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NOTES ON 4-DIMENSIONAL HYPER-PARA-KÄHLER MANIFOLDS

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ABSTRACT. In this paper, we prove that a 4-dimensional hyper-para-Kähler structure is locally SO(3)-deformable to a hyper Kähler structure.

1. INTRODUCTION

An almost hypercomplex manifold is a quadruple $M = (M; I, J, K)$, where I, J and K are almost complex structures on a smooth manifold M satisfying $K = IJ = -JI$. It is well-known that an almost hypercomplex manifold is 4m-dimensional and orientable. An almost hypercomplex manifold $M = (M, I, J, K)$ equipped with a Riemannian metric g is called an almost hyperhermitian manifold if $(I, g), (J, g)$ and (K, g) are simultaneously almost Hermitian structures on M, especially, if $(I, g), (J, g)$ and (K, g) are simultaneously Kähler structures on M, then $M = (M; I, J, K, g)$ is called a hyper -Kähler manifold. Concerning the integrability of almost hyperhermitian manifolds, the following results have been obtained ([2]).

Theorem 1.1. *Any hyper-quasi-Kähler manifold is a hyper-Kähler manifold.*

Theorem 1.2. *Any* $4m\ (m \geq 2)$*-dimensional hyper-generalized locally conformal almost Kähler manifold is a locally conformal hyper-Kähler manifold.*

The above Theorem **1.1** is a generalization of the result by Hitchin ([1]). Any 4-dimensional almost hypermitian manifold is regarded as a hyper-generali -zed locally conformal almost Kähler manifold. Recently, we have proved

Theorem 1.3. ([3]) *Let* $(M; I, J, K, g)$ *be a* 4*-dimensional compact almost hyperhermitian Einstein manifold. Then* (M, I, J, K, g) *is a Ricci-flat and*

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$*-$Ricci-flat hyper-para-Kähler manifold. Furthermore, (M, g) is either flat or its universal covering is a hyper-Kähler $K3$ surface.

Theorem 1.4. ([3]) *Let $M = (M; I, J, K, g)$ be a 4-dimensional compact hyperhermitian manifold with non-positive scalar curvature.. Then M is a hyper-Kähler manifold.*

In the present paper, we shall show that a 4-dimensional hyper-para-Kähler structure is locally $SO(3)$-deformable to a hyper Kähler structure. Namely, we shall prove the following.

Theorem 1.5. *Let $M = (M; I, J, K, g)$ be a 4-dimensional hyper-para-Kähler manifold. Then, for each point $p \in M$, there exists a local smooth $SO(3)$-valued function Θ such that the almost hyperhermitian structure $(\overline{I}, \overline{J}, \overline{K}, g)$ given by*

$$\begin{pmatrix} \overline{I} \\ \overline{J} \\ \overline{K} \end{pmatrix} = \Theta \begin{pmatrix} I \\ J \\ K \end{pmatrix}$$

is a hyper -Kähler structure near the point p.

Taking account of the above Theorem **1.5** and Theorem **1.3**, we have also the following.

Corollary 1. *Let $M = (M; I, J, K, g)$ be a 4-dimensional compact simply-connected almost hyperhermitian Einstein manifold. Then, M is a hyper-Kähler manifold and, furthermore, isometric to a $K3$-surface with Ricci-flat metric, and (I, J, K, g) is globally $SO(3)$-deformable to a hyper-Kähler structure on a $K3$-surface in the sense of Theorem* **1.5.**

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2. PRELIMINARIES

We prepare some fundamental terminologies of a 4-dimensional almost hyperhermitian manifold. Let$(M; I, J, K, g)$ be a 4-dimensional almost hyperhermitian manifold with almost Hermitian structures (I, g), (J, g) and (K, g) and we define Ω_I, Ω_J and Ω_K Kähler forms of M by $\Omega_I(X, Y) = g(IX, Y)$, $\Omega_J(X, Y) = g(JX, Y)$ and $\Omega_K(X, Y) = g(KX, Y)$, for $X, Y \in \mathfrak{X}(M)$, respectively ($\mathfrak{X}(M)$denotes the Lie algebra of all smooth vector fields on M). We assume that M is oriented by the volume form $dM = \frac{1}{2}\Omega_{I^2}$ $\left(\text{or } \frac{1}{2}\Omega_{J^2}, \frac{1}{2}\Omega_{K^2}\right)$. We denote by ∇ and R the Riemannian connection and the curvature tensor of M. The curvature tensor R is defined by

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z$$

for $X, Y, Z \in \mathfrak{X}(M)$. Now, for $X, Y \in \mathfrak{X}(M)$, we define ∇^2_{XY} by $\nabla^2_{XY} = \nabla_X(\nabla_Y) - \nabla_{\nabla_X Y}$. We see easily that $R(X, Y) = \nabla^2_{XY} - \nabla^2_{YX}$. For any local unit vector field e, we see that $\{e_1 = e, e_2 = Ie, e_3 = Je, e_4 = Ke\}$ is a local orthonormal frame field compatible with the orientation on M. We denote by e^i the dual basis of e_i. Then, we may easily observe that the Kähler forms Ω_I, Ω_J and Ω_K are given respectively by

$$\begin{aligned} \Omega_I &= e^1 \wedge e^2 + e^3 \wedge e^4 \\ \Omega_J &= e^1 \wedge e^3 - e^2 \wedge e^4 \\ \Omega_K &= e^1 \wedge e^4 + e^2 \wedge e^3 \end{aligned} \tag{2.1}$$

By (2.1) we see that vector bundle $\Lambda^2_+ M$ is spanned by $\{\Omega_I, \Omega_J, \Omega_K\}$. We denote by $\Lambda^2 M$ the vector bundle of real 2-forms over M. Then it is known that the vector bundle $\Lambda^2 M$ is decomposed in the following forms:

$$\begin{aligned} \Lambda^2 M &= \mathbf{R}\Omega_I + \Lambda^{1,1}_{0I}(M) + LM_I \\ \Lambda^2 M &= \mathbf{R}\Omega_J + \Lambda^{1,1}_{0J}(M) + LM_J \\ \Lambda^2 M &= \mathbf{R}\Omega_K + \Lambda^{1,1}_{0K}(M) + LM_K \end{aligned} \tag{2.2}$$

where $\Lambda^{1,1}_{0I}(M)$, $\Lambda^{1,1}_{0J}(M)$ and $\Lambda^{1,1}_{0K}(M)$ denote the vector bundles of real primitive I-invariant, J-invariant and K-invariant 2-forms, LM_I, LM_J and

LM_K the vector bundles of real primitive I-skew invariant, J-skew invariant and K-skew invariant 2-forms over M, respectively. The bundle LM_I (resp. LM_J, LM_K) is endowed with a natural complex structure (also denoted by I (resp. J, K)) which is defined by $(I\Phi_I)(X,Y) = -\Phi_I(IX,Y)$ (resp. $(J\Phi_J)(X,Y) = -\Phi_J(JX,Y)$ and $(K\Phi_K)(X,Y) = -\Phi_K(KX,Y)$), for any section Φ_I (resp. Φ_J, Φ_K) of M and $X, Y \in \mathfrak{X}(M)$.

In the above decompositions (2.2) of $\Lambda^2 M$, we see easily that $LM_I = \operatorname{span}\{\Omega_J, \Omega_K\}$, $LM_J = \operatorname{span}\{\Omega_K, \Omega_I\}$ and $LM_K = \operatorname{span}\{\Omega_I, \Omega_J\}$. A 4-dimensional almost hyperhermitian manifold $M = (M; I, J, K, g)$ is called a 4-dimensional hyper-para-Kähler manifold if M satisfies the conditions:

$$\begin{aligned} R(X,Y).I &= 0 \\ R(X,Y).J &= 0 \\ R(X,Y).K &= 0 \end{aligned} \tag{2.3}$$

for $X, Y \in \mathfrak{X}(M)$.

3. PROOF OF THEOREM 1.5

We shall prove the theorem 1.5.

Proof. Let $(M; I, J, K, g)$ be a 4-dimensional hyper-para-Kähler Einstein manifold. Let $\{e_1 = e, e_2 = Ie, e_3 = Je, e_4 = Ke\}$ ($\|e\| = 1$) be a local orthonormal frame field compatible with the orientation on M. Let $\{e^i\}$ be the dual basis of $\{e_i\}$. we define the fiber metric h of $\Lambda^2_+ M$ by

$$h(\alpha,\beta) = \sum \alpha_{ab}\beta_{cd}\{g(e_a,e_c)\,g(e_b,e_d) - g(e_a,e_d)\,g(e_b,e_c)\}, \tag{3.1}$$

where $\alpha = \sum \alpha_{ab} e^a \wedge e^b, \beta = \sum \beta_{cd} e^c \wedge e^d \in \Lambda^2_+ M$, $a, b, c, d = 1, \ldots, 4$. From (2.1) and (3.1), by direct calculation, we see that $\{\Omega_I/\sqrt{2}, \Omega_J/\sqrt{2}, \Omega_K/\sqrt{2}\}$ is an orthonormal frame field with respect to the fiber metric h.

We denote by $\tilde{\nabla}$ and ∇ the connection of $\Lambda^2_+ M$ and the Levi-Cevita connection of M. Taking into account of (3.1), we get

$$\tilde{\nabla} = \nabla \tag{3.2}$$

We denote by $\tilde{R}$ the curvature tensor with respect to $\tilde{\nabla}$. Since M is a hyperhermitian manifold, we see easily that (2.3) are equivalent to the following equations, respectively.

$$\begin{aligned} R(X,Y).\Omega_I &= 0 \\ R(X,Y).\Omega_J &= 0 \\ R(X,Y).\Omega_K &= 0 \end{aligned} \tag{3.3}$$

for $X,Y \in \mathfrak{X}(M)$. Taking into account of (3.2), from (3.3), we get easily

$$\begin{aligned} \tilde{R}(X,Y).\Omega_I &= 0 \\ \tilde{R}(X,Y).\Omega_J &= 0 \\ \tilde{R}(X,Y).\Omega_K &= 0 \end{aligned} \tag{3.4}$$

for $X,Y \in \mathfrak{X}(M)$, and hence $\tilde{R} = 0$ on M. Thus, for the point $p \in M$, there exists a $\tilde{\nabla}$-parallel orthonormal frame field near the point p. Let $\Omega_1/\sqrt{2}, \Omega_2/\sqrt{2}, \Omega_3/\sqrt{2}$ be a $\tilde{\nabla}$-parallel orthonormal frame field near the point p. From (3.2), we get easily

$$\nabla\Omega_1 = 0, \nabla\Omega_2 = 0, \nabla\Omega_3 = 0 \tag{3.5}$$

near the point p. Thus, we see that Ω_1, Ω_2 and Ω_3 are relatively orthogonal and are ∇-parallel self dual 2-form near the point p. Since $\{\Omega_I/\sqrt{2}, \Omega_J/\sqrt{2}, \Omega_K/\sqrt{2}\}$ is an orthonormal frame field with respect to the fiber metric h, we can put Ω_1, Ω_2 and Ω_3 as follows near the point p:

$$\begin{aligned} \Omega_1 &= a_1\Omega_I + a_2\Omega_J + a_3\Omega_K \\ \Omega_2 &= b_1\Omega_I + b_2\Omega_J + b_3\Omega_K \\ \Omega_3 &= c_1\Omega_I + c_2\Omega_J + c_3\Omega_K \end{aligned} \tag{3.6}$$

where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2$ and c_3 are smooth functions near the point p. From (3.1) and (3.6), we see that

$$\Theta := \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \in SO(3) \tag{3.7}$$

near the point p. From (3.6), for $X,Y \in T_pM$, we get

$$\begin{aligned} \Omega_1(X,Y) &= g((a_1I + a_2J + a_3K)X, Y) \\ \Omega_2(X,Y) &= g((b_1I + b_2J + b_3K)X, Y) \\ \Omega_3(X,Y) &= g((c_1I + c_2J + c_3K)X, Y) \end{aligned}$$

Now, we put $\overline{I}, \overline{J}$ and $\overline{K}$ as follows:

$$\begin{pmatrix} \overline{I} \\ \overline{J} \\ \overline{K} \end{pmatrix} = \Theta \begin{pmatrix} I \\ J \\ K \end{pmatrix} \tag{3.8}$$

near the point p. Then, from (3.7) and (3.8), by direct calculation, we get $\overline{I}^2 = \overline{J}^2 = \overline{K}^2 = -E, \overline{K} = \overline{IJ} = -\overline{JI}$. Thus, $\overline{I}, \overline{J}$ and $\overline{K}$ are almost complex structures near the point p respectively. By (3.5), we see that $(\overline{I}, \overline{J}, \overline{K}, g)$ is hyper-Kähler structure near the point p. $\square$

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