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ORIGINAL ARTICLE

Quantum states as realizations of groups ☆

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Abstract A quick review of some Lie algebras related to well-known groups is given. We start with the Heisenberg-Weyl algebra and after the definitions of the Fock states we give the definition of the coherent state of this group. This is followed by the exposition of the $SU(2)$ and $SU(1,1)$ algebras and their coherent states. From there we go on to describe the binomial states and their extensions as the finite dimensional pair coherent states and their nonlinear versions as realizations of the $SU(2)$ group. This is followed by considering the negative binomial states, the single mode and two-mode squeezed states and their variants as realizations of the $SU(1,1)$ group. Generation schemes based on physical systems are considered for some of these states.

☆ A tribute: This is a small tribute to the late Prof. Gamal M. Abd AlKader [1963–2009] one with whom I have a very fruitful and most interesting collaboration for almost two decades. His friendship and amicable personality, many of his colleagues and students as well as I really miss. May Allah accept him in His Mercy.

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1. Introduction

The use of the theory of groups in quantum mechanics started with the early days of that theory. The book titled The Theory of Groups and Quantum Mechanics of H. Weyl that was first published in German in 1928 (later its English translation appeared by Dover in 1950) (Weyl, 1950) is a standing witness to this. The Heisenberg-Weyl group was used at the very beginning for the study of some physical structures. Wider dimensions in various branches of physics such as high-energy physics, condensed matter, atomic and nuclear physics benefited greatly from the use of the group theory. With the advances in the field of quantum optics which began in the 60s, group theory started to infiltrate in this branch. Groups involving simple Lie algebras such as $SU(2)$ and $SU(1,1)$ and their simple generalization have been used to study different aspects in quantum optics.

In this article we review some states used in the field of quantum optics as realizations of the $SU(2)$ or $SU(1,1)$ groups. We start by some preliminaries about the annihilation and creation operators and the number operators which constitute the corner stones of the Heisenbeg-Weyl algebra, then their eigenstates are defined. The familiar algebras of the $SU(2)$ and $SU(1,1)$ are introduced. Then some of the quantum states which are realizations of the $SU(2)$ are reviewed in Section 3. Section 4 is devoted to states as realization of the $SU(1,1)$ group. Some comments are given about the generations of some of these states through physical processes.

2. Preliminaries

2.1. The harmonic oscillator

In the study of the harmonic oscillator, the following operators are introduced: the annihilation operator $\hat{a}$, the creation operator $\hat{a}^\dagger$, and the number operator $\hat{n} = \hat{a}^\dagger \hat{a}$. They satisfy the commutation relations

$$[a, a^\dagger] = I, \quad [n, a^\dagger] = a^\dagger, \quad [n, a] = -a. \tag{2.1}$$

The eigen-states $|n\rangle$ of the number operator $\hat{n}$ are called Fock states or number states. They satisfy

$$\hat{n}|n\rangle = n|n\rangle. \tag{2.2}$$

The non-negative integer n can be looked upon as the number of particles in the state. When $n = 0$ we call $|0\rangle$ the vacuum state with no particles present.

The operations of a and $a^\dagger$ on $|n\rangle$ are given by

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{2.3}$$

The states $\{|n\rangle\}$ form a complete set and resolve the unity

$$\sum_n |n\rangle\langle n| = I. \tag{2.4}$$

Another state, which is a superposition of infinite series of the Fock states with their distribution being Poissonian, is the coherent state $|\alpha\rangle$. It is given by its expansion in the number state as

$$|\alpha\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \quad C_n = e^{-\frac{1}{2}|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}}. \tag{2.5}$$

The state $|\alpha\rangle$ can be looked upon as an eigenstate of the operator a such that

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.6}$$

Also, it can be produced by applying the Glauber displacement operator

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a),$$

which is a unitary operator on the vacuum state $|0\rangle$. This is the coherent state of the Heisenberg-Weyl group.

Thus $|\alpha\rangle$ can be expressed as

$$|\alpha\rangle = D(\alpha)|0\rangle = \left(\exp -\frac{1}{2}|\alpha|^2\right)\left(\exp \alpha a^\dagger\right)\left(\exp -\alpha^\dagger a\right)|0\rangle \quad \Rightarrow \quad (2.5) \tag{2.7}$$

after using the Baker–Hausdorff disentanglement formula (Louisell, 1973; Milburn and Walls, 1991). This state describes to a great deal the laser field where the phase is fixed while the number is not. The states $\{|\alpha\rangle\}$ are overcomplete and they satisfy $\int |\alpha\rangle\langle\alpha| \frac{a^2\alpha}{\pi} = I$.

2.2. The angular momentum

The angular momentum defined as $\hat{r} \times \hat{p}$ as well as the spin, are described by the three operators $J_x,\ J_y,\ J_z$ which satisfy the commutation relations (we take $\hbar = 1$)

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y \tag{2.8}$$

with

$$J^2 = J_x^2 + J_y^2 + J_z^2,$$

which commutes with each component. Raising and lowering operator are introduced through the relations

$$J_\pm = J_x \pm iJ_y.$$

Hence the commutation relations (2.8) become

$$[J_z, J_\pm] = \pm J_\pm \quad \text{and} \quad [J_+, J_-] = 2J_z. \tag{2.9}$$

The simultaneous eignestates of the operators J_z and J^2 denoted by $|j,m\rangle$ are given from

$$J^2|j,m\rangle = j(j+1)|j,m\rangle \quad \text{and} \quad J_z|j,m\rangle = m|j,m\rangle \tag{2.10}$$

with $|m| \leqslant j, j$ half integers.

The operations of J_+ and J_- on $|j,m\rangle$ are given by

$$J_+|j,m\rangle = \sqrt{(j-m)(j+m+1)}|j,m+1\rangle,$$
$$J_-|j,m\rangle = \sqrt{(j+m)(j-m+1)}|j,m-1\rangle. \tag{2.11}$$

The operators J_α are the generators of the group $SU(2)$. The angular momentum coherent state is defined by the action of the rotation operator

$$\hat{R}(\theta,\phi) = \exp\left[\frac{1}{2}\theta\left(e^{-i\phi}J_+ - e^{i\phi}J_-\right)\right] \tag{2.12}$$

on the state $|j,-j\rangle$.

The angular momentum coherent state $|\theta,\phi\rangle$ is given by

$$|\theta,\phi\rangle = \hat{R}(\theta,\phi)|j,-j\rangle = \left(\cos\frac{1}{2}\theta\right)^{2j}\sum_{m=-j}^{j}\sqrt{\begin{pmatrix}2j\\ j+m\end{pmatrix}}\left(\tan\frac{1}{2}\theta e^{-i\phi}\right)^{j+m}|j,m\rangle. \tag{2.13}$$

They resolve the identity operator on the space with total angular momentum j as follows:

$$\frac{2j+1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\sin\theta\, d\theta\, d\phi|\theta,\phi\rangle\langle\theta,\phi| = I. \tag{2.14}$$

2.3. The SU(1,1) group

The notion of coherent states is not restricted to the e.m. field or the angular momentum, but can be extended to any set of operators obeying a Lie algebra. The $SU(1,1)$ is the simplest non-abelian noncompact Lie group with a simple Lie algebra (For a comprehensive review we may refer to Perelomov (1986) and the recent review book Dodonov and Man'ko (2003).

The $SU(1,1)$ algebra is spanned by the three operators K_1, K_2, K_3 which satisfy the commutation relations

$$[K_1,K_2] = -iK_3, \quad [K_2,K_3] = iK_1, \quad [K_3,K_1] = iK_2.$$

By using the operators $K_\pm = K_1 \pm iK_2$, hence

$$[K_3,K_\pm] = \pm K_\pm \quad \text{and} \quad [K_+,K_-] = -2K_3. \tag{2.15}$$

The Casimir operator $K^2 = K_3^2 - K_1^2 - K_2^2$ has the value $K^2 = k(k-1)I$ for any irreducible representation. Thus, representation is determined by the parameter k which is called the Bargmann number. The corresponding Hilbert space is spanned by the complete orthonormal basis $\{|k,n\rangle\}$ which are the eigenstates of K^2 and K_3, such that

$$\langle k,n|k,m\rangle = \delta_{nm} \quad \text{and} \quad \sum_{n=0}^{\infty} |k,n\rangle\langle k,n| = I.$$

The operations of the operators $K_\pm$ and K_3 on $|k,n\rangle$ are given by

$$\begin{aligned} K_+|k,n\rangle &= \sqrt{(n+1)(2k+n)}|k,n+1\rangle, \\ K_-|k,n\rangle &= \sqrt{n(2k+n-1)}|k,n-1\rangle, \\ K_3|k,n\rangle &= (k+n)|k,n\rangle. \end{aligned} \tag{2.16}$$

The ground state $|k,0\rangle$ satisfies $K_-|k,0\rangle = 0$ while

$$|k,m\rangle = \sqrt{\frac{\Gamma(2k)}{n!\Gamma(2k+m)}}K_+^m|k,0\rangle.$$

There are two sets of coherent states related to the $SU(1,1)$ group, namely:

(i) The Perelomov coherent states.
By applying the unitary operator

$$D_{Per}(\xi) = \exp(\xi K_+ - \xi^* K_-)$$

to the ground state $|k,0\rangle$ to get

$$|\alpha,k\rangle_{Per} = D_{Per}(\xi)|k,0\rangle = (1-|\alpha|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{n!\Gamma(2k)}}\alpha^n|k,n\rangle \tag{2.17}$$

with $\xi = |\xi|e^{i\theta}$, $\alpha = (\tanh|\xi|)e^{i\theta}$, we have used

$$D_{Per}(\xi) = \exp(\xi K_+ - \xi^* K_-) = \exp(\alpha K_+)(1-|\alpha|^2)^{K_3} \exp -\alpha^* K_-.$$

(ii) The Barut-Girardello coherent state.
It is defined as the eigenstate

$$K_-|\alpha,k\rangle_{BG} = \alpha|\alpha,k\rangle_{BG},$$

which can be expressed as

$$|\alpha,k\rangle_{BG} = \sqrt{\frac{\alpha^{2k+1}}{I_{2k-1}(2|\alpha|^2)}}\sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!\Gamma(n+2k)}}|k,n\rangle. \tag{2.18}$$

$I_v(x)$ is the modified Bessel function of the 1st kind.

After this very quick review of these preliminaries we look at some states which are realization of the $SU(2)$ and $SU(1,1)$ groups.

3. $SU(2)$ realizations

We look at some states which can be looked upon as realizations of the $SU(2)$ group

3.1. The single mode Binomial state

3.1.1. Definitions

These states are of the form (Stoler et al., 1985)

$$|M\cdot\eta\rangle = \sum_{n=0}^{M}\sqrt{\binom{M}{n}}\eta^n(1-|\eta|^2)^{\frac{M-n}{2}}|n\rangle, \tag{3.1}$$

$M \in \mathbb{Z}^+,\ \eta \in \mathbb{C},\ |\eta|^2 \leqslant 1.$

They have the photon-number distribution (probability of finding n photons) as

$$P(n) = \binom{M}{n}|\eta|^{2n}(1-|\eta|^2)^{M-n},$$

which is the binomial distribution.

They are the eigen states of the operator

$$B = \eta a^\dagger a + \sqrt{1-|\eta|^2}\sqrt{MI - a^\dagger a}a$$

with the eigen-value ηM, i.e.

$$B|M,\eta\rangle = \eta M|M,\eta\rangle. \tag{3.2}$$

We may note the following limiting cases:

(i) As $\eta \to 0$; $|M,\eta\rangle \to |0\rangle$ the vacuum state.
(ii) $\eta \to 1$; $|M,\eta\rangle \to |M\rangle$ the Fock state.
(iii) $M \to \infty,\ \eta \to 0$ such that $M|\eta|^2 \to |\alpha|^2$ constant; $|M,\eta\rangle \to |\alpha\rangle$ the coherent state.

Thus it can be understood as an intermediate state.

3.1.2. Group realization

As mentioned in (2.2) when we looked at the $SU(2)$ representations, the angular momentum operators $\underline{J}$ satisfy the relations (2.8); and the $SU(2)$ coherent states which are defined as the action of the rotation operator (2.12) on the ground state. Hence Eq. (2.13) is the $SU(2)$ coherent state. Thus when we take $\eta = \sin\frac{\theta}{2}e^{-i\phi}$ and take $n = j + m$ and $M = 2j$, the binomial state $|2j, \sin\frac{\theta}{2}e^{-i\phi}\rangle$ is the coherent state of the $SU(2)$ group.

3.1.3. Generation scheme

An atom under a classical magnetic field $\underline{B}$ has the interaction Hamiltonian $H = -\underline{J}\cdot\underline{B}$ with the field along the direction x. Under this Hamiltonian the state $|j,-j\rangle$ evolves to the binomial state. The evolution operator $U(t)$ is given by

$$U = \exp -itH = \exp itB(J_+ + J_-).$$

Then $|\psi(t)\rangle = U|j,-j\rangle$ is the coherent state (2.13) with $\theta = 2Bt$ and $\phi = -\frac{\pi}{2}$.

3.2. Finite dimensional pair coherent state

It may be termed as the two-mode binomial state $|\xi, q\rangle$. It has the following definition.

3.2.1. Definition

It can be defined as the eigen state of the operators $\left(a^\dagger b + \frac{\xi^{q+1}(ab^\dagger)^q}{(q!)^2}\right)$ and $(a^\dagger a + b^\dagger b)$ where a, b are annihilation operators for the two modes. The states satisfy the eigen value equations (Obada and Khalil, 2006)

$$\left(a^\dagger b + \frac{\xi^{q+1}(ab^\dagger)^q}{(q!)^2}\right)|\xi, q\rangle = \xi|\xi, q\rangle; \quad (a^\dagger a + b^\dagger b)|\xi, q\rangle = q|\xi, q\rangle \tag{3.3}$$

and take the form

$$|\xi, q\rangle = N_q \sum_{n=0}^{q} \xi^n \sqrt{\frac{(q-n)!}{q!n!}}|q-n, n\rangle, \quad N_q^{-2} = \sum_{n=0}^{q} |\xi|^{2n} \frac{(q-n)!}{q!n!}. \tag{3.4}$$

This is a type of the entangled states where we find $(q-n)$ particles in 1st mode and (n) particles in the 2nd mode.

3.2.2. Relation to the $SU(2)$ group

When we define

$$J_x = \frac{a^\dagger b + ab^\dagger}{2}, \quad J_y = \frac{a^\dagger b - ab^\dagger}{2i}, \quad J_z = \frac{a^\dagger a - b^\dagger b}{2}, \tag{3.5}$$

which are the generators of the $SU(2)$ group. Hence the raising and lowering operators are $J_+ = J_x + iJ_y = a^\dagger b$, $J_- = J_x - iJ_y = ab^\dagger$.

The unitary irreducible representation of the $SU(2)$ are just the familiar angular momentum state $|j, m\rangle$ satisfying the relations

$$C^2|j, m\rangle = \left\{J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+)\right\}|j, m\rangle = j(j+1)|j, m\rangle; \quad J_z|j, m\rangle = m|j, m\rangle.$$

When the following equation $\left(J_- + \frac{\xi^{2j+1}}{((2j)!)^2}(J_+)^{2j}\right)|\xi\rangle = \xi|\xi\rangle$ is solved, it results in

$$|\xi, q\rangle = N_j \sum_{m=-j}^{j} \xi^{j+m} \sqrt{\frac{(j-m)!}{(j+m)!(2j)!}}|j, m\rangle = N_j \sum_{m=0}^{2j} \xi^m \sqrt{\frac{(2j-m)!}{m!(2j)!}}|j, m-j\rangle. \tag{3.6}$$

This is the state (3.4) when we label $q = 2j$ and identify the states $\{|j, m-j\rangle\}$ as the states $\{|q-n, n\rangle\}$.

3.2.3. Exponential form

Write

$$|\xi, q\rangle = N_q \sum_{n=0}^{q} \xi^n \frac{(q-n)!}{q!n!} a^n b^{\dagger n} |q, 0\rangle;$$

which may be cast as

$$|\xi, q\rangle = N_q \sum_{n=0} \frac{\xi^n}{n!} [(n_a+1)^{-1} ab^\dagger]^n |q, 0\rangle = N_q \exp[\xi(n_a+1)^{-1} ab^\dagger]|q, 0\rangle \tag{3.7}$$

3.2.4. Generation scheme

A proposal is presented in Obada and Khalil (2006) of an experimental scheme in the vibronic motion of the center of mass of an ion trapped in real 2-dimensional space; by using three laser fields. Under certain specifications the Hamiltonian of the interaction is simplified to

$$H_{int} = \lambda\left(a^\dagger b + \frac{\xi^{q+1}(ab^\dagger)^q}{(q!)^2} - \xi\right)\underline{\sigma} + h.c \tag{3.8}$$

with $\lambda = -|\underline{\mu} \cdot \underline{E}_1|\eta_1\eta_2 \exp\left(-\frac{(\eta_1^2+\eta_2^2)}{2} + i\phi_1\right)$, $\xi = \frac{|\underline{\mu}\cdot\underline{E}_0| \exp i(\phi_1-\phi_2)}{|\underline{\mu}\cdot\underline{E}_1|\eta_1\eta_2}$ and $|\underline{\mu} \cdot \underline{E}_2| = \frac{\xi^{q+1}|\underline{\mu}\cdot\underline{E}_1|}{(1-)^{q+1}(\eta_1\eta_2)^{q-1}}$ with the electric-dipole moment μ and $\underline{E}_i$ the amplitudes of the electric fields of the laser fields and η_i are the Lamb-Dicke parameters. The vibronic eigen state that satisfies $H_{int}|\xi\rangle = 0$ belongs to the class of states (3.4).

It is a stationary solution of the master equation of the density operator

$$\frac{\partial\rho}{\partial t} = -i[H_{int}, \rho] + \frac{\gamma}{2}[2\sigma_-\rho_+\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-].$$

Some properties of these states may be found in Obada and Khalil (2006).

3.3. Nonlinear two-mode binomial state

3.3.1. Definition

An extension to the earlier state is performed by introducing the nonlinear finite dimensional pair coherent state as the eigen state satisfying

$$\left[f_1(n_a)a^\dagger b f_2(n_b) + \xi^{q+1}\frac{\left(a\frac{1}{f_1(n_a)}\cdot\frac{1}{f_2(n_b)}b^\dagger\right)^q}{(q!)^2}\right]|\xi, q\rangle_f = \xi|\xi, q\rangle_f \tag{3.9}$$

and

$$(n_a+n_b)|\xi, q\rangle = q|\xi, q\rangle$$

with the usual notation.

It is expanded in the Fock states for the two modes as (Khalil, 2006)

$$|\xi, q\rangle_f = N_q \sum_{n=0}^{q} \xi^n \sqrt{\frac{(q-n)!}{n!q!}} \frac{f_1(q-n)!}{f_1(q)!f_2(n)!} |q-n, n\rangle \tag{3.10}$$

with $f(n)! = f(0) \cdot f(1) \cdots f(n)$ and $f(0) = 1$.

3.3.2. Relation to the SU(2) group

Introduce the operators

$$\begin{aligned} J_x &= \frac{f_1(n_a)a^\dagger f_n(n_b)b + af_1^{-1}(n_a)b^\dagger f_2^{-1}(n_b)}{2}, \\ J_y &= \frac{f_1(n_a)a^\dagger f_n(n_b)b - af_1^{-1}(n_a)b^\dagger f_2^{-1}(n_b)}{2i}, \quad J_z = \frac{\hat{n}_a - \hat{n}_b}{2} \end{aligned} \tag{3.11}$$

which satisfy the relations $[J_x, J_y] = iJ_z,\ [J_y, J_z] = iJ_x,\ [J_z, J_x] = iJ_y$. Note that neither J_x or J_y is Hermitian, hence we define

$$J_+ = f_1(n_a)a^\dagger f_2(n_b)b, \quad J_- = a\frac{1}{f_1(n_a)}b^\dagger\frac{1}{f_2(n_b)}$$

consequently $[J_z, J_\pm] = \pm J_\pm,\ [J_+, J_-] = 2J_z$.

These operators can be thought of as generators of an extended $SU(2)$ group. The operator

$$C_2 = J_z^2 + \frac{1}{2}(J_+J_- + J_-J_+) = \left(\frac{n_a+n_b}{2}\right)\left(\frac{n_a+n_b}{2}+1\right)$$

commutes with the generating operators.

For the states $|j, m\rangle$ satisfying the relations

$$\begin{aligned} C_2|j,m\rangle &= j(j+1)|j,m\rangle, \quad J_z|j,m\rangle = m|j,m\rangle, \\ J_+|j,m\rangle &= |f_1(j+m+1)||f_2(j-m-1)|\sqrt{(j+m+1)(j-m)}|j,m+1\rangle, \\ J_-|j,m\rangle &= |f_1(j+m)|^{-1}|f_2(j-m)|^{-1}\sqrt{(j+m)(j-m+1)}|j,m-1\rangle, \\ |\xi,2j\rangle &= N_{2j}\sum_{n=0}^{2j}\xi^n\frac{f_1(2j-n)!}{f_1(2j)!f_2(n)!}\sqrt{\frac{(2j-n)!}{2j!n!}}|j,n-j\rangle, \\ |\xi,2j\rangle &= N_{2j}\sum_{n=-j}^{j}\frac{f_1(j-n)!}{f_1(2j)!f_2(n+j)!}\sqrt{\frac{(j-n)!}{2j!(n+j)!}}|j,n\rangle, \end{aligned}$$

which is the same as (3.11) when we take $2j = q$ and the states $\{|j, n-j\rangle\}$ to correspond to $\{|q-n, n\rangle\}$.

3.3.3. Exponential form

We can cast Eq. (3.11) in the form

$$|\xi, q\rangle_f = N_q \sum_{n=0}^{q} \xi^n \frac{(q-n)!a^n b^{\dagger n}}{q!n!} \frac{f_1(q-n)!}{f_1(q)!f_2(n)!} |q, 0\rangle, \tag{3.12}$$

which can be written as

$$|\xi, q\rangle_f = N_q \sum_{n=0} \frac{\left[ab^\dagger \frac{1}{f_1(n_a)f_2(n_b)n_a}\right]^n}{n!} |q, 0\rangle = N_q \exp\left(ab^\dagger \frac{\xi}{f_1(n_a)f_2(n_b)n_a}\right)|q, 0\rangle$$

giving an exponential form for the state considered some properties of these states may be found in Khalil (2006).

3.3.4. Generation scheme
A generation scheme is presented in Khalil (2006) essentially similar to that of the variant of section 3.2.

4. $SU(1,1)$ realizations

There are a large number of states that can be termed as realizations of the $SU(1,1)$ group reviewed in section 2.3. Here we mention some of these states.

4.1. The negative binomial states

4.1.1. Definition
This state is defined as the Fock state expansion (Joshi and Lawande, 1989)

$$|M, \xi\rangle_N = \sum_{n=0}^{\infty} \sqrt{\frac{(n+M)!}{n!M!}} \xi^n (1-|\xi|^2)^{\frac{M+1}{2}} |n\rangle. \tag{4.1}$$

This state follows the negative binomial distribution for the photon number distribution

$$P(n) = \frac{(M+n)!}{M!n!} |\xi|^{2n} (1-|\xi|^2)^{M+1}.$$

The special case of $M=0$ is the Pascal distribution or the thermal distribution. The state (3.3) interpolates between the pure thermal state and the coherent state ($\xi \to 0, M \to \infty, M|\xi|^2 \to |\alpha|^2$), hence it is termed as an intermediate state.

4.1.2. SU(1,1) realization
This can be achieved by introducing the operators

$$K_+ = a^\dagger \sqrt{MI + \hat{n}}, \quad K_- = \sqrt{MI + \hat{n}} a, \quad K_z = \frac{M}{2} I + \hat{n}, \tag{4.2}$$

which are the raising, lowering and generators of the $SU(1,1)$ group.

Thus the unitary evolution operator $D(\eta) = \exp(\eta K_+ - \eta^* K_-)$ can be disentangled and applied on the vacuum state to have the state

$$D(\eta)|0\rangle = \exp(\xi K_+)[1-|\xi|^2]^{\frac{M+1}{2}}\exp(-\xi K_-)|0\rangle$$
$$= [1-|\xi|^2]^{\frac{M+1}{2}}\sum_{n=0}^{\infty}\xi^n\sqrt{\frac{(M+n)!}{M!n!}}|n\rangle = |M,\xi\rangle_N,$$

where $\xi = \frac{\eta}{|\eta|}\tanh|\eta|$.

4.2. The non-linear negative binomial state

4.2.1. Definition

The nonlinear extension to the above state has been introduced (Abdalla et al., 2007). It amounts to deform the operator a to $A = af(n)$ where $f(n)$ is an operator valued function. Hence the state is given by

$$|M\cdot\xi\rangle_{Nf} = N_f\sum_{n=0}^{\infty}\xi^n(1-|\xi|^2)^{\frac{M+1}{2}}\sqrt{\frac{(M+n)!}{M!n!}}(f^{\dagger}(n)!)|n\rangle. \tag{4.3}$$

The commutation relation

$$[A,A^{\dagger}] = [af(m),f^{\dagger}(n)a^{\dagger}] = (n+1)|f(n+1)|^2 - n|f(n)|^2$$

becomes $[A,A^{\dagger}] = 1$ for $f^{\dagger}(n) = f^{-1}(n)$ i.e. unitary operator.

4.2.2. Group realization

For the operator $f(n)$ being unitary, the following $SU(1,1)$ generators are defined

$$K_+ = a^{\dagger}f^{\dagger}(n)\sqrt{MI+n},\quad K_+ = \sqrt{MI+n}f(n)a,\quad K_z = \frac{M}{2}I+n. \tag{4.4}$$

The state (4.3) is obtained by applying $D(\eta)$ of Section 4.1.1 but with the operators given by (4.4), on the vacuum state

$$D_f(\eta)|0\rangle = (1-|\xi|)^{\frac{M+1}{2}}\sum_{n=0}^{\infty}\frac{\xi^n}{n!}(K_+)^n|0\rangle$$
$$= (1-|\xi|^2)^{\frac{M+1}{2}}\sum\xi^n\sqrt{\frac{(M+n)!}{M!n!}}(f^{\dagger}(n)!)|n\rangle = |M,\eta\rangle_f, \tag{4.5}$$

which is the $SU(1,1)$ realization for the nonlinear negative binomial state.

4.2.3. Non-unitary case

For the case when $f^{\dagger}(n)\neq f^{-1}(n)$ we can still define a state by using a similar expansion, however, with different conjugate operators. Define the canonical conjugate operators

$$B_+ = a^{\dagger}\frac{1}{\sqrt{MI+n}f^{\dagger}(n)},\quad \therefore [K_-,B_+] = I$$

and

$$B_- = \frac{1}{\sqrt{MI+n}f^{\dagger}(n)}a,\quad \therefore [K_+,B_-] = I$$

Then define

$$D_I(\xi) = \exp(\xi K_+ - \xi^\dagger B_-),$$

which is not a unitary operator. Applying $D_I(\xi)$ to $|0\rangle$ results on

$$D_I(\xi)|0\rangle = e^{-\frac{1}{2}|\xi|^2} \exp(\xi K_+) \exp(-\xi^\dagger B_-)|0\rangle,$$
$$= e^{-\frac{1}{2}|\xi|^2} \sum_{n=0}^{\infty} \frac{\xi^n}{n!} (K_+)^n |0\rangle,$$

which is the state (4.3) apart from a normalization constant.

There is another state associated with the operator

$$D_{II}(\xi) = \exp(\xi B_+ - \xi^\dagger K_-),$$

when applied to the vacuum state we get

$$D_{II}(\xi)|0\rangle = e^{-\frac{1}{2}|\xi|^2} \sum_{s=0}^{\infty} \frac{\xi^s}{n!} \left(a^\dagger \frac{1}{f(n)\sqrt{MI + \hat{n}}} \right)^s |0\rangle$$
$$= e^{-\frac{1}{2}|\xi|} \sum_n \frac{\xi^n}{\sqrt{n!(n+M)!f(n)!}} |n\rangle,$$

which is another nonlinear form of the pair-coherent state.

Some properties of these states are found in Abdalla et al. (2007).

4.3. Single mode squeezed vacuum and 1st excited states

4.3.1. Definition

The squeezed vacuum state is defined as the eigenstate of the operator $b = \mu a + \nu a^\dagger$ with eigenvalue zero and $|\mu|^2 - |\nu|^2 = 1$ (Gerry, 1987; Agarwal, 1988; Nieto and Truax, 1993). It has the expansion

$$|\xi\rangle_0 = \sum_{n=0}^{\infty} \frac{\sqrt{2n!}}{2^n n!} \xi^n (1 - |\xi|^2)^{\frac{1}{4}} |2n\rangle \tag{4.6}$$

where $\xi = \tanh r e^{i\phi}$, $\mu = \cosh r$, $\nu = \sinh r \; e^{i\phi}$.

While the squeezed 1st excited state is obtained as the eigen state of the operator b^2 with eigenvalue 0. It has the form

$$|\xi\rangle_1 = \sum_{n=0}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!} \xi^n (1 - |\xi|^2)^{\frac{3}{4}} |2n+1\rangle. \tag{4.7}$$

This can be cast as a realization of the $SU(1,1)$ group by taking

$$K_+ = \frac{a^{\dagger 2}}{2}, \quad K_- = \frac{a^2}{2}, \quad K_3 = \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right). \tag{4.8}$$

The Casamir operator C_2 in this case

$$C_2 = k(k-1)I = \frac{-3}{16} I.$$

The state space associated with $k = \frac{1}{4}$ is the even Fock sub-space with $\{|2n\rangle\}$ and that associated with $k = \frac{3}{4}$ is the odd Fock subspace with $\{|2n+1\rangle\}$. The unitary operator (the squeeze operator)

$$S(z) = \exp(zK_+ - z^* K_-) = \exp\left(\frac{1}{2} z a^{\dagger 2} - \frac{1}{2} z^* a^2\right)$$

The $SU(1,1)$ coherent states are the single-mode squeezed sates. For $k = \frac{1}{4}$ we have squeezed vacuum

$$\left|\xi, \frac{1}{4}\right\rangle = S(z)|0\rangle = |\xi\rangle_0 \quad \text{of (4.6)} \;\; \xi = \frac{z}{|z|} \tanh |z|$$

for $k = \frac{3}{4}$ we have the squeezed one photon state

$$\left|\xi, \frac{3}{4}\right\rangle = S(\xi)|1\rangle = |\xi\rangle_1 \quad \text{of (4.7).}$$

4.4. Nonlinear squeezed states

The use of the following operators:

$$K_+ = \frac{1}{2} (f^\dagger(n) a^\dagger)^2, \quad K_- = \frac{1}{2} (af(n))^2. \tag{4.9}$$

For the unitary operator function $f^\dagger(n) = f^{-1}(n)$, we have

$$K_3 = \frac{1}{2}\left(a^\dagger a + \frac{1}{2}\right).$$

Under these operators, we have the nonlinear squeezing operator

$$S_f(z) = \exp \frac{1}{2} (z A^{\dagger 2} - z^\dagger A^2), \quad \text{where } A = af(n).$$

Consequently

$$\begin{aligned} \left|\xi, \frac{1}{4}\right\rangle_f &= (1 - |\xi|^2)^{\frac{1}{4}} \sum_{n=1}^{\infty} \frac{\sqrt{(2n)!}}{2^n n!} (f(2n)!) \xi^n |2n\rangle, \\ \left|\xi, \frac{3}{4}\right\rangle_f &= (1 - |\xi|^2)^{\frac{3}{4}} \sum_{n=1}^{\infty} \frac{\sqrt{(2n+1)!}}{2^n n!} (f(2n+1)!) \xi^n |2n+1\rangle. \end{aligned} \tag{4.10}$$

4.4.1. Non-unitary f

If $f^\dagger(n) \neq f^{-1}(n)$, we use the canonical conjugate operators for $A = af(n)$ the operator $B^\dagger = \frac{1}{f(N)} a^\dagger$ where $[A, B^\dagger] = I$.

One looks for the eigenvalue problems (Obada and Darwish, 2005; Obada and Abd-Al-Kader, 2007).

$$C_1|\psi_1\rangle_f = 0 \quad \text{and} \quad C_2|\phi_2\rangle = 0,$$

where $C_1 = \frac{1}{\sqrt{1-|\xi|^2}}(A - \xi_1 B^\dagger),\ C_2 = \frac{1}{\sqrt{1-|\xi|^2}}(B - \xi A^\dagger)$.

It is straightforward to find the expressions

$$|\psi_1\rangle_f = N_1 \sum_{m=0}^{\infty} \frac{\sqrt{2m!}(f(2m))^{-1}}{2^m m!} \xi_1^m |2m\rangle,$$

$$|\psi_2\rangle_f = N_2 \sum_{m=0}^{\infty} \frac{\sqrt{2m!}(f(2m))}{2^m m!} \xi_2^m |2m\rangle,$$

which are of the same form as these of $|\xi, \frac{1}{4}\rangle_f$.

While for the eigenvalue problems $C_i^2|\phi_i\rangle = 0, i = 1, 2$, we get

$$|\phi_1\rangle_f = N_1' \sum_{m=0}^{\infty} \frac{\sqrt{(2m+1)!}}{2^m (m+1)!(f(2m+1)!)} \xi^m |2m+1\rangle,$$

$$|\phi_2\rangle_f = N_2' \sum_{m=0}^{\infty} \frac{\sqrt{(2m+1)!}(f(2m+1)!)}{2^m (m+1)!} \xi^m |2m+1\rangle.$$

Similar to the forms of $\left|\xi, \frac{3}{4}\right\rangle_f$ of (4.10) but with different normalization constants.

4.5. Single mode squeezed coherent state

4.5.1. Definition

These states are the solutions of eigenvalue problem (Yuen, 1976; Schnmaker, 1986)

$$b|\beta\rangle = \mu a + v a^\dagger |\beta\rangle = \beta|\beta\rangle$$

with $\mu = \cosh r,\ \ v = \sinh r e^{i\phi}$.

If we write $\xi = -\frac{v}{\mu} = -e^{i\phi}\tanh r$, then the state

$$|\beta\rangle = |\beta, \xi\rangle = |\alpha, r\rangle$$

$$= (1 - |\xi|^2)^{\frac{1}{4}} \exp -\frac{1}{2}\left\{|\beta|^2 - \frac{|\xi|}{2}(\beta^2 e^{i\phi} + \beta^{*2} e^{-i\phi})\right\}$$

$$\times \sum_{m=0}^{\infty} \frac{(-\frac{1}{2}\xi)^{\frac{m}{2}}}{\sqrt{m!}} H_m\left(\frac{\beta\sqrt{1-|\xi|^2}}{\sqrt{-2\xi}}\right)|m\rangle \tag{4.11}$$

with $\beta = \mu\alpha + v\alpha^*$.

4.5.2. SU(1,1) realization

When we use the representation (4.8) for the operators $K_\pm,\ K_3$, the state (4.11) can be cast as the operation of the operator

$$S(\xi) = \exp(\xi K_+ - \xi^* K_-) = \exp\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)$$

on the state $|\beta\rangle$ of the form (2.5). Therefore we have

$$|\xi, \beta\rangle = S(\xi)|\beta\rangle = S(\xi)D(\beta)|0\rangle.$$

After using the disentanglement of the squeezing operator and applying it to the state $|\beta\rangle$ we get the expression (4.11). We may use the relation

$$S(z)D(\beta) = D(\alpha)S(z) \quad \text{with } \alpha = \mu\beta + \nu\beta^*.$$

Hence we get

$$|\xi, \beta\rangle = S(\xi)D(\beta)|0\rangle = D(\alpha)S(\xi)|0\rangle,$$

i.e. we displace the squeezed vacuum or squeeze the coherent state.

4.6. *Nonlinear squeezed coherent state*

4.6.1. *Unitary operator function*

The nonlinear operator A is defined as $A = af(n)$ and $A^\dagger = f^\dagger(n)a^\dagger$ where the operator valued function $f(n)$ is a unitary operator i.e. $f^\dagger = f^{-1}$. In this case we find $[A, A^\dagger] = I$.

The operators $K_\pm,\ K_3$ of the $SU(1,1)$ are defined as is (4.9). The nonlinear realization in this case is given by Satyanarayana (1985) and Kral (1990b)

$$\begin{aligned}|\xi, \beta\rangle_f &= S_f(\xi)D_f(\beta)|0\rangle = \exp\frac{1}{2}(\xi A_+^{\dagger 2} - \xi^* A_-^2)\exp(\beta A^\dagger - \beta^* A)|0\rangle \\ &= (1 - |\xi^2|)^{\frac{1}{4}} \exp -\frac{1}{2}\left\{|\beta|^2 - \frac{|\xi|}{2}(\beta^2 e^{i\phi} + \beta^* e^{-i\phi})\right\} \\ &\quad \times \sum_{m=0}^{\infty} \frac{1}{(f(n))!)'\sqrt{m!}}\left(-\frac{1}{2}\xi\right)^{\frac{m}{2}} H_m\left(\frac{\beta\sqrt{1-|\xi|^2}}{\sqrt{-2\xi}}\right)|m\rangle \end{aligned} \tag{4.12}$$

The appearance of the function $f(m)$ denotes the effect of the nonlinearity.

4.6.2. *Definition for nonunitary nonlinear function*

For the nonlinear operator valued function $f(n)$ we define the canonical conjugate operators B and $B^\dagger$ such that

$$B = a\frac{1}{f^\dagger(n)}, \quad B^\dagger = \frac{1}{f(n)}a^\dagger \quad \text{such that } [A, B^\dagger] = I, \quad [B, A^\dagger] = I$$

and we look for the eigen functions of the two operators

$$C_1 = \frac{1}{\sqrt{1-|\xi|^2}}(A - \xi_1 B^\dagger), \quad C_2 = \frac{1}{\sqrt{1-|\xi|^2}}(B - \xi_2 A^\dagger),$$

which satisfy the equations

$$C_i|\xi_i, \beta_i\rangle_f = \beta_i|\xi_i, \beta_i\rangle_f, \quad i = 1, 2.$$

The eigenfunctions for C_1 are of the same form of (4.12) except for the normalization constant. While the eigen function of C_2 are of the form

$$|\xi_2, \beta_2\rangle_f = c' \sum_m \frac{f(m)!}{\sqrt{m!}} \left(-\frac{1}{2}\xi\right)^{\frac{m}{2}} H_m \left(\frac{B_2\sqrt{1-|\xi_2|^2}}{\sqrt{-2\xi_2}} \right) |m\rangle.$$

Note the appearance of the nonlinearly function in the last form.

4.7. Squeezed displaced Fock states

These states are defined as application of the squeezed operators and the displacement operators on the Fock state $|m\rangle$ in the following form (Satyanarayana, 1985; Kral, 1990b).

$$|\alpha, \xi, m\rangle = S(\xi)D(\alpha)|m\rangle = D(\alpha_0)S(\xi)|m\rangle$$

with $\alpha_0 = \alpha \cosh r + \alpha^* \sinh r e^{i\phi} = \mu\alpha + v\alpha^*,\ \xi = re^{i\phi}$.

Its expansion in the Fock state space is given by

$$|\alpha, \xi, m\rangle = \sum_n \left(\frac{n!}{\mu m!}\right)^{\frac{1}{2}} \left(\frac{v}{2\mu}\right)^{\frac{n}{2}} \exp\left(-\frac{|\alpha|^2}{2} + \frac{v^*}{2\mu}\alpha^2\right)$$
$$\times \sum_{i=0}^{\min(n,m)} \binom{m}{i} \frac{\left(\frac{2}{\mu v}\right)^{\frac{1}{2}}}{(n-i)!} \left[\left(-\frac{v^*}{2\mu^*}\right)^{\frac{m-i}{2}}\right]$$
$$\times H_{n-i}\left(\frac{\alpha}{sqrt2v\mu}\right) H_{m-i}\left(\frac{-\alpha^*}{sqrt2v^*\mu^*}\right) \tag{4.13}$$

Here $S(\xi)$ is the unitary operator that could be expressed in terms of the generators of the $SU(1,1)$ group as defined in the Eq. (4.8).

4.8. Two-mode squeezed vacuum states

These states are obtained by applying the non-degenerate two-mode operator

$$S_2(\xi) = \exp(-\xi a^\dagger b^\dagger + \xi^* ab)$$

on the vacuum state $|0_1, 0_2\rangle$ with $\xi = re^{i\phi}$. These states are expressed in terms of the two-mode Fock states in the form

$$|\xi\rangle_2 = S_2(\xi)|0_1, 0_2\rangle = \frac{1}{\cosh r} \sum (\tanh re^{i\phi})^n |n, n\rangle. \tag{4.14}$$

These states are considered as a class of entangled states where the numbers of quanta in both modes are equal in each component.

4.8.1. SU(1, 1) realization

This state can be considered as realization of the $SU(1,1)$ as coherent states of this group. This is accomplished by defining the generators as follows:

$$K_+ = ab, \quad K_- = a^\dagger b^\dagger, \quad K_3 = \frac{1}{2}(n_a + n_b + 1). \tag{4.15}$$

The Casimir operator

$$C_2 = K_3^2 - \frac{1}{2}(K_+K_- + K_-K_+) = \frac{1}{4}[(n_a - n_b)^2 - 1]$$

Therefore the irreducible representation with $k = \frac{q+1}{2}$, $q = 0, 1, 2, \ldots$ is the eigenvalue of $(n_a - n_b)$ hance $|m,k\rangle \to |n+q,n\rangle$ with $n = 0, 1, 2, \ldots$

$$\left|\xi, k = \frac{1+q}{2}\right\rangle = \sum_{n=0}^{\infty}(1-|\xi|^2)^{\frac{q+1}{2}}\sqrt{\frac{(n+q)!}{n!q!}}\xi^n|n+q,n\rangle. \tag{4.16}$$

There is another coherent state of $SU(1,1)$ (Barut-Girardillo) which is the eigenfunction of the operator K_-

$$K_-|\alpha,k\rangle_{BG} = \alpha|\alpha,k\rangle_{BG} \quad \therefore |\alpha,k\rangle_{BG} = \sqrt{\frac{|\alpha|^{2k+1}}{I_{2k-1}(2|\alpha|)}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!\Gamma(n+2k)}}|n,k\rangle.$$

4.9. Nonlinear two mode squeezed vacuum state

As before we have $A = af_1(n_a), B = bf_2(n_b)$ and their Hermitian conjugates, consequently

$$[A, A^\dagger] = (n_a+1)f_1(n_a+1)f^\dagger(n_a+1) - n_a f_1^\dagger(n_a)f_1(n_a) \tag{4.17}$$

with a similar formula for $[B, B^\dagger]$.

4.9.1. The unitary case

For this case $f_i^\dagger = f_i^{-1}$, $\therefore f_i f_i^\dagger = I$. Consequently the nonlinear squeezing operator will be a unitary operator and we have

$$\begin{aligned}|\xi\rangle_{2f} &= S_{2f}(\xi)|0,0\rangle = \exp(-\xi A^\dagger B^\dagger + \xi^\dagger AB) \\ &= \frac{1}{\cosh r}\sum(\tanh re^{i\phi})^n f_1(n_a)!f_2(n_b)!|n_a,n_b\rangle,\end{aligned} \tag{4.18}$$

which has the same form as (4.13) apart from the appearance of the nonlinearity functions.

4.9.2. The nonunitary case

In this case $f_i^\dagger \neq f_i^{-1}$ we define in this case $A_1 = a\frac{1}{f_1^\dagger(n_a)}$, $B_1 = b\frac{1}{f_2^\dagger(n_b)}$ and their Hermitian conjugates.

These operators satisfy $[A, A_1^\dagger] = I$, $[B, B_1^\dagger] = I$ and we look for eigenfunctions of the operators

$$C_1 = \frac{1}{\sqrt{1 - |\xi_1|^2}}(AB - \xi_1 A_1^\dagger B_1^\dagger) \quad \text{and} \quad C_2 = \frac{1}{\sqrt{1 - |\xi_2|^2}}(A_1 B_1 - \xi_2 A^\dagger B^\dagger).$$

The eigen function $|\psi_i\rangle$ satisfying $C_i|\psi_i\rangle = 0, i = 1, 2$ give the function $|\psi_1\rangle$ in the same form of (4.18) apart from a normalization constant, where $|\psi_2\rangle$ is of the same form but with $(f_1(n)!)^{-1}(f_2(n)!)^{-1}$ instead of $(f_n(n)!)(f_2(n)!)$. Some of the properties of these states introduced in these subsections are presented in Abd-Al-Kader and Obada (2008a).

4.10. *SU*(1, 1) *Intelligent states*

For the two self-adjoint operator A, B, one obtains the uncertainty relation

$$\langle(\Delta A)^2\rangle\langle(\Delta B)^2\rangle \geqslant \frac{1}{4}|\langle[A, B]\rangle|^2$$

A state is called an intelligent state (IS) if it satisfies the strict equality. Such states must satisfy the eigen value equation

$$(A - i\lambda B)|\psi\rangle = \eta|\psi\rangle, \tag{4.19}$$

λ is a positive real parameter, η a complex number. When $[A, B] = cI$ with constant c, the minimum uncertainty states (MUS) coincide with the IS.

For the $SU(1, 1)$ the IS $|\psi\rangle$ are solutions of the eigenvalue problem

$$(K_1 - i\lambda K_2)|\psi\rangle = \eta|\psi\rangle$$

or

$$(\alpha_1 K_- + \beta_1 K_+)|\psi\rangle = \eta|\psi\rangle, \quad \alpha_1 = 1 + \lambda,\ \beta_1 = 1 - \lambda. \tag{4.20}$$

In the basis $|n, k\rangle$ of the $SU(1, 1)$ of Eq. (2.16) $|\psi\rangle$ is given by

$$|\psi\rangle = \sum_{n=0}^{\infty} c_n|n, k\rangle.$$

The coefficients c_n can be calculated to be related to the Pollaczek polynomial (Abd-Al-Kader and Obada, 2008b)

$$\begin{aligned} c_n &= \left(\frac{\beta_1}{\alpha_1}\right)^n P_n\left(\frac{\eta}{\sqrt{\alpha_1\beta_1}}, k\right) \\ &= i^n \left(\frac{\beta_1}{\alpha_1}\right)^{\frac{n}{2}} \left(\frac{\Gamma(n + 2k)}{n!\Gamma(2k)}\right)^{\frac{n}{2}} {}_2F_1\left(-m, k + \frac{i\eta}{\sqrt{\alpha_1\beta_1}}; 2k, 2\right) \end{aligned} \tag{4.21}$$

The polynomials $P_n(z, k)$ form a complete orthonormal set of polynomials on the real line with weight function

$$\rho_k(z) = (2)^{2k-1}\left[\frac{|\Gamma(k+iz)|^2}{\pi\Gamma(2k)}\right]$$

Some special cases are discussed in Abd-Al-Kader and Obada (2008b). For example:

(i) The one mode realization includes as special cases: the Barut-Girardillo state, the Perelomov C.S., and the nonlinear squeezed coherent states.
(ii) The two-mode realizations include as special cases; the pair coherent state as the correlated $SU(1,1)$ C.S., and nonlinear realizations.

5. Conclusion

In this article we had tried to review some quantum states and their relations to some algebraic groups. Starting from the harmonic oscillator and the algebra that underlies it, the annihilation and creation operators are defined. Some states related to these operators are constructed, especially the Fock or number states and the coherent states.

The angular momentum and its algebra which introduced us to the $SU(2)$ group and its Lie algebra are demonstrated. Then the $SU(1,1)$ algebras and some of the relations of their operators and representations, especially some of their coherent states, are mentioned. As realizations of these groups the discussion went on with the single mode binomial states, the finite dimensional pair coherent states, and their nonlinear variants. Then came the negative binomial states, single mode squeezed vacuum, squeezed coherent, squeezed displaced Fock states and their nonlinear variants. The two mode squeezed vacuum states and their nonlinear counterparts are discussed. Finally the intelligent states are mentioned.

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