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$A(\infty)$-SIMULTANEOUS RATIONAL APPROXIMATION

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ABSTRACT. We consider the simultaneous approximation of finite number of functions by a rational function using a mixture of Chebyshev norms. Existence and uniqueness results are obtained, and a particular characterization of the best simultaneous approximation from a set of rational functions is given.

1. INTRODUCTION

The simultaneous approximation problem of finite number of functions using a mixture of Chebyshev norms is equivalent to that of two functions (see Remark 2). Therefore we consider here the problem of approximating two functions by a rational function, whose setting is as follows. Let $C[a, b]$ be the space of all real-valued continuous functions defined on $[a, b]$. For $f \in C[a, b]$, let $||f|| = \max\{|f(x)| : x \in [a, b]\}$, $W = C[a, b] \times C[a, b]$, and for $F = (F_1, F_2) \in W$, consider the following norm:

$$||F||_{A(\infty)} = \max\{||F_1||, ||F_2||\}.$$

Let S be a set of rational functions defined as follows:

$S = \{p/q : p \in P_n, q \in Q_m, p/q$ is irreducible, and $q > 0$ on $[a, b]\}$ where P_n denotes the set of algebraic polynomials of degree $\leq n$, Q_m is the set of algebraic polynomials of degree $\leq m$, and $U = \{(s, s) : s \in S\}$. Assume that for each $F \in W$, there exists $u^* = (s^*, s^*)$ with $s^* \in S$ such that

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$$\begin{aligned}
\|F - u^*\|_{A(\infty)} &= \inf_{u \in U} \|F - u\|_{A(\infty)} \\
&= \inf_{s \in S} \max \{\|F_1 - s\|, \|F_2 - s\|\} \\
&= \|F_k - s^*\|, k \in A^*,
\end{aligned}$$

where $A^* \subset \{1, 2\}$.

Such an s^* is called a best $A(\infty)$-simultaneous approximation to F_1 and F_2 from S. The set of best $A(\infty)$-simultaneous approximations will be denoted by $P_S(F, \infty)$. In addition we will let $P_S(F_k)$ denote the usual best L_∞ approximation to F_k from S, $k = 1, 2$. the problem of minimizing $\|F\|_{A(\infty)}$ was considered by Dunham [5], who obtained a particular characterization of solutions for the case when S is a Haar subspace, which was extended by Asiry and Watson [2] to a weak Haar subspace. In this paper these results are extended to a set of rational functions. We also obtain an existence and a uniqueness result.

In recent work of Li and Watson [7] they characterized the best approximation from a general sunset, and our characterization result can be viewed as an application of that result provided we know that S is a sunset. However to show that S is a sunset we need the characterization result. As by product of our analysis we show that S is a sunset for simultaneous approximation.

2. EXISTENCE, CHARACTERIZATION, AND UNIQUENESS

We first present an existence result

Theorem 1. *For each $F \in W$, there exists at least one best $A(\infty)$-siultaneous approximation to F from S.*

Proof. Let $d = \text{dist}(F, U) = \inf_{u \in U} \|F - u\|_{A(\infty)}$, and let $\{sk\} \subset S$ be a minimizing sequence, that is $\|F - u_k\|_{A(\infty)} \to d$, where $u_k = (s_k, s_k)$. Then there exists a subsequence of $\{s_k\}$ (which we will not rename) such that $\|F_1 - s_k\| \geq \|F_2 - s_k\|$ or $\|F_2 - s_k\| \geq \|F_1 - s_k\|$ for all k.

Without loss of generality let us assume that $\|F_1 - s_k\| \geq \|F_2 - s_k\|$ for all k. We may write $s_k = p_k/q_k$ with $\|q_k\| = 1$. Going to a subsequence if necessary we may assume that $\|F_1 - s_k\| \leq d{+}1$ for all k. Consequently

$$\|s_k\| \leq \|F_1 - s_k\| + \|F_1\| \leq d + 1 + \|F_1\| \equiv \gamma.$$

Since

$$|p_k(x)| = |q_k(x)|\,|s_k(x)| \leq \|q_k(x)\|\,\|s_k(x)\| \leq \gamma,$$

the sequence $\{(p_k, q_k)\}$ lies in a compact set defined by the inequalities $\|p\| \leq \gamma$ and $\|q\| = 1$. Going to a subsequence if necessary we may assume that $p_k \to p \in P_n$ and $q_k \to q \in Q_n$. Clearly $\|q\| = 1$ and q can be have at most m zeros at points $x_i \in [a, b]$. Thus $p(x)/q(x)$ is well defined and $p_k/q_k \to p/q$ on $Y = [a, b] - \cup\{x_i\}$. This implies that $|p(x)| \leq \gamma|q(x)|$ for all x in Y, and by continuity this inequality is valid for all x in $[a, b]$. Thus any zero of q in $[a, b]$ is also a zero of p, and the linear factor corresponding to it may be canceled from p and q. The removal of such a linear factor does not affect the previous inequality, and so we may repeat this cancellation process until q is free of zeros on $[a, b]$. Let $s^* \in S$ be the resulting element. Thus $s_k \to s^*$ and $\|F_1 - s^*\| = d$. Clearly $\|F_1 - s^*\| \geq \|F_2 - s^*\|$, which implies that $s^* \in P_S(F, \infty)$.

For the characterization we need more definitions, so for $F = (F_1, F_2) \in W$ and $s \in S$, define the sets

$$\begin{aligned} E_1 &= \{x \in [a, b] : |F_1(x) - s(x)| = \|F_1 - s\|\}, \\ E_2 &= \{x \in [a, b] : |F_2(x) - s(x)| = \|F_2 - s\|\}, \end{aligned}$$

$$E = \{x \in [a, b] : |F_1(x) - s(x)| = \|F - u\|_{A(\infty)}$$

or

$$|F_2(x) - s(x)| = \|F - u\|_{A(\infty)}\},$$

where $u = (s, s)$. Clearly, $E = E_1, E = E_2$ or $E = E_1 \cup E_2$. Also for each $s \in S$, let

$$H_s = \{p + sq; p \in P_n \text{ and } q \in Q_m\}.$$

Replacing s by s^* in the above definitions the corresponding sets will be denoted by E^*, E_1^*, E_2^*, and H_{s^*}. Furthermore, let

$$\begin{aligned} \sigma_1(x) &= sign(F_1 - s^*)(x) \\ \sigma_2(x) &= sign(F_2 - s^*)(x) \end{aligned}$$

Theorem 2. *Let $F = (F_1, F_2) \in W \backslash U$. Then $s^* \in S$ is a best $A(\infty)$ approximation to F if and only if one of the following conditions is satisfied:*

(a) *If $E^* = E_1^*$, then* $\underset{x \in E_1^*}{Min}\, \sigma_1(x)\varphi(x) \leq 0$, for all $\varphi \in H_{s^*}$

(b) *If $E^* = E_2^*$, then* $\underset{x \in E_2^*}{Min}\, \sigma_2(x)\varphi(x) \leq 0$, for all $\varphi \in H_{s^*}$

(c) *If $E^* = E_1^* \cup E_2^*$, then*

$$Min\, \{\sigma_1(x)\varphi(x) : x \in E_1^*\} \cup \{\sigma_2(x)\varphi(x) : x \in E_2^*\} \leq 0, \quad \text{for all } \varphi \in H_{s^*}$$

Proof. If $s^* \in S$ is not a best $A(\infty)$ approximation to F , then there exists an $s_0 \in S$ such that:

$$\|F - u_0\|_{A(\infty)} < \|F - u^*\|_{A(\infty)}$$

where, $u_0 = (s_0, s_0)$ and $u^* = (s^*, s^*)$. Let $\varphi = q_0(s_0 - s^*)$. Then $\varphi \in H_{s^*}$. With out loss of generality, assume $\|F - u_0\|_{A(\infty)} = \|F_1 - s_0\|$.

If $\|F - u^*\|_{A(\infty)} = \|F_1 - s^*\| > \|F_2 - s^*\|$, that is $E^* = E_1^*$, then for each $x \in E_1^*$ we have;

$$\begin{aligned} \sigma_1(x)(F_1 - s_0)(x) &\leq \|F_1 - s_0\| \\ &< \|F_1 - s^*\| = \sigma_1(x)(F_1 - s^*)(x). \end{aligned}$$

Thus $\sigma_1(x)(s_0 - s^*)(x) > 0$, and

(1) $$\sigma_1(x)\varphi(x) > 0, \text{ for all } x \in E_1^*,$$

which implies that (a) is not satisfied.

If $\|F - u^*\|_{A(\infty)} = \|F_2 - s^*\| > \|F_1 - s^*\|$, that is $E^* = E_1^*$, then for each $x \in E_1^*$ we have;

$$\begin{aligned} \sigma_2(x)(F_1 - s_0)(x) &\leq \|F_2 - s_0\| \leq \|F_1 - s_0\| \\ &< \|F_2 - s^*\| = \sigma_2(x)(F_2 - s^*)(x). \end{aligned}$$

Thus $\sigma_2(x)(s_0 - s^*)(x) > 0$, and

$$\sigma_2(x)\varphi(x) > 0, \text{ for all } x \in E_2^*, \tag{2}$$

which implies that (b) is not satisfied.

And, if $\|F - u^*\|_{A(\infty)} = \|F_1 - s^*\| = \|F_2 - s^*\|$, that is $E^* = E_1^* \cup E_2^*$, then inequalities (1) and (2) hold simultaneously, which implies that (c) is not satisfied.

For the converse, assume first that $\|F - u^*\|_{A(\infty)} = \|F_1 - s^*\| > \|F_2 - s^*\|$, that is $E^* = E_1^*$, and let $\varphi_0 \in H_{s^*}$ such that $\sigma_1(x)\varphi_0(x) > 0$ for all $x \in E_1^*$.

Write $\varphi_0 = p_0 + s^* q_0, s^* = p^*/q^*$, and

$$s_\lambda = \frac{p^* + \lambda p_0}{q^* - \lambda q_0},$$

where $\lambda > 0$ is chosen so that $q^* - \lambda q_0 > 0$ on $[a, b]$. Define

$$\delta = \inf_{x \in E_1^*} \sigma_1(x)\varphi_0(x).$$

By continuity and compactness $\delta > 0$. Let $e_1 = F_1 - s^*$, and define sets

$$\begin{aligned} Y &= \left\{ x \in [a, b] : \sigma_1(x)\varphi(x) > \frac{1}{2}\delta \text{ and } |e_1(x)| > \frac{1}{2}\|e_1\| \right\}, \\ Z &= [a, b] \backslash Y \end{aligned}$$

It is clear that Y contains E_1^*, and Z is a compact set. Hence there is a number γ such that for each $x \in Z$ we have;

$$\begin{aligned} |F_1(x) - s_\lambda(x)| &\le |F_1(x) - s^*(x)| + |s^*(x) - s_\lambda(x)| \\ &\le \gamma + \|s^* - s_\lambda\|. \end{aligned}$$

Since $\|s^* - s_\lambda\| \to 0$ as $\lambda \to 0$, the last term is less than $\|e_1\|$ for sufficiently small λ.

Now taking λ small enough so that $F_1 - s_\lambda$ has the same sign as $F_1 - s^*$ on Y, then for $x \in Y$ we have

$$\begin{aligned} |F_1(x) - s_\lambda(x)| &= \sigma(x)(F_1 - s^*)(x) + \sigma(x)(s^* - s_\lambda)(x) \\ &\leq \|e_1\| - \frac{\lambda\sigma_1(x)\varphi(x)}{(q - \lambda q_0)(x)} \\ &\leq \|e_1\| - \frac{\lambda\delta}{2\|q - \lambda q_0\|} < \|e_1\| . \end{aligned}$$

If $\|F_2 - s_\lambda\| \geq \|e_1\|$, then taking the limit as $\lambda \to 0$ implies that $\|F_2 - s^*\| \geq \|F_1 - s^*\|$ which is a contradiction, thus

$$\|F - u_\lambda\|_{A(\infty)} < \|F - u^*\|_{A(\infty)} ,$$

If $\|F - u^*\|_{A(\infty)} = \|F_2 - s^*\| > \|F_1 - s^*\|$, that is $E^* = E_2^*$, and there exists a $\varphi_0 \in H_{s^*}$ such that $\sigma_2(x)\varphi_0(x) > 0$ for all $x \in E_2^*$. Then proceeding exactly the same as the first case where e_1 is replaced by $e_2 = F_2 - s^*$, and $\delta = \inf_{x \in E_2^*} \sigma_2(x)\varphi_0(x) > 0$, we get

$$\|F - u_\lambda\|_{A(\infty)} < \|F - u^*\|_{A(\infty)} ,$$

for small λ.

Finally, if $\|F - u^*\|_{A(\infty)} = \|F_1 - s^*\| = \|F_2 - s^*\|$, that is $E^* = E_1^* \cup E_2^*$, and there exists a $\varphi_0 \in H_{s^*}$ such that $\sigma_1(x)\varphi_0(x) > 0$ for all $x \in E_1^*$ and $\sigma_2(x)\varphi_0(x) > 0$, for all $x \in E_2^*$. Then from the above arguments, if λ is small enough, we have

$$\|F_1 - s_\lambda\| < \|F_1 - s^*\| = \|F_2 - s^*\| ,$$

and

$$\|F_2 - s_\lambda\| < \|F_2 - s^*\| = \|F_1 - s^*\| .$$

Thus $\|F - u_\lambda\|_{A(\infty)} < \|F - u^*\|_{A(\infty)}$, and this completes the proof.

It is well known (for example [4] page 162) that H_s is a Haar subspace in $C[a, b]$ of dimension $r = 1 + \max\{n + \partial q, m + \partial p\}$, where ∂ stands for degree *of.* Thus Theorem 2 can be stated in the following form:

Theorem 3. *Let $F = (F_1, F_2) \in W \backslash U$. Then $s^* \in S$ is a best $A(\infty)$ approximation to F if and only if there exist point sets $X_1 = \{x_i, i \in I_1\} \subset E_1^*$, $X_2 = \{x_i, i \in I_2\} \subset E_2^*$ and non-negative numbers λ_i, $i \in I_1, \mu_i, i \in I_2$ such that*

(a) *if $E^* = E_1^*$:*

$$\begin{aligned}\sum_{i\in I_1} \lambda_i \sigma_1(x_i)\varphi(x_i) &= 0, \text{ for all } \varphi \in H_{s^*},\\ \sum_{i\in I_1} \lambda_i &= 1,\end{aligned}$$

(b) *if $E^* = E_2^*$:*

$$\begin{aligned}\sum_{i\in I_2} \mu_i \sigma_2(x_i)\varphi(x_i) &= 0, \textit{for all } \varphi \in H_{s^*},\\ \sum_{i\in I_2} \mu_i &= 1,\end{aligned}$$

(c) *if $E^* = E_1^* \cup E_2^*$:*

$$\begin{aligned}\sum_{i\in I_1} \lambda_i \sigma_1(x_i)\varphi(x_i) + \sum_{i\in I_2} \mu_i \sigma_2(x_i)\varphi(x_i) &= 0, \textit{for all } \varphi \in H_{s^*},\\ \sum_{i\in I_1} \lambda_i + \sum_{i\in I_2} \mu_i &= 1.\end{aligned}$$

Definition 1. A point $t \in [a, b]$ is called a straddle point for two functions f and g in $C[a, b]$, if there exists $\sigma = \pm 1$ such that $\|f\| = \sigma f(t)$, $\|g\| = -\sigma g(t)$.

Definition 2. The functions f and g in $C[a, b]$ are said to have d alternations on $[a, b]$ if there exist $d+1$ distinct points $x_1 < ... < x_{d+1}$ in $[a, b]$ such that for some $\sigma = \pm 1$,

$$\begin{aligned}f(x_i) &= \sigma \|f\|, && \text{if } i \text{ is odd},\\ g(x_i) &= -\sigma \|g\|, && \text{if } i \text{ is even},\end{aligned}$$

or

$$\begin{aligned} g(x_i) &= \sigma \|g\|, && \text{if } i \text{ is odd}, \\ f(x_i) &= -\sigma \|f\|, && \text{if } i \text{ is even}, \end{aligned}$$

Remark 1. If $\|F_1 - s^*\| = \|F_2 - s^*\| = \frac{1}{2}\|F_1 - F_2\|$, then $s^* \in P_S(F, \infty)$. Note that we can not do better than this, so in this case we say that s^* is optimal. Now if $F_1 - s^*$ and $F_2 - s^*$ have a straddle point with $\|F_1 - s^*\| = \|F_2 - s^*\|$, then s^* is optimal.

As noted by Dunham [5] the problem of simultaneous approximation to a finite number of functions from S with respect to the $A(\infty, \infty)$ norm is equivalent to the approximation of two functions F_1 and F_2 from S with $F_1 \geq F_2$. Therefore the case when we have $F_1 \geq F_2$ has particular significance. this condition also permits some stronger results to be obtained, and therefore we will make the following assumption.

Assumption. In the rest of this paper, it will be assumed that

$$F = (F_1, F_2) \in W \text{ with } F_1 \geq F_2 \text{ on } [a, b].$$

Lemma 1. *If* $s^* \in P_S(F, \infty)$, *then* $\|F_1 - s^*\| = \|F_2 - s^*\|$.

Proof. Assume $s^* \in P_S(F, \infty)$, but $\|F_1 - s^*\| > \|F_2 - s^*\|$. Theorem 3(a) and the usual L_∞ approximation theory implies that, $s* \in P_S(F_1)$. Because H_{s^*} is a Haar subspace, there exists a point t in $[a, b]$ such that

$$s^*(t) - F_1(t) = \|F_1 - s^*\|.$$

Since $F_1 \geq F_2$,

$$s^*(t) - F_1(t) \leq s^*(t) - F_2(t),$$

which is a contradiction. A similar argument when F_1 and F_2 are exchanged completes the proof.

Theorem 4. *Let* $F = (F_1, F_2) \in W \backslash U$. *Then* $s^* \in S$ *is a best* $A(\infty)$ *approximation if and only if* $F_1 - s^*$ *and* $F_2 - s^*$ *have a straddle point, or have* r *alternations in* $[a, b]$, *with* $\|F_1 - s^*\| = \|F_2 - s^*\|$, *where* $r = 1 + \max\{n + \partial q^*, m + \partial p^*\}$, $s^* = p^*/q^*$.

Proof. Let $s^* \in S$ be a best $A(\infty)$ approximation, then $\|F_1 - s^*\| = \|F_2 - s^*\|$. Assume $F_1 - s^*$ and $F_2 - s^*$ do not have a straddle point.

By Theorem 3(c), and since H_{s^*} is a Haar subspace, there exist points $x_1 < ... < x_{r+1}$ in $[a,b]$ such that

$$\begin{aligned}\sum_{i=1}^{r+1} \lambda_i \sigma_i \varphi(x_i) &= 0, \text{ for all } \varphi \in H_{s^*},\\ \sum_{i=1}^{r+1} \lambda_i &= 1,\end{aligned}$$

where $\lambda_i \geq 0$, and σ_i is θ_i or μ_i, $i = 1, ..., r+1$. Furthermore $\sigma_i = -\sigma_{i+1}, i = 1, ..., r$. And since $F_1 \geq F_2$, then

$$\begin{aligned}(F_1 - s^*)(x_i) &= \|F_1 - s^*\| \text{ if } i \text{ is odd},\\ (s^* - F_2)(x_i) &= \|F_2 - s^*\| \text{ if } i \text{ is even}\end{aligned}$$

or

$$\begin{aligned}(F_1 - s^*)(x_i) &= \|F_1 - s^*\| \text{ if } i \text{ is even},\\ (s^* - F_2)(x_i) &= \|F_2 - s^*\| \text{ if } i \text{ is odd}\end{aligned}$$

For the other direction, if $F_1 - s^*$ and $F_2 - s^*$ have a straddle point, the result follows from the remark after theorem 3. Assume $F_1 - s^*$ and $F_2 - s^*$ have r alternations at the points $x_1 < ... < x_{r+1}$ in $[a,b]$. Because H_{s^*} is a Haar subspace, then there exist numbers, $\lambda_i \geq 0, i = 1, ..., r+1$, such that

$$\begin{aligned}\sum_{i=1}^{r+1} \lambda_i \sigma_i \varphi(x_i) &= 0, \text{ for all } \varphi \in H_{s^*},\\ \sum_{i=1}^{r+1} \lambda_i &= 1,\end{aligned}$$

where,

$$\sigma_i = \begin{cases} sign(F_1 - s^*)(x_i) & i \text{ even},\\ sign(F_2 - s^*)(x_i) & i \text{ odd}\end{cases}$$

or

$$\sigma_i = \begin{cases} sign(F_1 - s^*)(x_i) & i \text{ odd},\\ sign(F_2 - s^*)(x_i) & i \text{ even}.\end{cases}$$

Thus by Theorem 3, $s^* \in P_S(F, \infty)$.

Definition 3. A set S is a sunset for simultaneous approximation if for any $F = (F_1, F_2) \in W$, there exists at least one best simultaneous approximation $s^* \in S$. Furthermore $s^*$is a best simultaneous approximation to $F^t = u^* + t(F - u)$ for each $t > 1$.

Clearly the set $S = \{p/q : p \in P_n, q \in Q_m \, p/q$ is irreducible, and $q > o$ on $[a, b]\}$ is a sunset for the $A(\infty)$ approximation.

If $F_1 - s^*$and $F_2 - s^*$ have a straddle point, then we may not have uniqueness, as the following example shows.

Example. Let $F_1(x) = x^2, F_2(x) = -x^2, [a, b] = [0, 1]$, and $S = \{p/q : p \in P_1, q \in Q_m$, p/q is irreducible, and $q > o$ on $[0, 1]\}$. Then for each $\alpha \in [-1, 1], s_\alpha = -\alpha x + \alpha \in P_S(f, \infty)$.

But if $F_1 - s^*$ and $F_2 - s^*$ have r alternations in $[a, b]$, we have the following uniqueness result.

Theorem 5. *Let $s^* \in P_S(F, \infty)$ be such that $F_1 - s^*$ and $F_2 - s^*$ have r alternations in $[a, b]$, with $r = 1 + \max\{n + \partial q^*, m + \partial p^*\}$, $s^* = p^*/q^*$. Then s^* is unique.*

Proof. Let $s^* \in P_S(F, \infty)$ be such that $F_1 - s^*$ and $F_2 - s^*$ have r alternations at the points $x_1 < ... < x_{r+1}$ in $[a, b]$. Without loss of generality, assume;

$$\begin{aligned} (F_1 - s^*)(x_i) &= \|F_1 - s^*\| \text{ if } i \text{ is odd,} \\ (s^* - F_2)(x_i) &= \|F_2 - s^*\| \text{ if } i \text{ is even.} \end{aligned}$$

If $s = p/q$ is another approximation, then $\varphi = q(s - s^*) \in H_{s^*}$, and

$$\begin{aligned} (F_1 - s)(x_i) &\leq \|F_1 - s^*\| (x_i) \text{ if } i \text{ is odd,} \\ (s - F_2)(x_i) &\leq \|s^* - F_2\| (x_i) \text{ if } i \text{ is even.} \end{aligned}$$

which implies that

$$(-1)^i(s^* - s)(x_i) > 0, \ i = 1, ..., r + 1.$$

Thus φ has at least r zeros in $[a, b]$, which implies that $s = s^*$.

The following remark is contained in [5], we state it here for completeness.

Remark 2. Let $G(G_1, ...G_l)$, $G_i \in C[a,b]$ for all $i = 1, ..., l$. And let

$$\|G\|_{A(\infty)} = \underset{1\leq i\leq l}{Max} \{\|G_i\|\}$$

For $x \in [a,b]$, define

$$\begin{aligned} F_1(x) &= \underset{1\leq i\leq l}{Max}\, G_i(x), \\ F_2(x) &= \underset{1\leq i\leq l}{Min}\, G_i(x), \end{aligned}$$

then $F_1, F_2 \in C[a,b]$ with $F_1 \geq F_2$. Let $F = (F_1, F_2), S$ nonempty subset of $C[a,b]$, $u = (s,s)$, and $v = (s,s,...,s)$ l copies of s, where $s \in S$. Clearly

$$\|G - v\|_{A(\infty)} = \|F - u\|_{A(\infty)} \text{ for all } s \in S$$

Therefore

$$s^* \in P_S(F,\infty) \text{ if and only if } s^* \in P_S(G,\infty).$$

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