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SOME REDUNDANCY RESULTS FOR CONTINUUM STRUCTURE FUNCTIONS

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ABSTRACT. It has been well known among reliability engineers and theorists that redundancy at the component level is preferable to redundancy at the system level when only two states of performance are distinguished - functioning and failed. Further, the parallel system is the only system for which these two types of redundancy are equivalent. The extensions of this to the finite state case has been made in other papers. More recent attention has been on continuum structure functions. In this paper, it is shown that such results do not hold, in general, for continuum structure functions. However, various conditions are developed under which such results hold.

A standard way of improving the reliability of a complex system of components is by adding redundancy. Redundancy may be active or standby. We shall focus on active redundancy in this paper, meaning that the redundant components are simultaneously active and therefore each is subject to failure. A principle well-known to design engineers is that redundancy at the component level is at least as good as redundancy at the system level when performance is distinguished as either functioning or failed. Further, the only situation in which they are equally good is the case of a parallel system meaning that the system will work if and only if at least one component works. Extensions to the multistate case have been made when the state space is finite. Extensions to the continuum structure function case are obtained in this paper. The methods in this case are not the same as in the multistate case and the multistate results do not directly carry over.

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We begin by reviewing the binary case using the approach of Barlow and Proschan (1981). We denote the state of component i by x_i. That is, $x_i = 1$ if component i is functioning while $x_i = 0$ if component i is failed. Then the system state is given by $\phi(\boldsymbol{x}) = \phi(x_i, \ldots, x_n)$ where $\phi(\boldsymbol{x}) = 1$ if the system is functioning and $\phi(\boldsymbol{x}) = 0$ if the system is failed. The two natural conditions to be imposed on ϕ are that ϕ be nondecreasing in each coordinate and that each coordinate (component) be relevant in the sense that there is some assignment of states to the other coordinates (components) so that it matters whether component i is a 1 or a 0. So we have

Definition 1. A function $\phi : \{0,1\}^n \to \{0,1\}$ satisfying

(1) if $\boldsymbol{x} \leq \boldsymbol{y}$, then $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$, where $\boldsymbol{x} \leq \boldsymbol{y}$ means $x_i \leq y_i, i = 1, 2, \ldots, n$ and

(2) for any component i, there exist a vector $(\cdot_i, \boldsymbol{x})$ so that $\phi(1_i, \boldsymbol{x}) = 1$ while $\phi(0_i, \boldsymbol{x}) = 0$

is called a binary coherent system (where $(1_i, \boldsymbol{x})$ or $(0_i, \boldsymbol{x}) = 0$ are the vectors with a 1 or 0 in the i^{th} position, but otherwise agreeing with $\boldsymbol{x}$).

Let us write $\bigvee_{i=1}^{n} x_i = \max_{1 \leq i \leq n} x_i$ and $\bigwedge_{i=1}^{n} x_i = \min_{1 \leq i \leq n} x_i$ for any vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)$. Further for vectors $\boldsymbol{x}$ and $\boldsymbol{y}$, $\boldsymbol{x} \vee \boldsymbol{y}$ is the vector whose ith coordinate is $x_i \vee y_i$ and $\boldsymbol{x} \wedge \boldsymbol{y}$ is the vector whose ith coordinate is $x_i \wedge y_i$. We can formalize the engineering principle that redundancy at the component level is preferred to redundancy at the system level in the following theorem.

Theorem 1 (Barlow and Proschan). *Let $\phi(\boldsymbol{x})$ be a (binary) coherent system. Then*

(1) $\phi(\boldsymbol{x} \vee \boldsymbol{y}) \geq \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$, for all $\boldsymbol{x}$ and $\boldsymbol{y}$. Furthermore, equality holds for all $\boldsymbol{x}$ and $\boldsymbol{y}$ if and only if $\phi(\boldsymbol{x}) = \max_{1 \leq i \leq n} x_i$.

(2) *$\phi(\boldsymbol{x} \wedge \boldsymbol{y}) \leq \phi(\boldsymbol{x}) \wedge \phi(\boldsymbol{y})$, for all $\boldsymbol{x}$ and $\boldsymbol{y}$. Furthermore, equality holds for all x and y if and only if $\phi(\boldsymbol{x}) = \min_{1\leq i\leq n} x_i$.*

The extension to the multistate case has been done by El-Neweihi, Proschan, and Sethuraman (1978), Griffith (1980), Block and Savits (1982), and Natvig (1982) when the state space is finite. We adopt the approach of Griffith.

Definition 2. A function $\phi : \{0,1,\ldots,M\}^n \to \{0,1,\ldots,M\}$ is called a multistate coherent system if it satisfies

(1) if $\boldsymbol{x} \leq \boldsymbol{y}$, then $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$

(2) for every component i and for every state $j \geq 1$, there exists a vector $(\cdot_i, \boldsymbol{x})$ such that $\phi((j-1)_i, \boldsymbol{x}) < \phi(j_i, \boldsymbol{x})$.

(3) $\min_{1\leq i\leq n} x_i \leq \phi(\boldsymbol{x}) \leq \max_{1\leq i\leq n} x_i$ for all $\boldsymbol{x}$.

A redundancy result can be obtained

Theorem 2 (Griffith). *Let $\phi(x)$ be a multistate coherent system. Then*

(1) *$\phi(\boldsymbol{x} \vee \boldsymbol{y}) \geq \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$. Further equality will hold for all $\boldsymbol{x}$ and $\boldsymbol{y}$ if and only if $\phi(\boldsymbol{x}) = \max_{1\leq i\leq n} x_i$*

(2) *$\phi(\boldsymbol{x} \wedge \boldsymbol{y}) \leq \phi(\boldsymbol{x}) \wedge \phi(\boldsymbol{y})$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$. Further equality will hold for all $\boldsymbol{x}$ and $\boldsymbol{y}$ if and only if $\phi(\boldsymbol{x}) = \min_{1\leq i\leq n} x_i$.*

In this paper, we are interested in extensions to the case where the state space is an interval. This represents the case where component or system degradation occurs continuously rather than in discrete jumps. These extensions are more complicated than in the case of finitely many states. Block and Savits (1984) have used the interval $[0,\infty)$ while Baxter (1987) has used a finite interval. Within a stochastic framework, components will assume values outside a finite interval with arbitrarily small probabilities so a finite interval is not unduly restrictive. The finite interval can be rescaled to [0,1]. We have

Definition 3. A function $\phi : [0,1]^n \to [0,1]$ is called a continuum structure function if it satisfies

(1) if $\boldsymbol{x} \leq \boldsymbol{y}$, then $\phi(\boldsymbol{x}) \leq \phi(\boldsymbol{y})$

(2) $\phi(\mathbf{0}) = 0$ and $\phi(\mathbf{1}) = 1$.

We note that by this definition $\phi(\boldsymbol{x}) = \max_{1 \leq i \leq n} x_i$ is a continuum structure function and in this case, $\phi(\boldsymbol{x} \vee \boldsymbol{y}) = \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$. That is, redundancy at the component level and at the system level are equivalent for this particular continuum structure function. In fact, within one subclass of continuum structure functions $\phi(x) = \max_{1 \leq i \leq n} x_i$ is the only one in which redundancy at the system level is equivalent to redundancy at the component level. This subclass consists of the Barlow and Wu continuum structure functions. A structure function is of the subclass if and only if $\phi(x) = \max_{1 \leq j \leq r} \min_{i \in P_j} x_i$ where $P_1, P_2, \ldots, P_r$ are the minimal path sets of a binary coherent structure function. For binary theory, this subclass is the entire class of coherent structure functions. It was introduced by Barlow and Wu (1978) as a definition of a multistate coherent structure function in the finite state space case and is a proper subclass of those studied by other authors. Block and Savits (1984) have extended this definition to continuum structure functions and studied some aspects of this subclass.

To verify that the the only Barlow-Wu continuum structure function for which redundancy at the system and component level are equivalent is $\phi(\boldsymbol{x}) = \max_{1 \leq i \leq n} x_i$, we suppose that $\phi(\boldsymbol{x} \vee \boldsymbol{y}) = \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$ for all $\boldsymbol{x}$ and $\boldsymbol{y}$ and that some minimal path set P_k contains at least two elements. Let i and j be the designation of two of these components. Then define $\boldsymbol{x}$ by $x_m = 0$ if $m \notin P_k$, $x_m = 1$ if $m \in P_k \backslash \{i, j\}$, $x_i = 1$ and $x_j = 0$. Define $\boldsymbol{y}$ by $y_m = 0$ if $m \notin P_k$, $y_m = 1$ if $m \in P_k \backslash \{i, j\}$, $y_i = 0$ and $y_j = 1$. It is easy to verify that for $m \in P_k$, $x_m \vee y_m = 1$. Hence

$\phi(\boldsymbol{x} \vee \boldsymbol{y}) = 1$. However, $\phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y}) = 0$ since $\boldsymbol{x}$ and $\boldsymbol{y}$ were set up so that $\phi(\boldsymbol{x}) = \phi(\boldsymbol{y}) = 0$.

But there are other continuum structure functions outside the Barlow-Wu class in which $\phi(\boldsymbol{x})$ is not $\max_{1\le i\le n} x_i$ and for which the redundancies are equivalent. For example, $\phi(\boldsymbol{x}) = \max_{1\le i\le n} \sqrt{x_i}$ is a continuum structure function and $\max_{1\le i\le n}(\sqrt{x_i \vee y_i}) = \max_{1\le i\le n} \sqrt{x_i} \vee \sqrt{y_i}) = \max_{1\le i\le n} \sqrt{x_i} \vee \max_{1\le i\le n} \sqrt{y_i}$. A more general example is $\phi(x) = \max_{1\le i\le n} g(x_i)$ where g is nondecreasing and $g(0) = 0$ and $g(1) = 1$. Thus, the binary and finite space results do not carry over to continuum structure functions. This leads to the question of under what circumstances is the parallel system (i.e., $\max_{1\le i\le n} x_i$) the only one in which redundancies at the system and component levels are equivalent. The question is more complex than it was in the finite state space and the methods and results do not carry over to the continuum structure functions case. In the finite state space, the proof just exhausted the state space. This is not possible when the state space is a continuum. We will now investigate the question of when $\phi(\boldsymbol{x} \vee \boldsymbol{y}) = \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$ implies that $\phi(\boldsymbol{x}) = \max_{1\le i\le n} x_i$ in this case of general continuum structure functions. That is, we shall examine under what conditions the parallel system is the only one for which redundancy at the component level and redundancy at the system level are equivalent.

We begin with some lemmas.

Lemma 1. *Let $\phi(\boldsymbol{x})$ be a continuum structure function. Then*

(1) $\phi(\boldsymbol{x} \vee \boldsymbol{y}) = \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$, *for all* $\boldsymbol{x}, \boldsymbol{y}$ *if and only if* $\phi(\boldsymbol{x}) = \max_{1\le i\le n} \phi((x_i)_i, \mathbf{0})$ *for all* $\boldsymbol{x}$

(2) $\phi(\boldsymbol{x} \wedge \boldsymbol{y}) = \phi(\boldsymbol{x}) \wedge \phi(\boldsymbol{y})$, *for all* $\boldsymbol{x}, \boldsymbol{y}$ *if and only if* $\phi(\boldsymbol{x}) = \min_{1\le i\le n} \phi((x_i)_i, \mathbf{1})$ *for all* $\boldsymbol{x}$

Proof. First let us suppose that $\phi(\boldsymbol{x}) = \max_{1\le i\le n} \phi((x_i)_i, \mathbf{0})$ for all $\boldsymbol{x}$.

Then $$\phi(\boldsymbol{x} \vee \boldsymbol{y}) = \max_{1\le i\le n} \phi((x_i \vee y_i)_i, \mathbf{0})$$

$$
\begin{aligned}
&= \max_{1\le i\le n}(\phi((x_i)_i,\mathbf{0}) \vee \phi((y_i)_i,\mathbf{0})) \\
&= (\max_{1\le i\le n}(\phi((x_i)_i,\mathbf{0})) \vee (\max_{1\le i\le n}\phi((y_i)_i,\mathbf{0})) \\
&= \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y}).
\end{aligned}
$$

Conversely, now let us suppose that $\phi(\boldsymbol{x} \vee \boldsymbol{y}) = \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$, for all $\boldsymbol{x}$, $\boldsymbol{y}$. Note that $\phi(\boldsymbol{x}) = \phi(\bigvee_{i=1}^{n}((x_i)_i,\mathbf{0}))$. It is not difficult to show by induction that the desired result $\phi(\bigvee_{i=1}^{n}((x_i)_i,\mathbf{0})) = \max_{1\le i\le n}\phi((x_i)_i,\mathbf{0}))$ holds. A dual argument can be used to prove 2.

Lemma 2. *Let $\phi(\boldsymbol{x})$ be a continuum structure function. If $\phi(\boldsymbol{x})$ satisfies*

(1) $\phi((\alpha)_i,\mathbf{0})$ *is independent of $i \in \{1,2,\ldots,n\}$ for all $\alpha \in [0,1]$.*

(2) $\phi((\alpha x)_i,\mathbf{0}) = \alpha\phi((x)_i,\mathbf{0})$ *for all $\alpha, x \in [0,1]$*

(3) $\phi(\boldsymbol{x} \vee \boldsymbol{y}) = \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$, *for all $\boldsymbol{x}$ and $\boldsymbol{y}$,*

then $\phi(\boldsymbol{x}) = \max_{1\le i\le n} x_i$.

Proof. By an inductive generalizations of (3), one can show that $\phi(\mathbf{1}) = \max_{1\le i\le n}\phi(1_i,\mathbf{0})$. However, by (1), $\phi(1_i,\mathbf{0})$ is independent of i. Thus $\phi(1_i,\mathbf{0}) = 1$ since $\phi(\mathbf{1}) = 1$. Now applying (2), it follows directly that $\phi((\alpha)_i,\mathbf{0}) = \alpha\phi(1_i,\mathbf{0}) = \alpha$ for all i and for all $\alpha \in [0,1]$. We now observe, in light of lemma 1, that $\phi(\boldsymbol{x}) = \max_{1\le i\le n}\phi((x_i)_i,\mathbf{0}) = \max_{1\le i\le n} x_i$.

Lemma 3. *Let $\phi(\boldsymbol{x})$ be a continuum structure function. If $\phi(\boldsymbol{x})$ satisfies*

(1) $\phi((\alpha x)_i,\mathbf{1}) = \alpha\phi(x_i,\mathbf{1})$ *for all $\alpha, x \in [0,1]$*

(2) $\phi(\boldsymbol{x} \wedge \boldsymbol{y}) = \phi(\boldsymbol{x}) \wedge \phi(\boldsymbol{y})$, *for all $\boldsymbol{x}, \boldsymbol{y}$,*

then $\phi(\boldsymbol{x}) = \min_{1\le i\le n} x_i$

Proof. From (1), we may conclude that $\phi(\alpha_i, \mathbf{1}) = \alpha$ for all i and for all $\alpha \in [0,1]$ since $\phi(\mathbf{1}) = 1$. By lemma 1 we know that $\phi(\boldsymbol{x}) = \min_{1 \le i \le n} \phi((x_i)_i, \mathbf{1})$. It then follows immediately that $\phi(\boldsymbol{x}) = \min_{1 \le i \le n} x_i$.

We now prove a theorem which tells us that provided that the system performance lies between the extremes of component performances and that the components are relevant in a certain sense at every level, the only continuum structure function for which redundancy at the system and component levels are equivalent is the parallel system. This provides a definitive answer to the questions about the class of continuum structure functions for which the multistate redundancy results extend since these conditions can also be shown to be necessary.

Theorem 3. *Let $\phi(\boldsymbol{x})$ be a continuum structure function with $\min_{1 \le i \le n} x_i \le \phi(\boldsymbol{x}) \le \max_{1 \le i \le n} x_i$ for all $\boldsymbol{x}$. If*

(1) for all i, $\phi((\alpha')_i, \boldsymbol{\alpha}) > \alpha$ for all α', α with $\alpha \in [0,1)$ and $\alpha' > \alpha$

(2) $\phi(\boldsymbol{x} \vee \boldsymbol{y}) = \phi(\boldsymbol{x}) \vee \phi(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y}$,

then $\phi(\boldsymbol{x}) = \max_{1 \le i \le n} x_i$.

Proof. We will first prove that $\phi(\alpha_i, \mathbf{0}) = \alpha$ for any $\alpha \in [0,1]$ and any component i. It is obvious for $\alpha = 0$. Let us now focus our attention on $\alpha \in (0,1)$. Suppose that $\alpha \in (0,1)$ is arbitrary but fixed and let i be any component. Since $\min_{1 \le i \le n} x_i \le \phi(\boldsymbol{x}) \le \max_{1 \le i \le n} x_i$, for all $\boldsymbol{x}, 0 \le \phi(\alpha_i, 0)) \le \alpha$. Assume $\phi(\alpha_i, \mathbf{0}) = \beta < \alpha$. We now investigate $\phi(\alpha_i, \boldsymbol{\beta})$. By an inductive generalization of (2), $\phi(\alpha_i, \boldsymbol{\beta}) = \max\left\{\phi(\alpha_i, \mathbf{0}), \max_{j \ne i} \phi(\beta_j, \mathbf{0})\right\} = \beta \vee \max_{j \ne i} \phi(\beta_j, \mathbf{0})$. However, $0 \le \phi(\beta_j, \mathbf{0}) \le \beta$ for all j and thus $\phi(\alpha_i, \boldsymbol{\beta}) = \beta$. This violates (1). Consequently, $\phi(\alpha_i, \mathbf{0}) = \alpha$.

Next consider $\phi(1_i, \mathbf{0})$. By monotonicity $\phi(1_i, \mathbf{0}) \ge \phi(\alpha_i, \mathbf{0})$, for all $\alpha \in [0,1]$. If $\phi(1_i, \mathbf{0}) < 1$, we need only take $\alpha = \frac{1+\phi(1_i, \mathbf{0})}{2}$ to derive a contradiction since $\phi(\alpha_i, \mathbf{0}) = \alpha$. Therefore, we have now established that $\phi(\alpha_i, \mathbf{0}) = \alpha$ for all $\alpha \in [0,1]$ and all components i. Therefore

$\phi((\alpha x)_i, \mathbf{0}) = \alpha x = \alpha\phi((x)_i, \mathbf{0})$ for all $\alpha, x \in [0,1]$ and all i. We now apply lemma 2 to obtain the result.

We next obtain a dual proposition characterizing $\min_{1\leq i\leq n} x_i$ as a continuum structure function.

Theorem 4. *Let $\phi(\boldsymbol{x})$ be a continuum structure function, with $\min_{1\leq i\leq n} x_i \leq \phi(\boldsymbol{x}) \leq \max_{1\leq i\leq n} x_i$ for all $\boldsymbol{x}$. If*

(1) for all $i, \phi((\alpha')_i, \boldsymbol{\alpha}) < \alpha$ for all α' and α with $\alpha' < \alpha$ and $\alpha \in (0,1]$

(2) $\phi(\boldsymbol{x} \wedge \boldsymbol{y}) = \phi(\boldsymbol{x}) \wedge \phi(\boldsymbol{y})$, for all $\boldsymbol{x}, \boldsymbol{y}$,

then $\phi(\boldsymbol{x}) = \min_{1\leq i\leq n} x_i$.

Proof. We shall prove that $\phi(\alpha_i, \mathbf{1}) = \alpha$ for all $\alpha \in [0,1]$. From this, condition (1) of lemma 3 and the fact that $\phi(\boldsymbol{x}) = \min_{1\leq i\leq n} x_i$ will follow immediately.

For $\alpha = 1$, the result is trivial. Now assume $\alpha \in (0,1)$ and let i be any component. We note that $\alpha \leq \phi(\alpha_i, \mathbf{1}) \leq 1$. Suppose that $\phi(\alpha_i, \mathbf{1}) = \beta > \alpha$. Then $\phi(\alpha_i, \boldsymbol{\beta}) = \phi(\alpha_i, \mathbf{1}) \wedge \min_{j\neq i} \phi(\beta_j, \mathbf{1})$ by an inductive generalization of (2). Since $\beta \leq \phi(\beta_j, \mathbf{1}) \leq 1$ for all $j \neq i$, we must have $\phi(\alpha_i, \boldsymbol{\beta}) = \beta$, a violation of condition (1). Thus $\phi(\alpha_i, \mathbf{1}) = \alpha$ for $\alpha \in (0,1)$. Finally, for $\alpha = 0$, suppose $\phi(0_i, \mathbf{1}) > 0$. Then, let $\beta = \frac{\phi(0_i, \mathbf{1})}{2}$. By monotonicity $\beta = \phi(\beta_i, \mathbf{1}) \geq \phi(0_i, \mathbf{1})$. But, $\phi(0_i, \mathbf{1}) = 2\beta > \beta$, a contradiction.

REFERENCES

1. R.E. Barlow, and F. Proschan, *Statistical theory of reliability and life testing*, To Begin with , Silver Springs, Maryland, 1981.

2. R.E. Barlow, and A.S. Wu, *Coherent systems with multistate components*, Math. Opns. Res. **3** (1978), 275-281.

3. L.A. Baxter, *Continuum Structures I*, J. of Appl. Prob. **21**(1984), 802-815.

4. H.W. Block and T.H. Savits, *Continuous multistate structure functions*, Oper. Res. **32** (1984), 703-714.

5. H.W. Block and T.H. Savits, *A decomposition for multistate monotone systems*, J. of Appl. Prob. **19**(1982), 391-402.

6. E. El-Neweihi, F. Proschan, and J. Sethuraman, *Multistate coherent systems*, J. Appl. Prob. **15** (1978), 675-688.

7. W.S. Griffith, *Multistate reliability models*, J. of Appl. Prob. **17**(1980), 735-744.

8. L.M. Leemis, *Reliability: Probabilistic Models and Statistical Methods*, Prentice-Hall Inc., Englewood Cliffs 1995.

9. B. Natvig, *Two suggestions of how to define a multistate coherent system*, Adv. Appl. Prob. **14** (1982), 424-455.

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