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MULTIPLICITY IN ALGEBRA AND GEOMETRY

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ABSTRACT. Each of §§2,3,5,6 gives a brief description of four approaches to the definition of multiplicity of intersection. In that order they are geometrically via perturbations, algebraically using resultants, Hilbert-Samuel polynomials and the Euler-Poincaré characteristic of a Koszul complex. The final section contains the statement of a new result on this Euler-Poincaré characteristic in the graded case.

1. INTRODUCTION

The purpose of this article is to outline four of the approaches to the idea of multiplicity that have been used in algebraic geometry and in commutative algebra over the last fifty years. In geometry, multiplicity arises naturally in two connected contexts. First, two plane curves have a simple intersection at a common point when their tangents are distinct and a multiple intersection if not, and it is natural to try to extend to multiple intersections the classical theorem of Bézout, for example:

Two algebraic curves V_1, V_2 of the complex projective plane $P(2,\mathbb{C})$ with orders r_1, r_2, which only have simple intersections, meet in $r_1 r_2$ points.

Generally for algebraic varieties $V_1, \dots. Vq$ in $P(n,\mathbb{C})$ with orders $r_1, \dots r_q$ and dimensions $n - d_1, \dots, n - d_q$ which meet in varieties $U_1, \dots, U_t$ with orders $s_1, \dots, s_t$, all of dimension $n - \sum_{j=1}^{q} d_j$, it is natural to try to define a multiplicity of intersection $i(V_1, \dots, V_q : U_k)$ which is a positive integer for each k such that

$$\sum_{k=1}^{r} s_k . i(V_1, \dots, V_q : U_k) = \Pi_{j=1}^{q} r_j .$$

Second, it should be possible to extend the idea of a double or triple point on a plane curve to associate with every subvariety U of an algebraic variety V a multiplicity m in some meaningful way. Thus, when U is a point, m should be the order of the tangent cone to V at U and the intersection multiplicity of V with a general linear space through U of complementary dimension. In this way, a definition for the multiplicity of U on V follows from a satisfactory definition of intersection multiplicity.

2. PERTURBATIONS AND SPECIALIZATION

The geometric approach to multiple intersections is to perturb the intersecting varieties such that all intersections in the perturbed positions are simple then to count coincidences when the perturbation is reversed. The algebraic version of this procedure is to regard each of the intersecting varieties as a member of a (linear) system of varieties with parameters in the defining equations so that the generic members of the systems have simple intersections; the multiplicities can then be defined by specializing. The details required to put this approach on a sound algebraic footing can be found in the work of A. Weil [6].

For the purposes of this article two parts of Weil's arguments are worth high-lighting. First, he gives a list of properties for $i(V_1, ..., V_q : U)$, which characterize these intersection numbers [6; Appendix III], and second, he shows that without real loss of generality in defining $i(V_1, ..., V_q : U)$, it may be assumed that $V_2, ..., V_q$ are all hypersurfaces, i.e., given by a single equation [6; Chap VI].

3. RESULTANTS, ELIMINATION

When, for $i = 1, ..., n$, V_i is a hypersurface given by a single equation $f_i(x_1, ..., x_n) = 0$ in n-dimensional space and the V_i meet (properly) only in points then, intersection multiplicities can be obtained by eliminating the variables x_j[5; §83].

Adjoin a single generic linear polynomial $u = u_0 + \sum_{i=1}^{n} u_i x_i$ to the polynomials $f_1, \ldots, f_n$ and eliminate the x_j to form the u-resultant

$R(u, f_1, ..., f_n)$. R is a homogeneous polynomial of degree equal to the product of the degrees of the f_i which factorizes into powers of distinct linear factors

$$R = \Pi \left(\sum_{j=0}^{n} \lambda_j u_j \right)^{m(\lambda)}$$

where the common points of the V_i are the points $P(\lambda)$ with coordinates $(\lambda_1/\lambda_0, \ldots, \lambda_n/\lambda_0)$. Note that $\lambda_0 = 0$ signifies that $P(\lambda)$ is "at Infinity". The intersection multiplicity $i(V_1, ..., V_n; P(\lambda))$ may be defined as the index $m(\lambda)$.

4. LOCAL RINGS

The approaches to multiplicity outlined in §2,3 have two undesirable properties. First, both are difficult to use, both in examples and in proving theorems. Second, both approaches require global constructions whereas multiplicity is a local property. In the 1930's, Krull introduced the notion of a local ring [3] in order to treat local geometric phenomena in a formal algebraic manner. This section constitutes a digression to construct a local ring $Q(U, V)$ for an irreducible subvariety U of our algebraic variety V. The problem of defining the multiplicities of U as a proper intersection of V with other varieties then becomes a purely algebraic problem of defining the multiplicities of an ideal in a local ring.

For U, an irreducible subvariety of V in n-dimensional affine space over a field K we consider, in the polynomial ring $R - K[x_1, ..., x_n]$, the ideals A, P of those polynomials which vanish at all points of V, U, respectively. Then $P \supseteq A$ and P is a prime ideal of R. The factor ring R/A represents behaviour on V, and the ring R_p of equivalence classes of quotients $f/g (g \notin P)$ represents behaviour near to U. The image $PR_p = \{f/g | g \in P\}$ of P in R_p is the unique maximal ideal of R_p, so R_p is a local ring. Applying both operations in either order produces the same local ring $(R/A)_{P/A} = R_p/(AR_p)$ denoted by $Q(U, V)$ with maximal ideal $M(U, V)$. The difference d of dimensions of U and V is equal to the Krull dimension of Q, i.e. the maximal length of a chain of prime ideals of Q.

When U is part of the intersection of V with d hypersurfaces V_j with equations $f_j(x_1,\ldots,x_n) = 0 (j = 1,\ldots,d)$, the images b_j of $f_j(x_1,\ldots,x_n)$ in $Q(j = 1,\ldots,d)$ generate an ideal $B = \sum_{j=1}^{d} Qb_j$ of Q such that both, Q/B is a Q-module of finite length and, equivalently, B contains a power of M. The task of defining $i(V, V_1,\ldots,V_d : U)$ becomes the algebraic problem of defining the multiplicity of the d-generated ideal B in the d-dimensional local ring Q. The two sections that follow provide solutions to this algebraic problem.

5. HILBERT-SAMUEL POLYNOMIALS

For a d-generated ideal $B = \sum_{i=1}^{d} Qb_i$ in a d-dimensional local ring Q such that $\text{length}_Q\ (Q/B)$ is finite, a natural first suggestion for multiplicity $e(B,Q)$ is that length. This does not agree with the geometric notion as the following example due to Gröbner shows.

Let V be the cone of dimension 2 in 4-dimensional space given parametrically by

$$x_1 = u^4, x_2 = u^3v, x_3 = uv^3, x_4 = v^4.$$

The intersection of V with the hyperplanes V_1 with equation $x_1 = 0$ and V_2 with equation $x_4 = 0$ is the point U with coordinates (0,0,0,0). The multiplicity $i(V, V_1, V_2 : U)$ must be 4.

The local ring $Q(U,V)$ is the ring of rational functions. $f(u^4,u^3v,uv^3,v^4)/g(u^4,u^3v,uv^3,v^4)$ with $g(0,0,0,0) \neq 0$, and the composition series

$$Q \supset (u^4,u^3v,uv^3,v^4) \supset (u^4,u^3v,u^2v^6,v^4) \supset (u^4,u^6v^2,u^2v^2,v^4)$$

$$\supset (u^4,u^6v^2,v^4) \supset (u^4,v^4) = B$$

shows that $\text{length}_Q\ (Q/B) = 5$.

By making use of the following classical result of Hilbert, Samuel [4] gave, the correct modification of length for a satisfactory definition of multiplicity.

If $A \subseteq K[x_0, ..., x_n]$ is the (graded) ideal of all (homogeneous) polynomials which are zero at all points of a d-dimensional algebraic variety V in n-dimensional projective space, then with the linear K-space of all homogeneous polynomials of degree r denoted by F, there exist integers $a_0(>0), a_1, \ldots, a_d$ such that

$$\dim_k(F_r/F_r \cap A) = a_0 \binom{r+d}{d} + a_1 \binom{r+d-1}{d-1} + \ldots + a_d \binom{r}{0}$$

for all large r. In particular, the order of V is a_0.

In the case of a local ring Q of dimension d with maximal ideal M containing an ideal B such that length$_Q$ (Q/B) is finite, consider the graded ring $R(B) = \bigoplus_{i=0}^{\infty} B^i t^i$ and its ideal $BR(B) = \bigoplus_{i=0}^{\infty} B^{i+1} t^i$ with factor ring $R(B)/BR(B) = \bigoplus_{i=0}^{\infty} (B^i/B^{i+1})t^i$. When Q is the local ring of a point U on the variety V and $B = M, R(M)/MR(B)$ is the graded for the cone of tangents to V at U. $R(B)/BR(B)$ is a polynomial ring over the Artinian ring Q/B and the above result of Hilbert can be extended by replacing the field K by Q/B, to establish the existence of integers $e_0(>0), e_1, \ldots, e_d$ such that, for all r large,

(i) length$_Q$ $(B^r/B^{r+1}) = e_0 \binom{r+d-1}{d-1} + e_1 \binom{r+d-2}{d-1} + \ldots + e_{d-1} \binom{r}{0}$

and (ii) length$_Q$ $(Q/B^r) = e_0 \binom{r+d-1}{d-1} + e_1 \binom{r+d-2}{d-1} + \ldots + e_d \binom{r-1}{0}$

by summing.

Samuel's definition for the multiplicity $e(B, Q)$ of the ideal B in the local ring Q is the integer e_0.

When $Q = Q(U, V)$ and $B = M, e(M, Q)$ is the multiplicity of U as a subvariety of V.

When $Q = Q(U,V)$ and U is part of the intersection of V with d hypersurfaces V_j with equations $f_j = 0 (j = 1, ..., d)$ and B is generated by the images of $f_1, \ldots, f_d$, then $e(B,Q)$ is the intersection multiplicity $i(V, V_1, \ldots, V_d : U)$.

We end this section by noting that in the example above, for $r \geq 1$

$$\text{length}_Q\ (Q/B^r) = 4\begin{pmatrix} r+1 \\ 2 \end{pmatrix} + \begin{pmatrix} r+0 \\ 1 \end{pmatrix},$$

or $e_0 = 4$, $e_1 = 1$, $e_2 = 0$ and $e(B,Q) = 4 = i(V, V_1, V_2 : U)$ agrees with the geometric argument.

6. KOSZUL COMPLEX

A second way of modifying $\text{length}_Q(Q/B)$ to give the intersection multiplicity as in §5, when B is generated by d elements of the maximal ideal M of the d-dimensional local ring Q, was suggested by Serre and studied in detail by Auslander and Buchsbaum [1]. This makes use of the Koszul complex $K(a_1, ..., a_d|R)$ for elements $a_1, .., a_d$, in a commutative ring R.

$K(a_1, ..., a_d|R)$ can be defined by induction on d and consists of R-homomorphisms between direct sums R^i of i copies of R for various values of i. We regard the elements of R^i as column vectors and express the R-homomorphisms in matrix form. For low values of d we have

$K(|R)$ is $$0 \to R^1 \to 0$$

$K(a_1|R)$ is $$0 \to R^1 \xrightarrow{a_1} R^1 \to 0$$

$K(a_1, a_2|R)$ is $$0 \to R^1 \xrightarrow{\begin{pmatrix} -a_2 \\ a_1 \end{pmatrix}} R^2 \xrightarrow{(a_1, a_2)} R^1 \to 0$$

$K(a_1, a_2, a_3|R)$ is $0 \rightarrow R^1 \overset{\begin{pmatrix} a_3 \\ -a_2 \\ a_1 \end{pmatrix}}{\rightarrow} R^3 \overset{\begin{pmatrix} -a_2 & -a_3 & 0 \\ a_1 & 0 & -a_3 \\ 0 & a_1 & a_2 \end{pmatrix}}{\rightarrow} R^3 \overset{(a_1,a_2,a_3)}{\rightarrow} R^1 \rightarrow 0$

Note that the product of two neighbouring homomorphisms is zero, so each sequence is a complex.

In general for any complex C of R-modules and R-homomorphisms

$$0 \rightarrow M_d \overset{\delta_d}{\rightarrow} M_{d-1} \overset{\delta_{d-1}}{\rightarrow} M_{d-2} \ldots \rightarrow M_2 \overset{\delta_2}{\rightarrow} M_1 \overset{\delta_1}{\rightarrow} M_0 \rightarrow 0$$

we have $\delta_i \delta_{i+1} = 0$ and homology modules $H_i C = Ker\ \delta_i / Im\ \delta_{i+1}$ can be defined $(i = 0, 1, ..., d)$. Also for an element a of R we have a mapping cone

$$\begin{array}{cccccccc} 0 \rightarrow & M_d \overset{\delta_d}{\rightarrow} & M_{d-1} \overset{\delta_{d-1}}{\rightarrow} & M_{d-2} \ldots \rightarrow & M_2 \overset{\delta_2}{\rightarrow} & M_1 \overset{\delta_1}{\rightarrow} & M_0 \rightarrow 0 \\ & \downarrow a & \downarrow a & \downarrow a & \downarrow a & \downarrow a & \downarrow a \\ 0 \rightarrow & M_d \overset{\delta_d}{\rightarrow} & M_{d-1} \overset{\delta_{d-1}}{\rightarrow} & M_{d-2} \ldots \rightarrow & M_2 \overset{\delta_2}{\rightarrow} & M_1 \overset{\delta_1}{\rightarrow} & M_0 \rightarrow 0 \end{array}$$

and an associated new complex

$$0 \rightarrow M_d \overset{\begin{pmatrix} (-1)^d a \\ \delta_d \end{pmatrix}}{\rightarrow} \begin{array}{c} M_d \\ \bigoplus \\ M_{d-1} \end{array} \overset{\begin{pmatrix} \delta_d & (-1)^{d-1} a \\ 0 & \delta_{d-1} \end{pmatrix}}{\rightarrow} \begin{array}{c} M_{d-1} \\ \bigoplus \\ M_{d-2} \end{array} \ldots \rightarrow \begin{array}{c} M_2 \\ \bigoplus \\ M_1 \end{array} \overset{\begin{pmatrix} \delta_2 & -a \\ 0 & \delta_1 \end{pmatrix}}{\rightarrow} \begin{array}{c} M_1 \\ \bigoplus \\ M_0 \end{array} \overset{(\delta_1 a)}{\rightarrow} M_0 \rightarrow 0.$$

The Koszul complex $K(a_1, ..., a_{+1}|R)$ is derived in this way from the mapping cone of $K(a_1, ..., a_d|R)$ multiplied by a_{d+1}.

The two basic properties we require are

Theorem $a_j H_i K(a_1, \ldots, a_d|R) = 0$ for $i = 0, \ldots, d,\ j = 1, \ldots, d$.

Corollary If R is a Noetherian ring and $H_0 K(a_1, \ldots, a_d|R) = R / \sum_{j=1}^d Ra_j$ has finite length as an R-module, $H_i K(a_1, \ldots, a_d|R)$ has finite length as an R-module for all i.

In these circumstances, we define the multiplicity of $a_1, ..., a_d$ in R as the Euler-Poincaré characteristic

$$\chi(a_1, \ldots, a_d|R) = \sum_{i=0}^{d} (-1)^i \quad \text{length } H_i K(a_1, \ldots, a_d|R).$$

when R is a d-dimensional local ring Q and $B = \sum_{j=1}^{d} Ra_j$ satisfies $\subseteq B$ and $\text{length}_Q(Q/B)$ is finite, then the Krull dimension of Q is at most d. If the dimension is less than d, $\chi(a_1, \ldots, a_d|Q) = 0$, whereas if the dimension is d, then $\chi(a_1, \ldots, a_d|Q) = e(B, Q)$, the multiplicity of §5.

7. THE GRADED KOSZUL COMPLEX

In this final section, I take the opportunity to announce a recent result of the author and D. Rees [2] concerning a multiplicity associated with a sequence $a_1, ..., a_d$ of homogeneous polynomials of $R[x_1, ..., x_m] (= R[x]$ say) with positive degrees $r_1, \ldots, r_d$. This extends the consideration of §6, in the sense that §6 corresponds to the case $m = 1$.

Denote the ideal $\sum_{j=1}^{m} x_j R[x]$ by X and put $A = \{b \in R | bX^t \subseteq \sum_{i=1}^{d} a_i R[x]$ for some $t\}$.

Because the polynomials $a_1, \ldots, a_d$ are homogeneous, the Koszul complex $K(a_1, \ldots, a_d|R[x])$ splits as a direct sum of homogeneous components $K_s (s \geq 0)$. When R is Noetherian, $d \geq m - 1$ and R/A has finite length, all the homology modules $H_i K_s$ of K_s, for $s > \sum_{i=1}^{d}(r_i - 1)$, have finite length. So in this case, the Euler-Poincaré characteristic

$$\chi(K_s) = \sum_{i=0}^{d} (-1)^i \quad \text{length } H_i K_s$$

can be defined and can be shown to be independent of s for $s > \sum_{i=1}^{d}(r_i - 1)$.

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