$. It is clear that z_0 and z_1 lie in ∂B_p. A simple computation shows that $2 = |L(z_0)|^2 + |L(z_1)|^2 = 2.2^{-\frac{2}{p}}(|L(e_1)|^2 + |L(e_2)|^2) = 4.2^{-\frac{2}{p}}$. Hence, $p = 2$.

• Assume that p and q are real numbers in $[1, \infty[\setminus\{2\}$. According to [1], the conjugate balls $B_p^* = \{N_p^* < 1\}$ and $B_q^* = \{N_q^* < 1\}$ are linearly equivalent. Then, without loss of generality we may assume that $p \notin 2\mathbb{N}$. If $q \in 2\mathbb{N}$, the result is obvious. Hence we may assume that p and q are not in $2\mathbb{N}$. Since the boundary of B_p and B_q are smooth only outside $\partial\mathbb{K}$, we obtain $L(\partial\mathbb{K}) = \partial\mathbb{K} \subset \mathbb{K}$. By linearity $L(\mathrm{\mathbb{R}^n})$ is a vector space of dimension n include in $\mathbb{K}$. According to lemma 2.1, there exists a real number α such that $L(\mathrm{\mathbb{R}^n}) = \mathrm{e}^{i\alpha}\mathrm{\mathbb{R}^n}$. The mapping $h = e^{-i\alpha}L : \mathrm{\mathbb{R}^n} \to \mathrm{\mathbb{R}^n}$ verifies $||h(x)|| = ||x||$ for all x in $\mathrm{\mathbb{R}^n}$; then $h \in O(n, \mathbb{R})$ and $L \in S^1.O(n, \mathbb{R})$. But $S^1.O(n, \mathbb{R}) \subset \mathrm{Aut(B_p)}$, therefore $B_p = B_q$ and $p = q$.

• Assume that $q = \infty$ and p is real in $[1, \infty[\setminus\{2\}$. (Note that in view of the regularity of the boundary, it is clear the B_2 and B_∞ are not biholomorphic.) Since the boundary of B_∞ is smooth only outside $\partial\mathbb{H}_\infty$ and the boundary of B_p is smooth only outside $\partial\mathbb{K}$, we have $L(\partial\mathbb{K}) = \partial\mathbb{H}_\infty$ and hence $L(\mathbb{K}) = \mathbb{H}$. This is impossible; since $\mathbb{K}$ is a smooth manifold and $\mathbb{H}$ is a manifold with singularity at 0. □

3. PROOF OF THE THEOREM

Suppose that $p \neq \{1, 2\}$. First we will show that 0 is a fixed point of any automorphism of B_p. We denote by Orb(0) the orbit of 0 under the action of $Aut(B_p)$. According to [7], there exists a complex vector subspace V such that $\mathrm{Orb}(0) = V \cap B_p$. For $2 < p < \infty$, B_p is strongly pseudoconvex and not biholomorphic to the unit ball. Then by the Wong-Rosay theorem, $V = \{0\}$.

Now, let $p \in]1,2[\cup\{\infty\}$. Assume that $V \neq \{0\}$ and let $z \in V\backslash\{0\}$. The point $z/N_p(z)$ belongs to $V \cap \partial B_p$. Since B_p is strongly pseudoconvex only outside $\partial\mathbb{K}$ for $1 < p < 2$ and only outside $\partial\mathbb{H}_\infty$ for $p = \infty$, then again the Wong-Rosay theorem implies that $z/N_p(z) \in \partial\mathbb{K}$ for $1 < p < 2$ and $z/N_\infty(z) \in \partial\mathbb{H}_\infty$. Thus $V \subset \mathbb{K}$ for $1 < p < 2$ and $V \subset \mathbb{H}$ for $p = \infty$.
Let $z_o = a + ib \in \text{Orb}(0)\backslash\{0\}$. Consider the dimension of the orbit of z_o under the action of $S^1.O(n,\mathbb{R})$, denoted by E.

• Case 1 : $z_o \in \mathbb{H}$ (i.e., $|a| = |b|$ and $< a|b >= 0$). The orbit E of z_o under the action of $S^1.O(n,\mathbb{R})$ is a manifold of dimension $2n-3$. Indeed, E is homeomorphic to $S^1.O(n,\mathbb{R})/G_{z_o}$, where $G_{z_o} = \{f \in S^1.O(n,\mathbb{R}) : f(z_o) = z_o\}$. Let α be a real number and M be an orthogonal real matrix in $O(n,\mathbb{R})$. The equation $e^{i\alpha}M.z_o = z_o$ is equivalent to the system defined by :

$$\begin{cases} Ma &= \quad acos\alpha + bsin\alpha \\ Mb &= - asin\alpha + bcos\alpha \end{cases}$$

Let B $= (\frac{a}{|a|}, \frac{b}{|a|}, ...)$ be an orthogonal basis in $\mathbb{C}^n$. The matrix of M in the basis B has the form $\begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$ where $A = \begin{pmatrix} cos\alpha & sin\alpha \\ -sin\alpha & cos\alpha \end{pmatrix}$ and $C \in O(n-2,\mathbb{R})$. Then the group G_{z_o} can be identified to $S^1.O(n-2,\mathbb{R})$ and consequently

$$\dim E = 1 + \frac{n(n-1)}{2} - \left[1 + \frac{(n-2)(n-3)}{2}\right] = 2n - 3.$$

Thus the orbit of z_o under the action $S^1.O(n,\mathbb{R})$ is a manifold of dimension $2n-3$.

• Case 2 : $z_o \in \mathbb{K}$. In this case G_{z_o} has two connected components. These components can be identified to $O(n-1,\mathbb{R})$. Consequently

$$\dim E = 1 + \frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} = n.$$

Thus the orbit of z_o under the action $S^1.O(n,\mathbb{R})$ is a manifold of dimension n.

We need the following lemma whose proof is deferred until the end of this section.

Lemma 3.1. *Every complex vector subspace contained in* $\mathbb{H}$ *has dimension less or equal to* $n/2$ *and every complex vector subspace contained in* $\mathbb{K}$ *is either trivial or a complex line.*

By considering dimension of the orbit we obtain :

• If $z_o \in \mathbb{K}$, then $dimE \leq 2$, i.e. $n \leq 2$. In the case $n = 2$, the consideration of the dimension shows that the orbit of z_o under the action of $S^1.O(2,\mathbb{R})$ coincides with the Orb(0). Therefore there exist $\alpha \in \mathbb{R}$ and $g \in O(2,\mathbb{R})$ such that $e^{i\alpha}g(z_o) = 0$; it is a contradiction with the fact that $z_o \neq 0$.

• If $z_o \in \mathbb{H}$ then $dimE \leq [\frac{n}{2}]$. More precisely : $2n - 3 \leq [\frac{n}{2}]$, i.e. : $n \leq 2$. For $n = 2$, $\mathbb{H} = \{(\mathrm{z},\mathrm{w}) \in \mathbb{C}^2 : \mathrm{z} - \mathrm{iw} = 0\} \cup \{(\mathrm{z},\mathrm{w}) \in \mathbb{C}^2 : \mathrm{z} + \mathrm{iw} = 0\}$. As V is a vector space include in $\mathbb{H}$, then it is equal to one of these complex lines. It follows that the orbit of z_o under the action of $S^1.O(2,\mathbb{R})$ coincides with Orb(0). This is impossible as it was shown in the previous case. Thus, Orb(0) $= \{0\}$.

Now, by the classical Cartan's theorem, we deduce that any automorphism of B_p is $\mathbb{C}$-linear. To finish the proof of the theorem, we use the following lemma:

Lemma 3.2. *Any* $\mathbb{C}$*-linear automorphism preserving* B_p $1 < p \leq \infty$, $p \neq 2$ *belongs to* $S^1.O(n,\mathbb{R})$.

Proof. Since the boundary of B_p is not smooth on $\partial\mathbb{H}_\infty$ for $p = \infty$ and not smooth on $\partial\mathbb{K}$ for $1 < p < \infty$ ($p \neq 2$), the automorphisms of B_p preserve $\partial\mathbb{H}_\infty$ for $p = \infty$ and $\partial\mathbb{K}$ for $1 < p < \infty$ ($p \neq 2$). Let g be a $\mathbb{C}$-linear automorphism. If $g(\partial\mathbb{K}) = \partial\mathbb{K}$, then $g(\mathbb{K}) = \mathbb{K}$. In this case, as $g(\mathbb{R}^{\mathrm{n}})$ is a vector space of dimension n included in $\mathbb{K}$, according to lemma 1, there exists a real α such that $g(\mathbb{R}^{\mathrm{n}}) = \mathrm{e}^{\mathrm{i}\alpha}\mathbb{R}^{\mathrm{n}}$. The mapping h defined by $h = e^{-i\alpha}g$ verifies $||h(x)|| = ||x||$ for all $x \in \mathbb{R}^{\mathrm{n}}$, then $h \in O(n,\mathbb{R})$ and $g \in S^1.O(n,\mathbb{R}^{\mathrm{n}})$.
For the case $g(\partial\mathbb{H}_\infty) = \partial\mathbb{H}_\infty$, let $(e_j)_{1\leq j\leq n}$ be the canonical basis of $\mathbb{R}^{\mathrm{n}}$. For $j \neq k$ the vectors $e_j + ie_k$ and $e_j - ie_k \in \partial\mathbb{H}_\infty$, since g preserves $\partial\mathbb{H}_\infty$, we have:

$||g(e_j + ie_k)||^2 = ||g(e_j - ie_k)||^2 = 2$ and $g(e_j + ie_k)^2 = g(e_j - ie_k)^2 = 0$, i.e. :

$$||g(e_j)||^2 + ||g(e_k)||^2 = 2,\ Re < ig(e_k)|g(e_j) >= 0$$

and

$$g(e_j)^2 = g(e_k)^2 \text{ and } < g(e_j)|\overline{g(e_k)} >= 0.$$

Since e_j and e_k are in the boundary of B_∞, $g(e_j)$ and $g(e_k)$ belong also to ∂B_∞, i.e. : $||g(e_j)||^2 + ||g(e_j)^2|| = 2$ and $||g(e_k)||^2 + ||(g(e_k)^2|| = 2$.

Then $||g(e_j)|| \geq 1$ and $||g(e_k)|| \geq 1$; thus

$$||g(e_j)|| = ||g(e_k)|| = |g(e_j)^2| = |g(e_k)^2| = 1 \tag{3.1}$$

$$Im < g(e_k)|g(e_j) >= 0 \tag{3.2}$$

$$g(e_j)^2 = g(e_k)^2 \tag{3.3}$$

$$< g(e_j)|\overline{g(e_k)} >= 0 \tag{3.4}$$

The equation (3.1) shows that $g(e_j)$ and $g(e_k)$ belong to $\mathbb{K}$, then there exist then real numbers α_j and α_k such that the vectors $e^{-i\alpha_j}g(e_j)$ and $e^{-i\alpha_k}g(e_k)$ belong to $\mathbb{R}^n$. From (3.3), we have $e^{2i\alpha_j} = e^{2i\alpha_k}$ and we can assume then the existence of a real α and a family $(b_j)_j$ of vectors of $\mathbb{R}^n$ such that $g(e_j) = e^{i\alpha}b_j$ for all $j \in \{1,..,n\}$. The equation (3.4) involves that $< b_j|b_k >= 0$. Thus the family $(b_j)_j$ is an orthogonal family and the mapping $e^{-i\alpha}g$ belongs to $O(n, \mathbb{R})$. □

We conclude this paper with the proof of lemma 3.1. Let F be a complex subspace of $\mathbb{H}$ and let $(e_1, ..., e_m)$ be a basis of F. Then $(e_i + e_j) \circ (e_i + e_j) = 0$ for all $i \neq j$, i.e., $e_i \circ e_j = 0$. It follows that the family $(e_1, ...e_m, \bar{e}_1, ..., \bar{e}_m)$ is independent in $\mathbb{C}^n$ ($\bar{e}_j$ denotes the conjugate of the vector e_j). Then necessary $2m \leq n$. Now, let F be a complex vector subspace of $\mathbb{K}$ and $x, y \in F$. Since $\mathbb{K} = S^1.\mathbb{R}^n$, there exist λ_1, $\lambda_2 \in S^1$ and $a, b \in \mathbb{R}^n$ such that $x = \lambda_1 a$ and $y = \lambda_2 b$. Since F is a complex vector space, it follows that $a + ib \in F$. Then $a + ib = \lambda_3 c$ for a certain $\lambda_3 \in S^1$ and $c \in \mathbb{R}^n$. Thus x and y are dependent and so the complex dimension of F is ≤ 1.

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