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CHARACTERIZATION OF FUZZY T_0 AND R_0 TOPOLOGICAL SPACES

M.A. AMER

ABSTRACT. It is the purpose of this note to suggest new definitions of fuzzy T_0 and fuzzy R_0-spaces using the Wong defintion of fuzzy points. It will also be shown that these new definitions are equivalent to those introduced by Srivastava. Moreover, the properties of T_0-ness and R_0-ness are shown to be both productive and hereditary and that a topologically generated fuzzy topological space is T_0 or R_0 if the original topological space is T_0 or R_0, respectively.

1. INTRODUCTION

The fundamental concept of a fuzzy set was introduced by Zadeh in 1965 [9]. Since then, intensive studies of fuzzy sets have been developed. In particular, the definition of a fuzzy point was first given in 1974 by Wong [8]. It is notable that, with this definition, an ordinary point of a set is not a special case of a fuzzy point. In 1980, Pu and Liu [5] remedied this drawback by redefining fuzzy point in a way that can be used to develop the theory of fuzzy topology in a satisfactory way. In 1984, R. Srivastava, S.N. Lal, and A.K. Srivastava [5] studied the concept of fuzzy T_1-topological space using the Wong fuzzy point [8]. Latter in 1988, they introduced an equivalent definition (depending upon the ordinary points of a set) of a fuzzy T_1-space [7]. On the other hand, a fuzzy T_0-topological space has been defined and studied by Hutton and Reilly [2], Pu and Liu [4], R. Srivastava, S.N. Lal, and A.K. Srivastava [6]. Hutton [2] and Srivastava [6] studied, in addition, the concept of a fuzzy R_0 topological spaces in order to establish a satisfactory relationship between fuzzy T_1, T_0, and R_0-spaces. It can be seen that in papers [2,3,4] the authors investigated fuzzy T_0 and fuzzy R_0 -spaces depending upon the ordinary points of a set and not the fuzzy points. In 1990, Ali, Wuyts, and

Srivastava introduced and intensively studied fuzzy R_0-spaces [1]. It is the purpose of this note to suggest new definitions of fuzzy T_0 and fuzzy R_0-spaces using the Wong definition of fuzzy points [8]. These new definitions can be extended in a straightforward manner to the case of fuzzy T_1-spaces studied by Srivastava in [5]. It will be also shown that the new definition of fuzzy T_0-spaces is equivalent to that introduced by Srivastava [6]. The new definitions of fuzzy R_0-spaces are shown to be equivalent to that introduced by Ali in [1], but they are more general than that introduced by Srivastava in [6]. Moreover, the properties of T_0-ness and R_0-ness are shown to be both productive and hereditary and that a topologically generated fuzzy topological space is T_0 or R_0 iff the original topological space is T_0 or R_0, respectively.

2. BASIC DEFINITIONS AND PROPERTIES

A function U from a non empty set X to the unit interval [0,1] is called a fuzzy set in X. The fuzzy set that takes the value 0 at all points $x \in X$ is denoted by ϕ and that which takes the value 1 at all points of X is denoted by X itself. A fuzzy topology τ on X (in Lowen's sense [3]) is that contain, in addition to the above properties, all constant fuzzy sets. The term "fuzzy topological space" will be abbreviated as fts. A fuzzy point p in X is a fuzzy set in X such that $p(x_p) = t$ for $x = x_p$, and $p(x) = 0$, otherwise. The point x_p is called the support of p and t its value, $t \in (0,1)$. A fuzzy point p is said to belong to a fuzzy set U in $X (p \in U)$ iff $p(x_p) < U(x_p)$. Two fuzzy points are said to be distinct iff they have different supports. For a fuzzy point p in X and a fuzzy set U in X, we say that $p \cap U = 0$ iff $U(x_p) = 0$. We denote the characteristic function of a singleton set $\{x\}$ by 1_x and its closure by $\bar{1}_x$.

3. FUZZY T_0-TOPOLOGICAL SPACES

Definition 3.1. (Pu and Liu [4]). A fts (X, τ) is said to be fuzzy T_0 iff for any $s, t \in [0,1)$ and $x, y \in X$, $x \neq y$, $\exists U \in \tau$ such that $U(x) = s$ and $U(y) > t$, or $U(x) > s$ and $U(y) = t$.

Definition 3.2. (Hutton and Reilly [2]). A fts (X,τ) is said to be fuzzy T_0 iff each fuzzy set in X can be written as $\sup_i \inf_j U_{ij}$, where $i \in I$, $j \in J$, and each U_{ij} is fuzzy open or fuzzy closed in (X,τ).

Definition 3.3. (R. Srivastava, S.N. Lal, and A.K. Srivastava [9]). A fts (X,τ) is said to be fuzzy T_0 iff for all $x,y \in X$, $x \neq y$, $\exists U \in \tau$ such that either $U(x)=1$ and $U(y)=0$, or $U(x)=0$ and $U(y)=1$.

Now we introduce our new definition of a fuzzy T_0 topological space

Definition 3.4. A fts (X,τ) is said to be fuzzy T_0 iff for any two distinct fuzzy points p and q in X, $\exists U \in \tau$ such that either $p \in U$ and $q \cap U = 0$ or $q \in U$ and $p \cap U = 0$.

We now compare the above four definitions of fuzzy T_0-ness in the following theorem:

Theorem 3.1. *Consider the following statements for the fts (X,τ):*

(I) For any distinct fuzzy points p,q in X, $\exists U \in \tau$ such that $p \in U$ and $q \cap U = 0$, or $q \in U$ and $p \cap U = 0$.

(II) $\forall\, x,y \in X$, $x \neq y$, $\exists U \in \tau$ such that either $U(x)=1$ and $U(y)=0$, or $U(y)=1$ and $U(x)=0$.

(III) Each fuzzy set in X can be written in the form $\sup_i \inf_j U_{ij}$ where each U_{ij}, $i \in I, j \in J$, is a fuzzy open or a fuzzy closed set.

(IV) For any two distinct points $x,y \in X$ and for all $s,t \in [0,1)$, there exists $U \in \tau$ such that either $U(x)=s$ and $U(y)>t$, or $U(x)>s$ and $U(y)=t$.

We have the following implications:

(I) $\Leftrightarrow$ (II)

(I) $\Rightarrow$ (III)

(III) $\not\Rightarrow$ (I)

(I) $\Rightarrow$ *(IV)*

(IV) $\not\Rightarrow$ *(I)*

Proof. It suffices to prove that (I) $\Leftrightarrow$ (II). The remaining implications follow directly using [6, Theorem 2.1].

(I) $\Rightarrow$ (II). Let $x, y \in X$, $x \neq y$ and let p_n, q_n be fuzzy points in X with supports x, y, respectively, and such that $p_n(x) = q_n(y) = 1 - \frac{1}{2n}$, $n \in N$. Since $x \neq y$ then $p_n \neq q_n$ for every $n \in N$ and by (I) $\exists U_n \in \tau$ such that either $p_n \in U_n$ and $q_n \cap U_n = 0$, or $q_n \in U_n$ and $p_n \cap U_n = 0$. Assume that there is an infinite subset J of N such that $p_n \in U_n$ and $q_n \cap U_n = 0$ for all $n \in J$ (the other case can be treated similarly) then $U_n(x) > 1 - \frac{1}{2n}$, $U_n(y) = 0$ for every $n \in J$. Define $U = \bigcup_{n \in J} U_n$ then, $U \in \tau$ and $U(x) = 1$, $U(y) = \bigcup_{n \in J} U_n(y) = 0$. So we have (II).

(II) $\Rightarrow$ (I). Suppose that p, q are two distinct fuzzy points in X with supports x, y, and values $r, s \in (0,1)$, respectively, then $x \neq y$ and by (II) $\exists U \in \tau$ such that either $U(x) = 1$ and $U(y) = 0$, or $U(x) = 0$ and $U(y) = 1$. Assume that $U(x) = 1$ and $U(y) = 0$ (the other case can be treated similarly). Since $p(x) = r < 1$, and $q(y) = s > 0$, it follows that $p \in U$ and $q \cap U = 0$. So we have (I).

Remark 3.1. Definition 3.4 can be replaced by an equivalent definition where we replace the fuzzy open set U by a fuzzy closed set V. In this case all the implications of theorem 3.1 remain valid.

The following theorem shows that the property of T_0-ness of a fuzzy topological space is productive.

Theorem 3.2. *Let $\{(X_i, \tau_i) : i \in I\}$ be a family of fuzzy topological spaces, then the product space $(X, \tau) = \Pi_i(X_i, \tau_i)$ is fuzzy T_0 iff each coordinate fts is fuzzy T_0 (in the sense of Definition 3.4).*

Proof. Let (X_j, τ_j) be fuzzy T_0, for $j \in I$ and let p, q be two distinct fuzzy points in X, $p = < p_j >$, $q = < q_j >$. Then $p_i \neq q_i$ for at least one $i \in I$. Then $\exists\ U_i \in \tau_i$ such that $p_i \in U_i$ and $q_i \cap U_i = 0$ or $q_i \in U_i$ and $p_i \cap U_i = 0$. Suppose that $p_i \in U_i$ and $q_i \cap U_i = 0$ (the other case can be treated similarly). Let $U = \Pi_j U'_j$, where $U'_j = X_j$, for $j \neq i$, $U'_j = U_j$ for $j = i$. It is clear that $U \in \tau$ and $p \in U$, $q \cap U = 0$. Hence (X, τ) is fuzzy T_0. Conversely, let (X, τ) be fuzzy T_0 and consider any (X_i, τ_i), $i \in I$. Let p_i, q_i be two distinct fuzzy points in X_i and construct the two distinct fuzzy points $p = < p'_j >$, $q = < q'_j >$ in X where $p'_j = q'_j$ for $j \neq i$ and $p'_i = p_i$, $q'_i = q_i$. Then $\exists\, U \in \tau$ such that either $p \in U$ and $q \cap U = 0$, or $q \in U$ and $p \cap U = 0$. Suppose that $p \in U$ and $q \cap U = 0$ (the other case can be treated similarly). Then we can find a basic fuzzy open set $\Pi_j U_j$ such that $p \in \Pi_j U_j \subset U$. It follows that $p_i \in U_i$, and since $q \cap U = 0$ then $q \cap \Pi U_j = 0$ and hence $\Pi_j q_j \cap \Pi_j U_j = 0$. Since $q_j = p_j$ for $j \neq i$ and $p_j \in U_j$ then $q_j \cap U_j \neq 0$, for $j \neq i$. Hence, we must have that $q_i \cap U_i = 0$. This proves that (X_i, τ_i) is fuzzy T_0.

Using the definitions of a fuzzy subspace introduced by Pu and Liu [4, Definition 8.1] and the topologically generated fuzzy topological space (introduced by Lowen [3]) together with Definition 3.4 we can easily prove the following theorems.

Theorem 3.3. *Every fuzzy subspace of a fuzzy T_0-space is also a fuzzy T_0-space.*

Theorem 3.4. *Let (X, T) be a topological space. Then (X, T) is T_0 iff $(X, w(T))$ is fuzzy T_0, where $w(T)$ is the topologically generated fuzzy topology generated by the topology T [5, Definition 2.8].*

4. FUZZY R_0-TOPOLOGICAL SPACES

Fuzzy R_0-spaces have been defined by Hutton and Reilly [2], R. Srivastava, S.N. Lal, A.K. Srivastava [6], and D.M. Ali, P. Wuyts, and A.K. Srivastava [1] as follows:

Definition 4.1. (Hutton and Reilly [2]). An fts (X, τ) is said to be fuzzy R_0 iff each fuzzy open set can be written as a supremum of fuzzy closed sets.

Definition 4.2. (R. Srivastava, S.N. Lal, A.K. Srivastava [6]). A fts (X, τ) is fuzzy R_0 iff $\forall x, y\ X$, $x \neq y$, whenever there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(x) = 0$ and $V(y) = 1$.

Definition 4.3. (D.M. Ali, P. Wuyts, and A.K. Srivastava [1]). A fts (X, τ) is said to be fuzzy R_0^7 iff $\forall (x, y) \in X^{(2)}$ it follows that $\bar{1}_x(y) = \bar{1}_y(x) \in \{0, 1\}$.

It has been shown in [6] that Definitions 4.1, 4.2 are totally independent and that the latter definition is a good extension of the concept of an R_0 topological space. In 1990 Ali, Wuyts, and A.K. Srivastava [1] introduced and studied carefully many fuzzy R_0 topological spaces. We propose here more general definitions of fuzzy R_0 topological space and show that these new definitions are not implied by that introduced by Srivastava in [6], but are equivalent to R_0^7 introduced in [1].

Definition 4.4. A fts (X, τ) is said to be fuzzy R_0^a iff $\forall x, y \in X, x \neq y$ if $\bar{1}_y(x) < 1$ then $\bar{1}_x(y) = 0$.

Definition 4.5. A fts (X, τ) is fuzzy R_0 iff $\forall x, yX, x \neq y$, whenever there is a $U \in \tau$ such that $U(x) \neq 0$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(x) = 0$ and $V(y) = 1$.

Definition 4.6. A fts (X, τ) is said to be fuzzy R_0 iff for every two distinct fuzzy points p, q in X, whenever there is a $U \in \tau$ such that $p \in U$ and $q \cap U = 0$ there is also $V \in \tau$ such that $q \in V$ and $p \cap V = 0$.

Theorem 4.1. *For a fts (X, τ) consider the following statements:*

(1) $\forall x, y \in X, x \neq y$, whenever there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(x) = 0$ and $V(y) = 1$.

(2) $\forall (x, y) \in X^{(2)}$ it follows that $\bar{1}_x(y) = \bar{1}_y(x) \in \{0, 1\}$.

(3) $\forall x, y \in X, x \neq y$ if $\bar{1}_y(x) < 1$ then $\bar{1}_x(y) = 0$.

(4) $\forall x, y \in X, x \neq y$, whenever there is a $U \in \tau$ such that $U(x) \neq 0$ and $U(y) = 0$, there is also $V \in \tau$ such that $V(x) = 0$ and $V(y) = 1$.

(5) for every two distinct points p, q in X, whenever there is a $U \in \tau$

such that $p \in U$ and $q \cap U = 0$ there is also $V \in \tau$ such that $q \in V$ and $p \cap V = 0$. Then the following implications hold:

(i) (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (5)

(ii) (5) $\Rightarrow$ (1) and (1) $\not\Rightarrow$ (5)

(iii) (1) $\Leftrightarrow$ (4).

Proof. (i) (2) $\Rightarrow$ (3). Let $x, y \in X, x \neq y$ are such that $\bar{1}_y(x) < 1$, then by (2) it follows that $\bar{1}_y(x) = 0$, this implies that $\bar{1}_x(y) = 0$. Hence, (X, τ) satisfies (3).

(3) $\Rightarrow$ (2). Suppose that $x, y \in X, x \neq y$ are such that $\bar{1}_x(y) = 0$. Since $\bar{1}_x(y) = 0 < 1$ then, again by (3) we must have that $\bar{1}_y(x) = 0$. Therefore, $\bar{1}_y(x) = \bar{1}_x(y) = 0$. Now, suppose that $\bar{1}_y(x) = 1$. If $\bar{1}_x(y) < 1$ then by (3) it follows that $\bar{1}_y(x) = 0$ which is not true. So $\bar{1}_x(y)$ must be 1. Hence, $\bar{1}_y(x) = \bar{1}_x(y) = 1$.

This proves that (X, τ) satisfies (2).

(3) $\Rightarrow$ (5). Let (X, τ) be fuzzy R_0. Suppose that $x, y \in X, x \neq y$ are such that $\bar{1}_y(x) = \gamma < 1$. Choose the real number t such that $t + \gamma < 1$. Let p be a fuzzy point in X supported at x and with value t. $\forall \alpha < 1$, let q_α be a fuzzy point supported at y and with value α. Let $U = \text{co}\ (\bar{1}_y)$, then $U(y) = 1 - \bar{1}_y(y) = 0$, $U(x) = 1 - \bar{1}_y(x) = 1 - \gamma > t$. Therefore, $p \in U$, $q_\alpha \cap U = 0$, hence $\exists V_\alpha \in \tau$ such that $q_\alpha \in V_\alpha$, $p \cap V_\alpha = 0$. Take $V = \underset{\alpha}{\bigcup} V_\alpha$. It follows that $V(y) = 1$, $V(x) = 0$, Hence, $1_x \subseteq \text{co}\ (V)$, $\bar{1}_x \subseteq \overline{\text{co}(V)} = \text{co}(V)$. Therefore, $\bar{1}_x(y) \leq \text{co}(V)(y) = 0$, this implies that $\bar{1}_x(y) = 0$.

(5) $\Rightarrow$ (3). Let p, q be two distinct fuzzy points in X with supports $x, y \in X$ and values $r, s \in (0, 1)$, respectively. Let U be such that $p \in U$ and $q \cap U = 0$. Therefore, $\text{co}(U(y)) = 1$, $\bar{1}_y \subseteq \overline{\text{co}(U)} = \text{co}(U)$. Hence, $\bar{1}_y(x) \leq j\text{co}(U(x)) = 1 - U(x) < 1 - \gamma < 1$. So, by R_0^a we get $\bar{1}_x(y) = 0$. Take $V = \text{co}(\bar{1}_x)$. Hence, $V(x) = 1 - \bar{1}_x(x) = 0$, $V(y) = 1 - \bar{1}_x(y) = 1$. This implies that $q \in V$ and $p \cap V = 0$.

(ii) (5) $\Rightarrow$(1). Let $x, y \in X, x \neq y$, and suppose that there is a $U \in \tau$ such that $U(x) = 1$ and $U(y) = 0$. Let p_n and q_n be fuzzy points in X with supports x and y, respectively, and such that $p_n(x) = q_n(y) = 1 - \frac{1}{2n}$, $n \in N$. It is clear that $p_n \in U$ and $q_n \cap U = 0$ for all $n \in N$. Hence, by (1) $\exists V_n \in \tau$ such that $q_n \in V_n$ and $p_n \cap V_n = 0$, for all $n \in N$. Let $V = \bigcup_n V_n$. Then, $V(y) = 1$ and $V(x) = \bigcup_n V_n(x) = 0$. so we have (1).

(1) $\not\Rightarrow$ (5). Consider the following counterexample: Let $X = \{x, y\}$ be a set of two points x and y, and let $\tau = \{U : U$ is a fuzzy set such that $U(x) = U(y)$, or $\frac{1}{2} \geq U(x) > U(y)\}$. Clearly, (X, τ) is a fuzzy topology on X, in both Chang's sense and Lowen's sense. The fts (X, τ) satisfies (1) because, the premise $U(x) = 1$ and $U(y) = 0$ and the premise $U(y) = 1$ and $U(x) = 0$, are both impossible. On the other hand (X, τ) does not satisfy (5). Take the two distinct fuzzy points p and q in X such that $p(x) = \frac{1}{4}$, $p(y) = 0$ and $q(y) = \frac{1}{4}, q(x) = 0$. Then, there exists $U \in \tau$ such that $U(x) = \frac{3}{8} > p(x), U(y) = 0$. Clearly, $p \in U$ and $q \cap U = 0$. But, for all $V \in \tau$, if $q \in V$, then $V(x) \geq V(y) > \frac{1}{4}$, and so $p \cap V = 0$.

(iii) (1) $\Leftrightarrow$ (4). This can be shown easily by applying the same technique used in (ii).

If the fuzzy open sets U and V in Definition 4.6 are replaced by fuzzy closed sets U' and V', respectively, then the statement of Theorem 4.1 remains valid.

Theorem 4.2. *Let $\{(X_i, \tau_i), i \in I\}$ be a family of fuzzy topological spaces. Then their product space $(X, \tau) = \Pi_i(X_i, \tau_i)$ is fuzzy R_0 iff each coordinate fts is fuzzy R_0*

Proof. Let (X_j, τ_j) be fuzzy R_0 for all $j \in I$ and let p, q be any two distinct fuzzy points in $X, p =< p_j >, q =< q_j >$ (see Srivastava [7]) and $U \in \tau$ such that $p \in U$, $q \cap U = 0$. Then p_i and q_i are distinct for at least one $i \in I$, and there exists a basic fuzzy open set $\Pi_j U_j$ such that $U = \Pi_j U_j$. Therefore, $p_j \in U_j$, $j \in I$. Also, $q_i \cap U_i = 0$, for at least one $i \in I$. Then $\exists V_i \in \tau_i$ such that $q_i \in V_i$ and $p_i \cap V_i = 0$. Construct $V = \Pi_j V_j'$, where $V_j' = X_j$, for $j \neq i$, and $V_i' = V_i$. Clearly, $q \in V$, and

$p \cap V = 0$. Hence, (X, τ) is fuzzy R_0.

Conversely, let (X, τ) be fuzzy R_0. Suppose that p_i and q_i are distinct fuzzy points of X_i, $U_i \in \tau_i$, $p_i \in U_i$ and $q_i \cap U_i = 0$. Construct $p =< p'_j >, q =< q'_j >, p'_j = q'_j, j \neq i$, $p'_i = p_i, q'_i = q_i$, and $U = \Pi_j U'_j, U'_j = X_j$, for $j \neq i$, and $U'_i = U_i$. Then p and q are distinct fuzzy points of X, and $p \in U$, $q \cap U = 0$. Then $\exists V = \Pi_j V_j \in \tau$ such that $q \in V$, $p \cap V = 0$. Then by the construction of p and q we must have that $q_j \in V_j$, $j \in I$, and $p_i \cap V_i = 0$. This proves that (X_i, τ_i) is fuzzy R_0.

Using Definition 4.6, Theorem 3.1 of [6] and Proposition 3.2 of [7], we can easily prove the following theorems:

Theorem 4.3. *A fuzzy subspace of a fuzzy R_0 space is also fuzzy R_0.*

Theorem 4.3. *A topological space (X,T) is R_0 iff the fts $(X, w(T))$ is fuzzy R_0.*

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DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, FACULTY OF SCIENCE, UNITED ARAB EMIRATES UNIVERSITY, P.O. BOX 17551, AL-AIN, UNITED ARAB EMIRATES

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