Arab J. Math. Sc.
Volume 6, Number 2, December 2000
pp. 41-47

INTEGRAL INEQUALITY OF GRONWALL-BELLMAN TYPE

JAMES ADEDAYO OGUNTUASE

ABSTRACT. In this paper, we establish some new Gronwall–Bellman type integral inequalities. These inequalities generalize in some cases the existing ones which have a wide range of applications in the study of the properties of solutions of differential and integral equations.

1. INTRODUCTION

The inequalities obtained in this paper originated from the celebrated Gronwall-Bellman inequality which plays a vital role in the study of differential and integral equations. On the basis of various motivations, the Gronwall-Bellman inequality has been generalized and used in various contexts (see for example [1] and the references cited therein). In this note, we establish some new Gronwall-Bellman type integral inequalities which in some special cases reduce to certain generalizations of the well-known Gronwall-Bellman inequalities given in [1, 2, 3, 6].

2. MAIN RESULTS

The main results of this paper are given in Theorems 2.1 and 2.2. There appear to be no results of this kind for Gronwall-type inequalities. We shall need the following Lemma in the proof of our main results.

2000 *Mathematics Subject Classification.* 26D10, 26D15.

Lemma 2.1 [1]. *Let f(t) and g(t) be nonnegative continuous functions for $t \geq a$, let v(t) be a differentiable function for $t \geq a$ and suppose*

$$v'(t) \leq g(t)v(t) + f(t), \qquad t \geq a, \quad v(a) = v_0.$$

Then, for $t \geq a$, we have

$$v(t) \leq v_0 \exp\left(\int_a^t g(s)ds\right) + \int_a^t f(s) \exp\left(\int_s^t g(\tau)d\tau\right) ds.$$

Theorem 2.1 *Let u(t) and g(t) be nonnegative, nondecreasing continuous functions in the real interval $I = [a, b]$. Suppose that k(t,s) and its partial derivative $k_t(t,s)$ are nonnegative continuous functions for $t, s \in I$ and assume further that k(t,s) is nondecreasing in its first variable. If for $1 \leq p < \infty$ and for every $t, s \in I$, $a \leq s \leq t \leq b$,*

$$u(t) \leq g(t) + \int_a^t k(t,s)u^p(s)ds \tag{1}$$

then

$$u(t) \leq g(t)[1 - (p-1)\int_a^t g^{p-1}(s)(k(s,s) + \int_a^s k_s(s,\tau)d\tau)ds]^{1/(1-p)}$$

where

$$\beta_p = Sup\{t \in I : (p-1)\int_a^t k(t,s)g^{p-1}(s)ds < 1\}; \qquad a \leq s \leq t \leq \tau < \beta_p.$$

Proof. Let $v(t) = \int_a^t k(t,s)u^p(s)ds$. Then (1) becomes

$$u(t) \leq g(t) + v(t) \tag{2}$$

Differentiating $v(t)$ with respect to t we have

$$\begin{aligned}
v'(t) &= k(t,t)u^p(t) + \int_a^t k_t(t,s)u^p(s)ds \\
&\leq [k(t,t) + \int_a^t k_t(t,s)ds]u^p(t) \\
&\leq [k(t,t) + \int_a^t k_t(t,s)ds][g(t) + v(t)]^p \\
&= [k(t,t) + \int_a^t k_t(t,s)ds][g(t) + v(t)]^{p-1}[g(t) + v(t)] \\
&\leq [k(t,t) + \int_a^t k_t(t,s)ds][g(t) + v(t)]^{p-1}[g(\tau) + v(t)],
\end{aligned}$$

where we have used the non-decreasing property of u and g.

Therefore

(3) $$v'(t) \leq A(t)[g(\tau) + v(t)]$$

where

(4) $$A(t) = [k(t,t) + \int_a^t k_t(t,s)ds][g(t) + v(t)]^{p-1}$$

Comparing (3) with Lemma 2.1 and noting that

$$A(t) \equiv g(t), \quad v(t) + g(\tau) \equiv v(t) \quad \text{and} \quad f(t) \equiv 0,$$

we have

$$v(t) + g(\tau) \leq g(\tau)\exp(\int_a^t A(s)ds); \qquad a \leq t \leq \tau.$$

If $t = \tau$ then

(5) $$v(t) + g(t) \leq g(t)\exp(\int_a^t A(s)ds).$$

$$[v(t) + g(t)]^{p-1} \leq [g(t)\exp(\int_a^t A(s)ds)]^{p-1}.$$

Therefore

(6) $$[v(t) + g(t)]^{p-1} \leq g^{p-1}(t)\exp(\int_a^t (p-1)A(s)ds).$$

Substitute (6) into (4), to obtain

(7) $$A(t) \leq [k(t,t) + \int_a^t k_t(t,s)ds]g^{p-1}(t)\exp(\int_a^t (p-1)A(s)ds).$$

Multiplying through by $(p-1)$ and denoting $(p-1)A(t)$ by $B(t)$ we have,

$$B(t) \leq (p-1)[k(t,t) + \int_a^t k_t(t,s)ds]g^{p-1}(t)\exp(\int_a^t B(s)ds).$$

Hence

$$B(t)\exp(-\int_a^t B(s)ds) \leq (p-1)[k(t,t) + \int_a^t k_t(t,s)ds]g^{p-1}(t).$$

That is,

$$\frac{\partial}{\partial t}[-\exp(-\int_a^t B(s)ds)] \leq (p-1)[k(t,t) + \int_a^t k_t(t,s)ds]g^{p-1}(t).$$

Integrating from a to t we obtain

$$\left[-\exp(-\int_a^t B(s)ds)\right]_a^t \leq \int_a^t (p-1)[k(s,s)+\int_a^s k_s(s,\tau)d\tau]g^{p-1}(s)ds.$$

This gives

$$1-\exp(-\int_a^t B(s)ds) \leq \int_a^t (p-1)[k(s,s)+\int_a^s k_s(s,\tau)d\tau]g^{p-1}(s)ds.$$

Replacing $B(t)$ by $(p-1)A(t)$ we have

$$1-\exp(-\int_a^t (p-1)A(s)ds) \leq \int_a^t (p-1)[k(s,s)+\int_a^s k_s(s,\tau)d\tau]g^{p-1}(s)ds.$$

Hence

$$\begin{aligned}\exp(\textstyle\int_a^t A(s)ds) \leq \{1-(p-1)\textstyle\int_a^t[k(s,s)\\ +\textstyle\int_a^s k_s(s,\tau)d\tau]g^{p-1}(s)ds\}^{1/(1-p)}\end{aligned} \tag{8}$$

From (8), (5) and (2) we have

$$u(t) \leq g(t)\{1-(p-1)\int_a^t [k(s,s)+\int_a^s k_s(s,\tau)d\tau]g^{p-1}(s)ds\}^{1/(1-p)}.$$

This completes the proof of the theorem.

Remark 2.1 If the kernel $k(t,s)$ is separable, that is $k(t,s) = h(t)f(s)$, then theorem 2.1 yields a result which is more general than Theorem 1.1 of Stachurska [6].

Corollary 2.1 *Under the assumptions of Theorem* 2.1, *if* $k(t,s) = h(t)$ *and* $h(t)$ *is nondecreasing, then theorem* 2.1 *becomes: If*

$$u(t) \leq g(t)+h(t)\int_a^t u^p(s)ds, \qquad t,s \in I, \quad a \leq s \leq t \leq b.$$

Then

$$u(t) \leq g(t)[1-(p-1)\int_a^t g^{p-1}(s)h(s)ds]^{1/(1-p)}; \qquad a \leq t \leq \tau < \beta_p.$$

where

$$\beta_p = Sup\{t,s \in I : (p-1)\int_a^t h(s)g^{p-1}(s)ds < 1\}; \qquad a \leq t \leq \tau < \beta_p.$$

Lemma 2.2 [1]. *Let $A, B \geq 0$, then the following inequality holds*

$$(A+B)^p \leq \begin{cases} A^p + B^p, & 0 < p \leq 1 \\ 2^{p-1}\left(A^p + B^p\right), & 1 \leq p < \infty. \end{cases}$$

Theorem 2.2 *Let $u(t)$ and $g(t)$ be nonnegative, nondecreasing continuous functions for $t \geq a$ in real interval $I = [a, b]$. Suppose that $k(t,s)$ and its partial derivatives $k_t(t, s)$ are nonnegative continuous functions for $a \leq s \leq t \in I$ and that $k(t,s)$ is nondecreasing in its first variable. Let $p > 0$ be a constant and suppose that for $t, s \in I$, the following inequality holds*

$$u(t) \leq g(t) + \left(\int_a^t k(t,s) u^p(s) ds\right)^{1/p}.$$

Then

$$u(t) \leq g(t) + \left[c_p \int_a^t R(s) g^p \exp\left(c_p \int_s^t R(\tau) d\tau\right) ds\right]^{1/p}$$

where

$$R(t) = k(t,t) + \int_a^t k_t(t,s) ds,$$

and

$$c_p = \begin{cases} 1 & \text{if } 0 < p \leq 1 \\ 2^{p-1} & \text{if } 1 \leq p \leq \infty. \end{cases}$$

Proof. Let $v(t) = \int_a^t k(t,s) u^p(s) ds$,

Hence

(9) $$u(t) \leq g(t) + v^{1/p}(t)$$

Differentiate $v(t)$ with respect to t we have

$$\begin{aligned} v'(t) &= k(t,t) u^p(t) + \int_a^t k_t(t,s) u^p(s) ds \\ &\leq \left[k(t,t) + \int_a^t k_t(t,s) ds\right] u^p(t) \end{aligned}$$

Since $v(t)$ is nondecreasing, we obtain

$$\begin{aligned} v'(t) &\leq \left[k(t,t) + \int_a^t k_t(t,s) ds\right] u^p(t) \\ &\leq R(t)\left[g(t) + v^{1/p}(t)\right]^p \\ &\leq c_p R(t)\left[g^p(t) + v(t)\right] \qquad \text{by Lemma 2.2} \end{aligned}$$

where c_p is equal to 1 (respectively, 2^{p-1}) for $0 < p \leq 1$ (respectively, $p \geq 1$).

Hence Lemma 2.1 gives

$$v(t) \leq c_p \int_a^t R(s) g^p(s) \exp\left(c_p \int_s^t R(\tau) d\tau\right) ds.$$

Hence by (9) we have

$$u(t) \leq g(t) + \left[c_p \int_a^t R(s) g^p(s) \exp\left(c_p \int_s^t R(\tau) d\tau\right) ds\right]^{1/p}.$$

This completes the proof of the theorem.

We now give the following specific cases of our result.

Remark 2.2 If we set $p = 1$, then our result yields Theorem 1.8 obtained in [1].

Remark 2.3 Unlike Theorem 2.1 in [3] which is only valid for $p > 1$, our estimate in Theorem 2.2 is valid for all $p > 0$.

Remark 2.4 If we set $p = 1$ and $k(t, s) = f(t)$, we obtain the original inequality of Gronwall-Bellman in [1].

Acknowledgement. I wish to thank the referee for his comments and helpful suggestions which have improved the final version of this paper.

REFERENCES

1. D. Bainov and P. Simeonov, *Integral inequalities and Applications*, Academic Publishers, Dordrecht, 1992.

2. Sh. T. Gamidov, *Some integral inequalities for boundary value problems for differential equations*, Diff. Eq **5** (3)(1969), 463-472.

3. J. Norbury and A.M. Stuart, *Volterra integral equations and a new Gronwall inequality Part I: the linear case*, Proc. Royal Soc. Edinburgh **106A** (1987), 361-373.

4. J.A. Oguntuase, *A note on Gronwall-Bellman integral inequality,* Zbornik Radova University of Kragujevac **20**(1998),77-81.

5. J.A. Oguntuase, *Remarks on Gronwall type inequalities,* Stiint. An. Univ. "Al. I. Cuza" t.**45**(1999) (in press).

6. B. Stachurska, *On a nonlinear integral inequality,* Zeszyty Nauk. Univ. Jagiettonsk. 252 Prace Mat. (1971), 151– 157.

DEPARTMENT OF MATHEMATICAL SCIENCES,
COLLEGE OF NATURAL SCIENCES, UNIVERSITY OF AGRICULTURE, ABEOKUTA, NIGERIA.
e-mail : adedayo@unaab.edu.ng

Date received April 2, 1999.