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ON THE OPTIMALITY OF A REPLENISHMENT POLICY FOR AN INVENTORY MODEL WITH DETERIOTATING ITEMS AND TIME-VARYING DEMAND AND SHORTAGES

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ABSTRACT. In a recent paper, the author suggested a policy for finding an optimal replenishment schedule for an inventory model with deteriorating items and time-varying demand and shortages over a known and finite planning horizon. No formal proof regarding the optimality of the policy suggested was given. In this paper, we prove the optimality of the policy using the concept of diagonally dominant matrices borrowed from linear Algebra. It turns out that the proof can be adapted to answer some unsettled questions related to similar existing models in the literature.

1. INTRODUCTION

Inventory models for deteriorating items have received considerable interest in the literature: see Raafat (1991) for his excellent review. In these models the traditional assumption of infinite shelf life of products is dropped and a more realistic approach to modeling the decaying phenomena in the life of products is adopted. However, existing inventory mathematical models seem to circle around special cases, where demand is assumed to be linear or exponential. Benkherouf and Mahmoud (1995), Benkherouf (1995) have addressed the problem of modeling inventory models with deteriorating items in a general manner. In these two papers, a procedure was put forward for finding the optimal replenishment schedule for inventory models for deteriorating items and monotonically increasing (or decreasing) demand rate over a known and finite planning horizon. Another complication was added by assuming shortages.

No proof of the optimality of the procedure was given. It was thought, at that time, that the Hessian matrix presents a big challenge in terms of mathematical manipulation. In this short note, we give a rigorous proof of the optimality of the algorithm suggested by Benkherouf and Mahmoud (1995). It turns out that the proof can be easily adapted to answer some unsettled questions regarding the optimality of previous algorithms suggested for some special cases of the model treated in this paper.

In the next section, we present the model and the algorithm of Benkherouf and Mahmoud (1995). The last section is concerned with the proof of the optimality of the algorithm.

2. MATHEMATICAL MODELS AND ALGORITHM

In the model of Benkherouf and Mahmoud (1995) it is assumed that single item is held in stock over a known and finite planning horizon, that is, H units long. The item deteriorates with a fixed rate θ, $\theta \geq 0$. The demand rates $D(t)$ are increasing with time and $D(t)/D'(t)$ is non decreasing in t, where $D'(t)$ denotes the first derivative of $D(.)$ with respect to t. Here $D(t) > 0$ for all $t \geq 0$. We also assume that shortages are allowed during the planning horizon. Further, the cost structure of the model is as follows:

a. a fixed ordering cost, K,

b. a holding cost per unit in stock per unit of time, c_1.

c. a cost per unit item, c_2,

d. a shortage cost per unit of item per unit of time, c_3,

In addition, the following notation is used in our mathematical model.

n:	The number of replenishment cycles during the planning horizon.
$I(t)$:	The inventory level at time t.
$S(t)$:	The shortage level at time t.
T_j:	The total time that elapses up to and including the jth cycle ($j = 1, 2, \ldots, n$) where $T_n = H, T_0 = 0$
t_j:	The total time that elapses up to and including the jth shortage ($j = 1, 2, \ldots, n$).
$I(T_{j-1}, t_j)$:	Amount of inventory carried during the jth cycle, $j = 1, \ldots, n$.
$S(t_j, T_j)$:	Amount of shortage carried during the jth cycle, $j = 1, \ldots, n$.
$TC(n)$:	Total variable costs when n orders are placed during the planning horizon.

First, let us consider the level of inventory at time $t, I(t)$, during the jth cycle. In the first part of the cycle, that is, $T_{j-1} \leq t < t_j$, the inventory is depleted by the combined effect of demand and deterioration. Therefore, the variation of $I(t)$ with respect to time is governed by the following differential equation:

$$\frac{dI(t)}{dt} = -D(t) - \theta I(t); \qquad T_{j-1} \leq t < t_j, \tag{2.1}$$

with boundary condition $I(t_j) = 0$.

The level of shortage, $S(t)$, during the jth cycle may be represented by the following differential equation, since no deterioration occurs in the interval $t_j \leq t < T_j$.

$$dS(t)/dt = -D(t), \qquad t_j \leq t < T_j, \tag{2.2}$$

with initial condition $S(t_j) = 0$.

The solution to (2.1) may be represented by

$$I(t) = e^{-\theta t} \int_t^{t_j} e^{\theta u} D(u) du. \tag{2.3}$$

The amount of inventory during the jth cycle is then given by:

$$I(T_{j-1},t_j)=\int_{T_{j-1}}^{t_j} e^{-\theta t}\left\{\int_t^{t_j} e^{\theta u}D(u)du\right\}dt.$$

After integrating by parts, the above reduces to:

$$I(T_{j-1},t_j)=\frac{1}{\theta}\int_{T_{j-1}}^{t_j}\left\{e^{\theta(u-T_{j-1})}-1\right\}D(u)du. \tag{2.4}$$

A similar argument as above can be used to show that (2.2) gives:

$$S(t_j,T_j)=\int_{t_j}^{T_j}(T_j-u)D(u)du. \tag{2.5}$$

The number of deteriorated items in Period j is equal to:

$$I(T_{j-1})-\int_{T_{j-1}}^{t_j} D(u)du$$

which can be shown using (2.3) and (2.4) to be equal to $\theta I(T_{j-1},t_j)$.

Next, suppose that n replenishments orders are received during the planning horizon. Then, the total variable cost, which is defined as the sum of ordering; holding; deterioration and shortage cost, is given by

$$TC(n)=nK+(c_1+\theta c_2)\sum_{j=1}^{n} I(T_{j-1},t_j)+c_3\sum_{j=1}^{n} S(t_j,T_j). \tag{2.6}$$

Here, $I(T_{j-1},t_j)$ and $S(t_j,T_j)$ are given by (2.4) and (2.5) respectively and $T_n=H$.

Expression (2.6) may be rewritten using relations (2.4) and (2.5) as:

$$\begin{aligned} TC(n) &= nK+\frac{c_1+\theta c_2}{\theta}\sum_{j=1}^{n}\int_{T_{j-1}}^{t_j}\left\{e^{\theta(u-T_{j-1})}-1\right\}D(u)du \\ &\quad +c_3\sum_{j=1}^{n}\int_{t_j}^{T_j}(T_j-u)D(u)du. \end{aligned} \tag{2.7}$$

Assuming that n is known, then the problem of finding the optimal replenishment strategy reduces to the problem of minimizing $TC(n)$ in (2.7) subject to the constraints:

$$T_j > t_j, t_j > T_{j-1}, T_0 = 0, T_n = H, \quad j = 1, 2, \ldots, n. \tag{2.8}$$

Let us first consider minimizing (2.7) by ignoring the constraints $T_j > t_j, t_j > T_{j-1}, j = 1, 2, \ldots, n-1$, we get:

$$\frac{\partial TC(n)}{\partial t_j} = \frac{c_1 + \theta c_2}{\theta} \left\{ e^{\theta(t_j - T_{j-1})} - 1 \right\} D(t_j) - c_3(T_j - t_j) D(t_j) = 0, \tag{2.9}$$

$$\frac{\partial TC(n)}{\partial T_j} = -(c_1 + \theta c_2) \left\{ \int_{T_j}^{t_{j+1}} e^{\theta(u - T_j)} D(u) du \right\} + c_3 \int_{t_j}^{T_j} D(u) du = 0, \quad j = 1, \ldots, n-1. \tag{2.10}$$

The following two theorems are due to Benkherouf and Mahmoud (1995).

Theorem 1. *The system of nonlinear equations described by (2.9) and (2.10) has a unique solution satisfying (2.8).*

Theorem 2. *Let $t_1, \ldots t_{n-1}$ and $T_1, \ldots T_{n-1}$ be vectors satisfying (9) and (10), then we always have*

(i) $T_{j+1} - t_{j+1} < T_j - t_j, \qquad j = 1, \ldots, n-1.$

(ii) $t_{j+1} - T_j < t_j - T_{j-1}, \qquad j = 1, \ldots, n-1.$

The basic idea of the algorithm of Benkherouf and Mahmoud (1995) is as follows:

Let $TC^*(n)$ be the optimal value of $TC(n)$ for a fixed n, where $TC^*(n)$ is found from solving the system of nonlinear equations described by (2.9) and (2.10). Also, let $\Delta TC^*(n) = TC^*(n+1) - TC^*(n)$. Then the optimal value of n^* is the smallest nonnegative integer such that:

$$\Delta TC^*(n-1) \leq 0 \leq \Delta TC^*(n).$$

In the next section, we show that the solution of the system of equations given by (2.9) and (2.10) gives the optimal solution of $TC^*(n)$.

3. OPTIMALITY OF THE ALGORITHM

Before, we prove our main result, we introduce some preliminary results and a couple of definitions.

Definition 1. Let Ω be a subspace of $\Re^n$, $\Omega \subset \Re^n$ and let A be an $n \times n$ real symmetric matrix. Then A is said to positive definite on the subspace Ω if and only if $x^t A x > 0$ for all $x \in \Omega$.

Definition 2. Let A be an $n \times n$ real symmetric matrix. $A = (a_{ij})$, then A is said to be diagonally dominant if and only if $a_{ii} \geq \sum_{j \neq i} |a_{ij}|$, and $a_{ii} > 0$ for all $i = 1, \ldots, n$.

Here $|a|$ denotes the absolute value of a.

It is well known, see for example Stewart (1973) that a symmetric matrix A which is strictly diagonally dominant (the inequality in Definition 2 is replaced by strict inequality) is positive definite. However, as we shall see the Hessian matrix of our problem is not strictly diagonally dominant. This means that we cannot use the above result. On the other hand, we shall show that the Hessian matrix is diagonally dominant. We also show that it is positive definite over our solution space. This should guarantee optimality of the procedure suggested by Benkherouf and Mahmoud. Next, we prove a general result which we shall find helpful later on.

Theorem 3. *Let* $\Omega = \{x = (x_1, \ldots, x_n \in \Re^n$ such that $x_i \neq x_j$ for all $i \neq j\}$. *Also, let* $A = (a_{ij})$ *be a real* $n \times n$ *symmetric matrix which is diagonally dominant. Then* A *is positive definite over the subspace* Ω.

Proof. Let $x = (x_1, \ldots, x_n) \in \Omega$. Consider $x^t Ax$, this is equal to:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j a_{ij},$$

which in turn, is equal to

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} x_i x_j a_{ij} + \sum_{i=1}^{n} x_i^2 a_{ii}.$$

By using the fact that A is diagonally dominant, we see that the above is greater that

$$\sum_{\substack{i=1 \\ i \neq j}}^{n} \sum_{j=1}^{n} x_i x_j a_{ij} + \sum_{i=1}^{n} x_i^2 \times \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|$$

Now, we use the fact that A is symmetric with some algebra to get that the last quantity is equal to

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} \{x_i^2 |a_{ij}| + 2x_i x_j a_{ij} + x_j^2 |a_{ij}|\},$$

which is > 0 when $x_i \neq x_j$. This completes the proof.

Now, we go back to our problem. First, we note that our solution space has $T_n > t_n > T_{n-1} > t_{n-1} > \ldots > t_n > 0$.

Direct computations from (2.9) and (2.10) give:

$$\begin{aligned}
\frac{\partial^2 TC(n)}{\partial t_j^2} &= \{(c_1 + \theta c_2) e^{\theta(t_j - T_{j-1})} + c_3\} D(t_j), \\
\frac{\partial^2 TC(n)}{\partial t_j \partial T_j} &= -c_3 D(t_j), \\
\frac{\partial^2 TC(n)}{\partial t_j \partial T_{j-1}} &= -(c_1 + \theta c_2) e^{\theta(t_j - T_{j-1})} D(t_j), \\
\frac{\partial^2 TC(n)}{\partial T_j^2} &= (c_1 + \theta c_2) \{D(T_j) + \theta \int_{T_j}^{t_{j+1}} e^{\theta(u - T_j)} D(u) du\} + c_3 D(t_j),
\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 TC(n)}{\partial T_j \partial t_j} &= -c_3 D(t_j),\\ \frac{\partial^2 TC(n)}{\partial T_j \partial t_{j+1}} &= -(c_1 + \theta c_2) e^{\theta(t_{j+1} - T_j)}.\end{aligned}$$

Note that

$$\frac{\partial^2 TC(n)}{\partial t_j^2} > 0 \text{ and } \frac{\partial T^2 C(n)}{\partial T_j^2} > 0.$$

Also,

$$\frac{\partial^2 TC(n)}{\partial t_j^2} = |\frac{\partial T^2 C(n)}{\partial t_j \partial T_j}| + |\frac{\partial^2 TC(n)}{\partial t_j \partial T_{j-1}}|.$$

We next show that

$$\frac{\partial^2 TC(n)}{\partial T_j^2} \geq |\frac{\partial^2 TC(n)}{\partial T_j \partial t_j}| + |\frac{\partial^2 TC(n)}{\partial t_j \partial T_{j+1}}|.$$

Note that

$$D(T_j) + \theta \int_{T_j}^{t_{j+1}} e^{\theta(u - T_j)} D(u) du \geq e^{\theta(t_{j+1} - T_j)} D(t_{j+1}),$$

by Lemma 1 of Benkherouf and Mahmoud (1995). Also, $D(T_j) \geq D(t_j)$ since $D(.)$ is increasing and $T_j > t_j$. It follows that the Hessian matrix of $TC(n)$ is diagonally dominant. Therefore, it is positive definite over our solution space by Theorem 3. Now, the fact that the algorithm of Benkherouf and Mahmoud (1995) is optimal is immediate.

This result extends in a straightforward manner to the model of Benkherouf (1995).

To summarize, in this paper, we have proved in a rigorous way that the procedure suggested by Benkherouf and Mahmoud (1995) is optimal. This result of the paper though specifically shown for the model of this paper. It extends directly to earlier algorithms suggested by Donaldson (1973), Hariga (1993), Hariga and Benkherouf (1994).

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