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EIGENVALUES OF THE LAPLACE OPERATOR ON COMPACT RIEMANNIAN MANIFOLDS

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ABSTRACT. In this survey article we are interested in reporting recent advances on the estimates of the eigenvalues of the Laplace operator on compact Riemannian manifolds of constant scalar curvature and compact Einstein-like manifolds.

1. INTRODUCTION

The geometry of the Laplace operator has been a subject of interest from different aspects for the last four decades, and some of the developments are recorded in [2], [3], and [11]. One of the interesting aspects of this study is reading the spectrum of the Laplace operator using the geometric data of the Riemannian manifold. In particular, obtaining lower bounds for the eigenvalues of the Laplace operator on a Riemannian manifold, using its curvature information, was first initiated by Lichnerowricz (cf. Thm 4.70, p. 210 in [9]). This result states that if M is an n-dimensional compact Riemannian manifold where the Ricci curvature satisfies $Ric \geq (n-1)k$ for a constant k, then the first nonzero eigenvalue λ_1 of the Laplace operator satisfies $\lambda_1 \geq nk$; and later Obata [12] has shown that equality holds if and only if M is isometric to the sphere $S^n(k)$. If we denote by $\mathcal{C}$ the class of Riemannian manifolds of constant scalar curvature, by $\mathcal{E}$ the class of Einstein manifolds, and by $\mathcal{A}$ the class of Einstein-like manifolds (Riemannian manifolds in which the Ricci curvature satisfies $(\nabla_X Ric)(X,X) = 0$, $X \in \mathfrak{X}(M)$ and $\mathfrak{X}(M)$ is the Lie-algebra

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of smooth vector fields on M), then it is known that

$$\mathcal{E} \subset \mathcal{A} \subset \mathcal{C},$$

and the inclusion is strict (cf. [10]). These classes of Riemannian manifolds are of significant importance from the viewpoint of geometry as well as physics (cf. [1]), and therefore studying the spectrum of the Laplace operator on manifolds in these classes is also interesting and important. For a Riemannian manifold in $\mathcal{E}$ that is, for a compact Einstein manifold M, Simon [14] has shown that either it is isometric to a sphere or else each non-zero eigenvalue λ of the Laplace operator satisfies $\lambda > 2nk_0$, k_0 being the infimum of the sectional curvature of M and $n = \dim M$. However there was no mechanism to estimate the non-zero eigenvalues, like Simon's estimate of the Laplace operator on the product $S^m(c_1) \times S^n(c_2)$, $(c_1 \neq c_2)$, which is in class $\mathcal{C}$. Similarly there are compact Riemannian manifolds which are not in $\mathcal{E}$ but in $\mathcal{A}$ (for instance the nearly Kaehler 3-symmetric space $U(4)/U(2) \times U(1) \times U(1)$). Although Lichnerowicz's estimate applies to these Riemannian manifolds, yet, Simon's result on compact Einstein manifolds (compact Riemannian manifold with additional restriction) gives more information on eigenvalues than Lichnerowicz's result. Similarly the manifolds in $\mathcal{A}$ and $\mathcal{C}$ are Riemannian manifolds with additional restrictions, so one expects of get results on the estimates on eigenvalues of the Laplace operator on the manifolds in classes $\mathcal{A}$ and $\mathcal{C}$ and, as $\mathcal{E} \subset \mathcal{A} \subset \mathcal{C}$, Simon's result should follow from these results. This question is considered in [6], [7], [8].

2. PRELIMINARIES

Let (M, g) be an n-dimensional compact Riemannian manifold without boundary and with Riemannian connection ∇. We denote by $\mathfrak{X}(M)$ the Lie-algebra of smooth vector fields on M. The Riemannian curvature tensor field R is defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad X, Y, Z, W \in \mathfrak{X}(M) \quad (2.1)$$

and $R(X, Y; Z, W) = g(R(X,Y)Z, W)$, $X, Y, Z, W \in \mathfrak{X}(M)$. For a local orthonormal fame $\{e_1, \ldots, e_n\}$ on M, the Ricci tensor field Ric and scalar curvature

S of M are defined by

$$Ric(X,Y) = \sum_{i=1}^{n} R(e_i, X; Y, e_i), \; S = \sum_{i=1}^{n} Ric\,(e_i, e_i)\,, \quad X, Y \in \mathfrak{X}(M). \quad (2.2)$$

The Ricci operator Q is a symmetric tensor field of type $(1,1)$ defined by

$$Ric\,(X,Y) = g(Q(X), Y), \quad X, Y \in \mathfrak{X}(M). \quad (2.3)$$

The Bianchi identities for a Riemannian manifold (M, g) are

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0, \; X, Y, Z \in \mathfrak{X}(M), \quad (2.4)$$

$$(\nabla_X R)(Y,Z)W + (\nabla_Y R)(Z,X)W + (\nabla_Z R)(X,Y)W = 0, \; X, Y, Z \in \mathfrak{X}(M). \quad (2.5)$$

For a tensor field A of type $(1,1)$ on (M, g) the Ricci identity is

$$\left(\nabla^2 A\right)(X,Y,Z) - \left(\nabla^2 A\right)(Y,X,Z) = R(X,Y)AZ - AR(X,Y)Z, \quad (2.6)$$

$X, Y, Z \in \mathfrak{X}(M)$, where the covariant derivatives $(\nabla A)(X,Y)$, $(\nabla^2 A)(X,Y,Z)$ are defined by

$$\begin{aligned} (\nabla A)(X,Y) &= \nabla_X AY - A\nabla_X Y \\ (\nabla^2 A)(X,Y,Z) &= \nabla_X (\nabla A)(Y,Z) - (\nabla A)(\nabla_X Y, Z) - (\nabla A)(Y, \nabla_X Z). \end{aligned}$$

For a smooth function f on a Riemannian manifold (M, g) it's gradient ∇f is defined by

$$g(\nabla f, X) = X(f), \; X \in \mathfrak{X}(M), \quad (2.7)$$

and for a smooth vector field $X \in \mathfrak{X}(M)$ it's divergence $divX$ is defined by

$$\operatorname{div} X = \sum_i g(\nabla_{e_i} X, e_i), \quad (2.8)$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. For a smooth function f on (M, g) the Laplacian of f is defined as

$$\Delta f = -\operatorname{div}(\nabla f), \quad (2.9)$$

which satisfies

$$\Delta\,(fh) = f\Delta h + h\Delta f - 2g\,(\nabla f, \nabla h) \quad (2.10)$$

Let $L^2\,(M)$ be the space of measurable functions on M which satisfy

$$\int_M |f|^2\, dv < \infty,$$

where dv is the volume element of the Riemannian manifold (M,g). On $L^2(M)$ we have the inner product and norm defined by

$$\langle f,h\rangle = \int_M fh\; dv, \quad \|f\|^2 = \langle f,f\rangle, \quad f,h\in L^2(M), \tag{2.11}$$

and with this inner product $L^2(M)$ is a Hilbert space. The Laplace operator Δ satisfies

$$\langle \Delta f,h\rangle = \langle f,\Delta h\rangle,$$

for all smooth smooth functions in $L^2(M)$, that is, Δ is a self-adjoint operator.

A smooth function f on a compact Riemannian manifold (M,g) is said to be an eigenfunction of the Laplace operator Δ on M if

$$\Delta f = \lambda f,$$

where λ is a constant, called an eigenvalue of Δ corresponding to the eigenfuntion f. The space V_λ of eigenfunctions of Δ corresponding to an eigenvalue λ is a finite dimensional vector space. Indeed we have the following theorem.

Theorem 2.1 ([3]). *Let (M,g) be a compact Riemannian manifold. Then the set of eigenvalues of Δ consists of a sequence*

$$0<\lambda_1<\lambda_2<\cdots<\infty$$

and each eigenspace is finite dimensional. Eigenspaces corresponding to distinct eigenvalues are orthogonal in $L^2(M)$ and $L^2(M)$ is the direct sum of all the eigenspaces.

A Riemannian manifold (M,g) is said to have constant scalar curvature if the scalar curvature S is a constant function. Moreover (M,g) is said to be an Einstein manifold if

$$Ric = cg$$

for a constant c, called the Einstein constant. If $\mathcal{C}$ denotes the class of Riemannian manifolds of constant scalar curvature and $\mathcal{E}$ that of the Einstein manifolds, then $\mathcal{E}\subset\mathcal{C}$ and the inclusion is strict. There is an intermediate class $\mathcal{A}$ of Riemannian manifolds whose Ricci tensor salsifies

$$(\nabla_X Ric)(X,X)=0, \quad X,Y\in\mathfrak{X}(M),$$

called Einstein-like manifolds, and we have

$$\mathcal{E}\subset\mathcal{A}\subset\mathcal{C}.$$

Again the inclusion in strict (cf. [10]).

Regarding the first nonzero eigenvalue λ_1 of the Laplace operator on a compact Riemannian manifold (M, g) we have the following result due to Lichnerowicz (cf. [9] p.210).

Theorem 2.2. *Let (M, g) be an n-dimensional compact Riemannian manifold. Suppose that the Ricci curvature of M satsfies*

$$Ric \geq (n-1)\, k,$$

for some constant $k > 0$. Then the first nonzero eigenvalue λ_1 of the Laplace operator on M salsifies

$$\lambda_1 \geq nk.$$

Moreover the equality holds if and only if M is isometric to a standard sphere $S^n(k)$.

the last part of the above theorem, when the bound is achieved, is due to Obata [13].

Also for the eigenvalue λ_1, Li and Yau [12] have established a relationship between λ_1 and the diameter of the compact Riemannian manifolds (M, g).

Theorem 2.3 ([12]). *Let (M, g) be an n-dimensional compact Riemannian manifold whose Ricci curvature is nonnegative. Then the first nonzero eigenvalue λ_1 of the Laplace operator on M satisfies*

$$\lambda_1 \geq \frac{\pi^2}{(a+1)\, d^2},$$

where d is the diameter of M and the constant $a + 1 = \sup_M \varphi, \varphi$ being an eigenfunction corresponding to the eigenvalue λ_1.

Furthermore, these authors conjectured that $\lambda_1 \geq \frac{\pi^2}{d^2}$ and this conjecture was finally proved in [15]. Regarding the eigenvalues of the Laplace operator on an Einstein manifold, Simon [14] has proved the following.

Theorem 2.4 ([14]). *Let (M, g) be an n-dimensional compact Einstein manifold with $n \geq 3$, whose sectional curvatures are bounded below by a constant k_0. Then either M is isometric to a sphere or else each nonzero eigenvalue λ of the Laplace operator on M satisfies $\lambda > 2nk_0$.*

3. INTEGRAL FORMULAS

In this section, we prepare the tools needed to estimate the eigenvalues of the Laplace operator on Riemannian manifolds in classes $\mathcal{A}$ and $\mathcal{C}$. For a smooth function f on a Riemannian manifold (M,g) there is a symmetric operator $A:\mathfrak{X}(M)\to\mathfrak{X}(M)$ defined by

$$A(X)=\nabla_X\nabla f,\quad X\in\mathfrak{X}(M),$$

which satisfies

$$trA=-\Delta f,\ g(AX,Y)=g(X,AY),\ X,Y\in\mathfrak{X}(M) \tag{3.1}$$

$$(\nabla A)(X,Y)-(\nabla A)(Y,X)=R(X,Y)\nabla f,\ X,Y\in\mathfrak{X}(M) \tag{3.2}$$

$$\begin{aligned}\left(\nabla^2 A\right)(X,X,Y)-(\nabla^2 A)(Y,X,X) &= 2R(X,Y)AX \\ &\quad -AR(X,Y)X+(\nabla_X R)(X,Y)\nabla f.\end{aligned} \tag{3.3}$$

The tensor field H_f defined by $H_f(X,Y)=g(AX,Y)$ is called the Hessian of the smooth function f. Using equations (3.1) and (3.2) we immediately get the following:

Lemma 3.1 ([7]). *For a local orthonormal frame $\{e_1,..,e_n\}$ on M,*

$$\sum_i\left(\nabla A\right)(e_i,e_i)=-\nabla\Delta f+Q(\nabla f)$$

holds.

Lemma 3.2 ([7]). *Let (M,g) be a compact Riemannian manifold and f a smooth function on M. Then*

$$\int_M\{(\Delta f)^2-g(\nabla f,\nabla(\Delta f))\}dv=0$$

Proof. Using $\Delta f=-dv(\nabla f)$ and equation (2.10), we get

$$\Delta(f\Delta f)=f\Delta^2 f+(\Delta f)^2-2g(\nabla f,\nabla(\Delta f)).$$

Integrating this equation, and using the fact that Δ is self adjoint, together with Stokes theorem, we get the desired result. □

Lemma 3.3 ([7]). *Let A be the operator corresponding to a smooth function f on a compact Riemannian manifold (M,g). Then*

$$\int_M\{\|A\|^2-(\Delta f)^2+Ric(\nabla f,\nabla f)\}dv=0$$

Proof. For a local orthonormal fame $\{e_1, \ldots, e_n\}$ on M, we compute

$$\operatorname{div}(A(\nabla f)) = g(\nabla f, \ \sum_i (\nabla A)(e_i, e_i)) + \|A\|^2 .$$

Integrating this last equation and using Lemma 3.1, we get the result. □

Suppose f is an eigenfunction of the Laplace operator corresponding to the eigenvalue λ_1, that is, $\Delta f = \lambda_1 f$ on a compact Riemannian manifold (M, g). Then Lemma 3.2 gives

$$\int_M \{(\Delta f)^2 - \lambda \|\nabla f\|^2\} dv = 0 \tag{3.4}$$

Using the Schwartz inequality $\|A\|^2 \geq \frac{1}{n}(\Delta f)^2$ and equation (3.4) in Lemma 3.3, we arrive at

$$\int_M \{-\frac{(n-1)}{n}\lambda_1 \|\nabla f\|^2 + Ric(\nabla f, \nabla f)\} dv \leq 0 \tag{3.5}$$

with the equality holding if and only if $A = -\frac{\lambda_1}{n} fI$. If the Ricci curvature satisfies. $Ric \geq (n-1)\, k$ for a constant $k > 0$, then, by the inequality (3.5), we get $\left(k - \frac{\lambda_1}{n}\right) \int_M \|\nabla f\|^2 \leq 0$. Thus $\lambda_1 \geq nk$ and the equality holds if and only if $A = -\frac{\lambda_1}{n} fI$, that is, $\nabla_X \nabla f = -\frac{\lambda_1}{n} fX$, $X \in \mathfrak{X}(M)$, that is, if and only if M is isometric to a sphere $S^n(c)$ by Obata's theorem [13]. This gives a short proof of Lichnerowicz's theorem 2 as stated in section 2.

Lemma 3.4 ([7]). *Let f be a smooth function on a Riemannian manifold (M, g). Then the Hessian $H_{\Delta f}$ of the smooth function Δf satisfies*

$$H_{\Delta f}(X, Y) = -\sum_i g((\nabla^2 A)(X, e_i, e_i), Y), \ X, Y \in \mathfrak{X}(M),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M.

Proof. Lemma 3.1, gives

$$-g(Y, \nabla \Delta f) = \sum_i g(Y, (\nabla A)(e_i, e_i)) - g(Q(\nabla f), Y), \ Y \in \mathfrak{X}(M),$$

which together with the definition of the Hessian $H_{\Delta f}$ gives

$$\begin{aligned} - H_{\Delta f}(X, Y) = XY(\Delta f) - \nabla_X Y(\Delta f) = \sum_i g(Y, (\nabla^2 A)(X, e_i, e_i)) \\ - g(\nabla_X Q(\nabla f), Y \end{aligned}$$

and this proves the lemma. □

On a Riemannian manifold (M,g), for a $X\in\mathfrak{X}(M)$, we define

$$\|R_X\|^2=\sum_{ij}g(R(e_i,e_j)X,\ R(e_i,e_j)X),\tag{3.6}$$

where $\{e_1,\dots,e_n\}$ is a local orthonormal frame on M, called the square of the length of the curvature operator in the direction of the vector field X. It is easy to check that for an n-dimensional Riemannian manifold (M,g) of constant curvature c, we have $\|R_X\|^2=2(n-1)c^2\|X\|^2$, $X\in\mathfrak{X}(M)$.

Lemma 3.5 ([7]). *Let (M,g) be an n-dimensional compact Riemannian manifold. Then, for a smooth function f on M, we have*

$$\int_M\{\|\nabla A\|^2-\|\nabla\Delta f\|^2+2Ric(\nabla f,\nabla\Delta f)-\|Q(\nabla f)\|^2-\frac{1}{2}\|R_{\nabla f}\|^2+\sum_{i<j}(\lambda_i-\lambda_j)^2K_{ij}\}dv=0$$

where $\lambda_1,\dots,\lambda_n$ are eigenvalues of the operator A with respect to a local orthonormal frame $\{e_1,\dots,e_n\}$, and K_{ij} is the sectional curvature of the plane section spanned by $\{e_i,e_j\}$.

Proof. Consider a vector field $X=\sum_i(\nabla A)(e_i,e_i)$, where $\{e_1,\dots,e_n\}$ is a local orthonormal frame. Then a straightforward computation shows.

$$\operatorname{div}(A(\nabla\Delta f)))=-Ric(\nabla f,\nabla\Delta f)+\|Q(\nabla f)\|^2+\sum_i g(\nabla_{e_i}AQ(\nabla f),Ae_i)\tag{3.7}$$

$$\operatorname{div}(A(\nabla\Delta f))=-\|\nabla\Delta f\|^2+Ric(\nabla f,\nabla\Delta f)+\sum_i g(\nabla_{e_i}\nabla\Delta f,Ae_i)\tag{3.8}$$

$$\operatorname{div}X=\sum_{ij}g((\nabla A)(e_j,e_i),(\nabla A)(e_i,e_j))+\sum_{ij}g(Ae_i,(\nabla^2A)(e_j,e_i,e_j)).\tag{3.9}$$

Using Ricci's identity (2.6) and lemma 3.5, we arrive at

$$\begin{aligned}\sum_{ij}g(Ae_i,(\nabla^2A)(e_j,e_i,e_j)) &= -\sum_i H_{\Delta f}(e_i,Ae_i)\\ &\quad+\sum_i g(\nabla_{e_i}Q(\nabla f),Ae_i)+\sum_{ij}R(e_j,e_i,Ae_j,Ae_i)\\ &\quad-\sum_{ij}R(e_j,e_i,e_j,A^2e_i)\end{aligned}\tag{3.10}$$

Using a local orthonormal frame $\{e_1, \ldots, e_n\}$ that diagonalizes A, that is, $A(e_i) = \lambda_i e_i$, we have

$$\begin{aligned} \sum_{ij} R(e_j, e_i, Ae_j, Ae_i) - \sum_{ij} R(e_j, e_i, e_j, A^2 e_i) &= -\sum_{ij} \lambda_j \lambda_i K_{ij} + \sum_{ij} \lambda_i^2 K_{ij} \\ &= \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} \end{aligned}$$

where K_{ij} is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$. Thus the equation (3.10) takes the form

$$\begin{aligned} \sum_{ij} g(Ae_i, (\nabla^2 A)(e_j, e_i, e_j)) &= \sum_i g(\nabla_{e_i} Q(\nabla f), Ae_i) \\ &\quad - \sum_i g(\nabla_{e_i} \nabla \Delta f, Ae_i) + \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} \end{aligned} \tag{3.11}$$

where we used the definition of the Hessian $H_{\Delta f}$.

Using equation (3.2), we get

$$\begin{aligned} \|R_{\nabla f}\|^2 &= \sum_{ij} g(R(e_i, e_j)\nabla f, R(e_i, e_j)\nabla f) = 2\|\nabla A\|^2 \\ &\quad -2\sum_{ij} g(\nabla A)(e_i, e_j), (\nabla A)(e_j, e_i)). \end{aligned}$$

Consequently,

$$\sum_{ij} g((\nabla A)(e_i, e_j), (\nabla A)(e_j, e_i)) = \|\nabla A\|^2 - \frac{1}{2}\|R_{\nabla f}\|^2.$$

Using this equation and equation (3.11) in (3.9), we arrive at

$$\begin{aligned} \operatorname{div} X &= \|\nabla A\|^2 - \frac{1}{2}\|R_{\nabla f}\|^2 - \sum_i g(\nabla_{e_i} \nabla f, Ae_i) \\ &\quad + \sum_i g(\nabla_{e_i} Q(\nabla f), Ae_i) + \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij}. \end{aligned} \tag{3.12}$$

Finally we use equations (3.7), (3.8) and (3.12) to arrive at

$$\begin{aligned} \operatorname{div} X + \operatorname{div}(A(\nabla \Delta f)) - \operatorname{div}(A(Q(\nabla f))) &= \|\nabla A\|^2 - \|\nabla \Delta f\|^2 + 2Ric(\nabla f, \nabla \Delta f) \\ &\quad - \|Q(\nabla f)\|^2 - \frac{1}{2}\|R_{\nabla f}\|^2 + \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij}. \end{aligned}$$

Integrating this equation and using Stoke's theorem we get the result. □

Lemma 3.6 ([7]). *Let f be a smooth function on a compact Riemannian manifold (M,g). Then*

$$\int_M \{\sum_{ij} g((\nabla_{e_i}R)(e_j,e_i)\nabla f, Ae_i\}dv = \int_M \{-\frac{1}{2}\left\|R_{\nabla f}\right\|^2 + \sum_{ij} R(e_i,e_j;Ae_j,Ae_i)\}dv.$$

Proof. Let $\{e_1,\ldots,e_n\}$ be a local orthonormal frame on M.

Define $X=\sum_i R(\nabla f, Ae_i)e_i$, and for this vector field we have

$$\begin{aligned}\operatorname{div} X &= \sum_{ij} g(\nabla_{e_i} R(\nabla f, Ae_i)e_i, e_j) = -\sum_{ij}(\nabla_{e_j}R)(e_j,e_i,\nabla f, Ae_i)\\ &\quad + \sum_{ij} R(e_i,e_j,Ae_j,Ae_i) + \sum_{ij} R(e_i,e_j,\nabla f,(\nabla A)(e_j,e_i)),\end{aligned}$$

which, together with equation (3.2), gives

$$\begin{aligned}\operatorname{div} X &= -\sum_{ij} g((\nabla_{e_j}R)(e_j,e_i)\nabla f, Ae_i) + \sum_{ij} R(e_i,e_j;Ae_j,Ae_i)\\ &\quad + \sum_{ij} g((\nabla A)(e_i,e_j);(\nabla A)(e_i,e_j)) - \left\|\nabla A\right\|^2.\end{aligned}$$

Finally using $\frac{1}{2}\left\|R_{\nabla f}\right\|^2 = \left\|\nabla A\right\|^2 - \sum_{ij} g((\nabla A)(e_i,e_j),(\nabla A)(e_j,e_i))$, which is an outcome of equation (3.2), in the above equation and integrating over M, we get the result. □

Lemma 3.7 ([7]). *Let f be a smooth function on are n-dimensional Riemannian manifold (M,g). Then*

$$\left\|\nabla A\right\|^2 \geq \frac{1}{n}\left\|\nabla \Delta f\right\|^2$$

and for positively curved M the equality holds if and only if $A(X) = -\frac{1}{n}(\Delta f)X$, $X \in \mathfrak{X}(M)$.

Proof. Define a symmetric tensor field B by

$$B(X,Y) = g(AX,Y) + \frac{1}{n}(\Delta f)g(X,Y),\ X,Y \in \mathfrak{X}(M).$$

Then

$$(\nabla B)(X,Y,Z) = g((\nabla A)(X,Y),Z) + \frac{1}{n}X(\Delta f)g(Y,Z),\ X,Y,Z \in \mathfrak{X}(M)$$

and consequently, for a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M, we have

$$\|\nabla B\|^2 = \|\nabla A\|^2 + \frac{1}{n}\|\nabla \Delta f\|^2 + \frac{2}{n}\sum_{ij} g((\nabla A)(e_i, e_j), e_j) g(\nabla f, e_i)$$

Using equation (3.2) and lemma (3.1) in the above equation we arrive at

$$\|\nabla B\|^2 = \|\nabla A\|^2 + \frac{1}{n}\|\nabla \Delta f\|^2 - \frac{2}{n}\|\nabla \Delta f\|^2 = \|\nabla A\|^2 - \frac{1}{n}\|\nabla \Delta f\|^2,$$

which proves the inequality $\|\nabla A\|^2 \geq \frac{1}{n}\|\nabla \Delta f\|^2$. If the equality holds, we should have $\nabla B = 0$ for a symmetric tensor field B and, as M is irreducible (being positively curved), we get $B(X,Y) = cg(X,Y)$ for some constant c. This, together with the definition of B, gives $A(X) = (c - \frac{1}{n}\Delta f)(X)$, and, as $trA = -\Delta f$, we get $c = 0$. Therefore $A(X) = -\frac{1}{n}(\Delta f)X$, $X \in \mathfrak{X}(M)$. □

Lemma 3.8 ([7]). *Let f be a smooth function on an n-dimensional compact Riemannian manifold and $\{e_1, \ldots, e_n\}$ be a local orthonormal frame that diagonalizes A with $Ae_i = \lambda_i e_i$. Then*

$$\int_M \left\{\sum_{i<j}(\lambda_i - \lambda_j)^2\right\} dv = (n-1)\int_M (\Delta f)^2 dv - n\int_M Ric(\nabla f, \nabla f) dv$$

holds.

Proof. We have

$$\begin{aligned}\sum_{ij}(\lambda_i - \lambda_j)^2 &= \sum_{ij}\lambda_i^2 + \sum_{ij}\lambda_j^2 - 2\sum_{ij}\lambda_i\lambda_j = 2n\|A\|^2 - 2\sum_j(-\Delta f)\lambda_j \\ &= 2n\|A\|^2 - 2(\Delta f)^2\end{aligned}$$

and $\sum_{ij}(\lambda_i - \lambda_j)^2 = 2\sum_{i<j}(\lambda_i - \lambda_j)^2$. Thus we have $\sum_{i<j}(\lambda_i - \lambda_j)^2 = \|A\|^2 - (\Delta f)^2$. Integrating this equation, and in view of lemma (3.3), we arrive at the result. □

Finally as a last tool, we define the a tensor field $\mathcal{F}$ of type (1,2) by

$$\mathcal{F}(X,Y) = \sum_{i=1}^n (\nabla_{e_i} R)(X,Y)e_i, \; X, Y \in \mathfrak{X}(M),$$

called the divergence of the curvature tensor field, where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. Using equations (2.4) and (2.5) it is easy to verify that the tensor field $\digamma$ satisfies.

$$\digamma(X,Y) = -\digamma(Y,X),\ \digamma(X,fY) = f\digamma(X,Y) = \digamma(fX,Y) \tag{3.13}$$

and

$$g(\digamma(X,Y),Z) + g(\digamma(Y,Z),X) + g(\digamma(Z,X),Y) = 0,\ X,Y,Z \in \mathfrak{X}(M) \tag{3.14}$$

and f a smooth function on M. Using the definition of the Ricci curvature and scalar curvature in equation (2.2) and equation (2.5), we obtain

$$(\nabla Q)(X,Y) - (\nabla Q)(Y,X) = -\digamma(X,Y),\ X,Y \in \mathfrak{X}(M) \tag{3.15}$$

and

$$\frac{1}{2}\nabla S = \sum_j (\nabla Q)(e_j, e_j), \tag{3.16}$$

for a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M. Also, using equation (3.16), we see that the Hessian of the scalar curvature function S is given by

$$\frac{1}{2}H_S(X,Y) = \sum_i g((\nabla^2 Q)(X, e_i, e_i), Y),\ X,Y \in \mathfrak{X}(M) \tag{3.17}$$

4. EIGENVALUES OF MANIFOLDS IN CLASS $\mathcal{A}$

In this section, we are interested in the eigenvalues of the Laplace operator on an Einstein-like manifold, which is a manifold in class $\mathcal{A}$ characterized by the following condition on it's Ricci tensor

$$(\nabla_X Ric)(X,X) = 0,\ X \in \mathfrak{X}(M). \tag{4.1}$$

Naturally Einstein manifolds are Einstein-like manifolds, but there are Einstein-like manifolds which are not Einstein manifolds (cf. [10]).

First, we have the following result.

Theorem 4.1 ([6]). *Let (M,g) be an n-dimensional compact and connected Einstein-like manifold. If all the sectional curvatures of M are bounded below by a constant k_0 and the square of the length of the curvature tensor field $\|R_X\|^2$ in the direction of $X \in \mathfrak{X}(M)$ satisfies $\|R_X\|^2 \leq 2k_0^2(n-1)\|X\|^2 L$ for some constant L, then either M is isometric to an n-sphere or else each eigenvalue λ of the Laplace operator satisfies one of the following inequalities*

(i) $\lambda > \frac{2(3n-2)}{3}k_0$, *(ii)* $\lambda < nk_0 L$, *(iii)* $\frac{2(3n-2)}{3}k_0 < \lambda < nk_0 L$

Proof. Note that the defining condition in equation (4.1) for Einstein-like manifolds is equivalent to

$$(\nabla_X Ric)(Y,Z) + (\nabla_Y Ric)(X,Z) + (\nabla_Z Ric)(X,Y) = 0, \;\; X,Y,Z \in \mathfrak{X}(M). \tag{4.2}$$

Using properties of the curvature tensor and equation (2.5), we get

$$\sum_{ij} g((\nabla_{e_j} R)(e_j, e_i)\nabla f, Ae_i) = -\sum_i (\nabla_{\nabla f} Ric)(Ae_i, e_i) + \sum_i (\nabla_{Ae_i} Ric)(\nabla f, e_i)$$

where A is an operator corresponding to a smooth function f on M.

Now, using (4.2) in the above equation, we arrive at

$$\sum_{ij} g((\nabla_{e_j} R)(e_j, e_i)\nabla f, Ae_i) = 2\sum_i (\nabla_{Ae_i} Ric)(\nabla f, e_i) + \sum_i (\nabla_{e_i} Ric)(\nabla f, Ae_i). \tag{4.3}$$

Since $(\nabla_X Ric)(Y,Z) = g((\nabla Q)(X,Y),Z)$, we get

$$\begin{aligned}\sum_i (\nabla_{Ae_i} Ric)(\nabla f, e_i) &= \sum_{ik} g(Ae_i, e_k) g(\nabla Q)(e_k, \nabla f), e_i) \\ &= \sum_k g((\nabla Q)(e_k, \nabla f), Ae_k) = \sum_k (\nabla_{e_k} Ric)(\nabla f, Ae_k).\end{aligned}$$

Using this in equation (4.3), we conclude that

$$\begin{aligned}\frac{1}{3}\sum_{ij} g((\nabla_{e_j} R)(e_j, e_i)\nabla f, Ae_i) &= \sum_i g((\nabla Q)(e_i, \nabla f), Ae_i) \\ &= \sum_i g(\nabla_{e_i} Q(\nabla f), Ae_i) - Ric(Ae_i, Ae_i)\end{aligned}$$

Integrating this equation and using lemma (3.7), we get

$$\begin{aligned}\int_M \left\{\sum_i g(\nabla_{e_i} Q(\nabla f), Ae_i)\right\} dv &= \frac{1}{3}\int_M \left\{\sum_{ij} R(e_i, e_j; Ae_j, Ae_i)\right\} dv \\ &- \frac{1}{6}\int_M \|R_{\nabla f}\|^2 \, dv + \int_M \left\{\sum_{ij} Ric(Ae_i, Ae_i)\right\} dv. \end{aligned} \tag{4.4}$$

Now define a smooth function $\Psi : M \to R$ by $\Psi = \frac{1}{2}\|A\|^2$, where A is an operator corresponding to smooth function f on M. Then a straightforward computation leads to

$$-\Delta\Psi = \|\nabla A\|^2 + \sum_{ij} g((\nabla^2 A)(e_i, e_i, e_j), Ae_j). \tag{4.5}$$

We use equation (3.2) and (3.3) to compute.

$$(\nabla^2 A)\,(e_i, e_i, e_j) = (\nabla^2 A)(e_j, e_i, e_i) + 2R(e_j, e_i)Ae_j - AR(e_j, e_i)e_j \\ + (\nabla_{e_i} R)(e_i, e_j)\nabla f.$$

Using this equation and the following outcome of Lemma (3.1)

$$\sum_i (\nabla^2 A)(e_j, e_i, e_i) = -\nabla_{e_j}\nabla\Delta f + \nabla_{e_j} Q(\nabla f)$$

in equation (4.5) we arrive at

$$- \Delta\Psi = \|\nabla A\|^2 + \sum_j [-g(\nabla_{e_j}\nabla\Delta f, Ae_i) + g(\nabla_{e_j} Q(\nabla f), Ae_j)] \\ + \sum_{ij} [2R(e_i, e_j, Ae_i, Ae_j) - R(e_i, e_j; e_i, A^2 e_j)] \\ + \sum_{ij} g((\nabla_{e_i} R)(e_i, e_j)\nabla f, Ae_j) \quad (4.6)$$

Substituting the value of $\sum_j g(\nabla_{e_j}\nabla\Delta f, Ae_j)$ from equation (3.8) in the above equation, integrating the resulting equation, and using lemma 3.7 and equation (4.4), we arrive at

$$\int_M \Big\{ \|\nabla A\|^2 - \|\nabla\Delta f\|^2 + Ric(\nabla f, \nabla\Delta f) + \\ \frac{1}{3}\sum_{ij} [4R(e_j, e_i, Ae_i, Ae_j) - 2\,\|R_{\nabla f}\|^2] + \sum_i Ric(Ae_i, Ae_i) \\ + \sum_{ij} [2R(e_i, e_j; Ae_i, Ae_j) - R(e_i, e_j; e_i, A^2 e_j)] \Big\}\, dv = 0$$

Taking $Ae_i = \lambda_i e_i$ in the above integral and noting that

$$\sum_i Ric(Ae_i, Ae_i) - \sum_{ij} R(e_j, e_i; Ae_i, Ae_j) = \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij},$$

where $K_{ij} = R(e_i, e_j; e_j, e_i)$ is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$, we arrive at

$$\int_M \Big\{ \|\nabla A\|^2 - \|\nabla\Delta f\|^2 + Ric(\nabla f, \nabla f) + \frac{2}{3}\sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} \\ + \frac{4}{3}\sum_i Ric(Ae_i, Ae_i) - \frac{2}{3}\,\|R_{\nabla f}\|^2 \Big\}\, dv = 0. \quad (4.7)$$

Since all the sectional curvatures are bounded below by k_0, using Lemmas (3.3), (3.9) we get

$$\int_M \{\sum_i Ric(Ae_i, Ae_i)\}dv \geq k_0(n-1)\int_M \|A\|^2 \, dv$$
$$= k_0\frac{(n-1)}{n}\int_M (\Delta f)^2 dv + \frac{k_0(n-1)}{n}\int_M \{\sum_{i<j}(\lambda_i - \lambda_j)^2\}dv. \quad (4.8)$$

Now suppose $\Delta f = \lambda f$, $\lambda > 0$, then using lemma 3.9, we compute

$$\int_M Ric(\nabla f, \nabla \Delta f)dv = \frac{\lambda(n-1)}{n}\int_M (\Delta f)^2 dv - \frac{\lambda}{n}\int_M \{\sum_{i<j}(\lambda_i - \lambda_j)^2 dv. \quad (4.9)$$

Substituting from (4.8) and (4.9) into Equation (4.7) and using $\|R_{\nabla f}\|^2 \leq 2k_0^2(n-1)L\|\nabla f\|^2$ and $\int_M (\Delta f)^2 = \lambda \int_M \|\nabla f\|^2$, an outcome of Lemma (3.2), we conclude that

$$\int_M (\|\nabla A\|^2 - \frac{1}{n}\|\nabla \Delta f\|^2)dv + \frac{2}{3n}\left((3n-2)\,k_0 - \frac{3\lambda}{2}\right)\int_M \{\sum_{i<j}(\lambda_i - \lambda_j)^2\}dv$$
$$+ \frac{4}{3}k_0(n-1)[\frac{\lambda}{n} - Lk_0]\int_M \|\nabla f\|^2 \, dv \leq 0.$$

If both $\frac{\lambda}{n} - Lk_0 \geq 0$ and $(3n-2)k_0 - \frac{3\lambda}{2} \geq 0$, then we must have, by Lemma (3.8), $\|\nabla A\|^2 = \frac{1}{n}\|\nabla A f\|^2$ and $\lambda = nLk_0 > 0$ (as f is a non-constant function with $\Delta f = \lambda f$ $\lambda > 0$), which again by Lemma (3.8) implies that $A(X) = -\frac{\lambda f}{n}X$, that is $\nabla_X \nabla f = -\frac{\lambda f}{n}X$, $X \in \mathfrak{X}(M)$, for a non-constant function f on M. By Obata's theorem this implies M is isometric to a sphere (cf. [13]). Otherwise we have one of the following possibilities from equation (4.10) (i) $(3n-2)k_0 - \frac{3\lambda}{2} < 0$, (ii) $\frac{\lambda}{n} - Lk_0 < 0$, (iii) $(3n-2)k_0 - \frac{3\lambda}{2} < 0$ and $\frac{\lambda}{n} - Lk_0 < 0$, and this proves the theorem. □

Theorem 4.1 is improved in [8] as follows:

Theorem 4.2 ([8]). *Let (M, g) be an n-dimensional compact Einstein-like manifold and let k_0 be the infimum of its sectional curvatures. If the Ricci curvature of M satisfies $Ric \leq (n-1)k_0\delta$ for a constant $\delta \geq 1$, then either M is isometric to the n-sphere $S^n(k_0)$ or else each nonzero eigenvalue λ of the Laplace operator on M satisfies*

$$\lambda^2 - nk_0\,(7-4\delta)\,\lambda + 2nk_0^2\delta\,(n-(n-1)\,(1-\delta)) > 0.$$

Note that sectional curvatures of a compact manifold lie in a closed interval of R and so do the Ricci curvatures. Thus there are real numbers m_1 and m_2 such that $(n-1)m_1 \leq Ric \leq (n-1)m_2$. But k_0 is the infimum of the sectional curvatures of M, $m_1 = k_0$ and we choose $\frac{m_2}{m_1} = \delta \geq 1$. consequently the constraint on the Ricci curvature in the statement of the above theorem is not an additional condition but the outcome of the compactness of M. We also note that if the Riemannian manifold (M, g) is an Einstein manifold then $k_0 = k_0\delta$, that is $\delta = 1$,and consequently the inequality satisfied by the eigenvalue λ in the above theorem takes the form.

$$\lambda^2 - 3nk_0\lambda + 2n^2k_0 > 0,$$

that is, $(\lambda - nk_0)(\lambda - 2nk_0) > 0$. The Lichnerowicz theorem gives $\lambda > nk_0$ (for (M, g) not isometric to $S^n(k_0)$), and consequently the above inequality gives $\lambda > 2nk_0$. Thus Simon's Theorem 4 in section 2 is a particular case of Theorem 4.2 .

Finally are shall use the length $\|R_X\|$ of the curvature tensor field in the direction of $X \in \mathfrak{X}(M)$ to obtain a characterization of spheres. Note that for a sphere $S^n(k_0)$ we have $\|R_X\|^2 = 2(n-1)k_0^2 \|X\|^2$ and the first nonzero eigenvalue $\lambda_1$of the Laplace operator on $S^n(k_0)$ is given by $\lambda = nk_0$. Thus

$$\|R_X\|^2 = \frac{(n-1)}{2n}\lambda_1(4k_0) \|X\|^2 = \frac{n-1}{2n}\lambda_1 \left[(3n+4)k_0 - 3\lambda_1\right] \|X\|^2 .$$

This raises a question: "Is a compact and connected positively curved n-dimensional Riemannian manifold (M, g) in which

$$\|R_X\|^2 = \frac{(n-1)}{2n}\lambda_1[(3n+4)k_0 - 3\lambda_1] \|X\|^2 ,$$

where k_0 is the infimum of the sectional curvatures of M, isometric to $S^n(k_0)$?". We show that the answer to this question is in the affirmative for Riemannian manifolds in class $\mathcal{A}$, that is, for Einstein-like manifolds.

Theorem 4.3 ([6]). *Let (M, g) be a compact and connected Einstein-like manifold with $k_0 > 0$ the infimum of the sectional curvatures of M. If the first nonzero eigenvalue λ_1 of the Laplace operator on M satisfies.*

$$\|R_X\|^2 \leq \frac{(n-1)\lambda_1}{2n} \left[(3n+4)k_0 - 3\lambda_1\right] \|X\|^2 ,$$

then M is isometric to $S^n(k_0)$.

Proof. Let $\Delta f = \lambda_1 f$, where f is a non-constant smooth function on M. Then equation (4.7) takes the form

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla\Delta f\|^2\}dv + \int_M \{(-(\frac{n-1}{n})\lambda_1^2 + (n-1)k_0\lambda_1)\|\nabla f\|^2 - \frac{2}{3}\|R_{\nabla f}\|^2\}dv + \frac{4}{3}\int_M \{\sum_i Ric(Ae_i, Ae_i)\}dv + \frac{2}{3}k_0\int_M \{\sum_{i<j}(\lambda_i - \lambda_j)^2\}dv \leq 0$$

and equation (4.8) takes the form

$$\int_M \{\sum_i Ric(e_i, e_i)\}dv \geq k_0\frac{(n-1)}{n}\lambda_1\int_M \|\nabla f\|^2 dv + \frac{k_0(n-1)}{n}\int_M \{\sum_{i<j}(\lambda_i - \lambda_j)^2\}dv.$$

The two inequalities above combine to imply that

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla\Delta f\|^2\}dv + \frac{2}{3}\int_M \{\lambda_1\frac{(n-1)}{2n}[(3n+4)k_0 - 3\lambda_1] - \|R_{\nabla f}\|^2\}dv + \frac{2}{3}\frac{(3n-2)}{n}k_0\int_M \{\sum_{i<j}(\lambda_i - \lambda_j)^2\}dv \leq 0. \quad (4.11)$$

If the given condition holds, then as the integrand in the first integral in (4.11) is nonnegative, we have form the last integral $A = \mu I$. This gives $\mu = -\frac{\lambda_1 f}{n}$ and consequently $\nabla_X\nabla f = -\frac{\lambda_1}{n}fX$, $X \in \mathfrak{X}(M)$, which is Obata's differential equation. Hence M is isometric to the sphere $S^n(c)$ where $\|R_{\nabla f}\|^2$, according to equation (4.11), is given by

$$\|R_{\nabla f}\|^2 = \frac{(n-1)}{2}c[(3n+4)k_0 - 3nc]\|\nabla f\|^2$$

as $\lambda_1 = nc$. However for $S^n(c)$, $\|R_{\nabla f}\|^2 = 2(n-1)c^2\|\nabla f\|^2$, using this in the above equation we get $c = k_0$ and consequently M is isometric to $S^n(k_0)$. □

5. EIGENVALUES OF MANIFOLDS IN CLASS $\mathcal{C}$

In this section we shall generalize Simon's theorem for manifolds in class $\mathcal{C}$, that is , for Riemannian manifolds of constant scalar curvature. First we prove the following:

Theorem 5.1 ([7]). *Let (M, g) be an n-dimensional compact Riemannian manifold of constant scalar curvature. Then for a smooth function f on M*

$$\int_M \{2Ric(\nabla f, \nabla \Delta f) - \|Q(\nabla f)\|^2 - 3\sum_i Ric(Ae_i, Ae_i) - \sum_i R(e_i, \nabla f; \nabla f, Q(e_i)) + \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} + \frac{1}{2}\|R_{\nabla f}\|^2\} dv = 0$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A with respect to a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M, and K_{ij} is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$.

Proof. Using Equations (3.15), (3.17) and the Ricci's identity for Q,

$$\begin{aligned} \operatorname{div}((\nabla Q)(\nabla f, \nabla f)) &= \sum_i g(\nabla_{e_i}(\nabla Q)(\nabla f, \nabla f)), e_i) = \sum_i g(Ae_i, (\nabla Q)(e_i, \nabla f) \\ &- \mathcal{F}(\nabla f, e_i)) + \sum_i g(\nabla f, (\nabla^2 Q)(e_i, \nabla f, e_i) + (\nabla Q)(Ae_i, e_i) = \\ &2\sum_i g((\nabla Q)(e_i, Ae_i), \nabla f) - \sum_i g(\mathcal{F}(\nabla f, e_i), Ae_i) \\ &- \sum_i g(\mathcal{F}(Ae_i, e_i), \nabla f) + \|Q(\nabla f)\|^2 - \sum_i R(e, \nabla f; \nabla f, Q(e_i)), \end{aligned} \tag{5.1}$$

where we used $g((\nabla Q)(X, Y), Z) = g(Y, (\nabla Q)(X, Z))$, $X, Y, Z \in \mathfrak{X}(M)$. This follows from the symmetry of the operator Q, and $H_S = 0$ for constant scalar curvature. Furthermore,

$$\begin{aligned} \sum_i g((\nabla Q)(e_i, Ae_i), \nabla f) &= \sum_i g(\nabla Q)(e_i, \nabla f), Ae_i) = \sum_i g(\nabla_{e_i} Q(\nabla f), Ae_i) \\ &- \sum_i g(\nabla_{e_i} \nabla f, Q(Ae_i)) = \sum_i g(\nabla_{e_i} Q(\nabla f), Ae_i) - Ric(Ae_i, Ae_i). \end{aligned} \tag{5.2}$$

Choosing a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M that diagonalizes A, with $Ae_i = \lambda_i e_i$, we get

$$\sum_i g(\mathcal{F}(Ae_i, e_i), \nabla f) = 0 \tag{5.3}$$

and

$$\begin{aligned}\sum_i g(F(\nabla f, e_i), Ae_i) &= -\sum_i \lambda_i g(F(e_i, \nabla f), e_i) \\ &= -\sum_{ij} \lambda_i g((\nabla_{e_j} R)(e_i, \nabla f)e_j, e_i) \\ &= \sum_{ij} \lambda_i (\nabla_{e_j} R)(e_j, e_i; \nabla f, e_i) \\ &= \sum_{ij} g((\nabla_{e_j} R)(e_j, e_i)\nabla f, Ae_i) \qquad (5.4)\end{aligned}$$

Thus, using Equations (3.7), (5.2), (5.3) and (5.4) in Equation (5.1), we get

$$\begin{aligned}&\operatorname{div}((\nabla Q)(\nabla f, \nabla f)) - 2\operatorname{div}(AQ(\nabla f)) = 2Ric(\nabla f, \nabla \Delta f) - \|Q(\nabla f)\|^2 \\ &- 2\sum_i Ric(Ae_i, Ae_i) - \sum_i R(e, \nabla f; \nabla f, Q(e_i)) - \sum_{ij} g((\nabla_{e_i} R)(e_j, e_i)\nabla f, Ae_i)\end{aligned}$$

Integrating this equation and using Lemma (3.7), we get

$$\begin{aligned}\int_M \{2Ric(\nabla f, \nabla \Delta f) - \|Q(\nabla f)\|^2 - 2\sum_i Ric(Ae_i, Ae_i) \\ - \sum_i R(e_i, \nabla f; \nabla f, Q(e_i)) \\ - \sum_{ij} R(e_j, e_i; Ae_i, Ae_j) + \frac{1}{2}\|R_{\nabla f}\|^2\} = 0. \quad (5.5)\end{aligned}$$

We have

$$\sum_i Ric(Ae_i, Ae_i) - \sum_{ij} R(e_j, e_i; Ae_i, Ae_j) = \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij},$$

using this in Equation (5.5), we get the result. □

Next using Lemma (3.6) and Theorem (5.1), we have the following:

Theorem 5.2 ([7]). *Let (M, g) be an n-dimensional compact Riemannian manifold of constant scalar curvature. Then, for a smooth function f on M,*

$$\begin{aligned}\int_M \{\|\nabla A\|^2 - \|\nabla \Delta f\|^2 + 4Ric(\nabla f, \nabla \Delta f) - 2\|Q(\nabla f)\|^2 - 3\sum_i Ric(Ae_i, Ae_i) \\ - \sum_i R(e_i, \nabla f; \nabla f, Q(e_i)) + 2\sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij}\}dv = 0,\end{aligned}$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M, λ_i are eigenvalues of A, and K_{ij} is the sectional curvature of the plane section spanned by $\{e_i, e_j\}$.

As a consequence of the above theorem we get the following corollary, which is essentially Simon's result [14].

Corollary 1. *Let (M, g) be an n-dimensional compact Einstein manifold, $n \geq 3$, whose sectional curvatures are bounded below by a constant k_0. Then either M is isometric to a sphere or else each nonzero eigenvalue λ of the Laplace operator on M satisfies $\lambda > 2nk_0$.*

Proof. Since M is an Einstein manifold of dimension ≥ 3, its scalar curvature S is a constant and $Ric(X,Y) = \frac{S}{n} g(X,Y)$, $X, Y \in \mathfrak{X}(M)$. For $\Delta f = \lambda f$, $\lambda > 0$, we have $\sum_i Ric(Ae_i, Ae_i) = \frac{S}{n} \|A\|^2$, $\sum_i R(e_i, \nabla f, \nabla f, Q(e_i)) = \frac{S^2}{n^2} \|\nabla f\|^2$, $\|Q(\nabla f)\|^2 = \frac{S^2}{n^2} \|\nabla f\|^2$, consequently theorem (5.2) gives

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n} \|\nabla \Delta f\|^2\} + \int_M \{-\frac{(n-1)}{n} \lambda (\Delta f)^2 + (\frac{4\lambda S}{n} - \frac{3S^2}{n^2}) \|\nabla f\|^2 - \frac{3S}{n} \|A\|^2$$
$$+ 2 \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij}\} dv = 0$$

where we have used Lemma (3.2). Now, using Lemmas (3.3) and (3.9) in the above equation, we get

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n} \|\nabla \Delta f\|^2\} dv + (2k_0 - \frac{\lambda}{n}) \int_M \{\sum_{i<j} (\lambda_i - \lambda_j)^2\} dv \leq 0.$$

If $2k_0 - \frac{\lambda}{n} \geq 0$, that is $k_0 \geq \frac{\lambda}{2n} > 0$, then the above inequality, together with Lemma (3.8), implies $A(X) = -\frac{\lambda f}{n} X$, $X \in \mathfrak{X}(M)$. By Obata's result this implies that M is isometric to a sphere, otherwise $\lambda > 2nk_0$. □

Now we prove the following theorem which generalizes Simon's result to the manifolds in class $\mathcal{C}$.

Theorem 5.3 ([7]). *Let (M, g) be an n-dimensional compact Riemannian manifold of constant scalar curvature. If the sectional curvatures of M are bounded below by a constant $k_0 > 0$ and the Ricci curvature satisfies $Ric \leq (n-1)k_0\delta$, $\delta \geq 1$, then either M is isometric to $S^n(k_0)$ or else each nonzero eigenvalue λ of the Laplace operator on M satisfies*

$$\lambda^2 - 3nk_0(2-\delta)\lambda + 2nk_0^2\delta(1 + (n-1)\delta) > 0.$$

Proof. Let f be an eigenfunction corresponding to the eigenvalue λ of the Laplace operator on M. Then, by theorem (5.2), we have

$$\int\{\|\nabla A\|^2 - \frac{1}{n}\|\nabla\Delta f\|^2\}dv + \int_M\{-\frac{(n-1)}{n}\lambda(\Delta f)^2 + 4\lambda Ric(\nabla f,\nabla f) - 2\|Q(\nabla f)\|^2 - 3\sum_i Ric(Ae_i, Ae_i) - \sum_i R(e_i,\nabla f,\nabla f,Q(e_i)) + 2\sum_{i<j}(\lambda_i-\lambda_j)^2K_{ij}\}dv = 0. \quad (5.6)$$

Since $(n-1)k_0 \le Ric \le (n-1)k_0\delta$, we choose a local orthonormal frame $\{e_1,\dots,e_n\}$ that diagonalizes Q. With $Q(e_i) = \mu_i e_i$, we have $\mu_i = Ric(e_i,e_i)$, that is, $(n-1)k_0 \le \mu_i \le (n-1)k_0\delta$ and consequently

$$\sum_i R(e_i,\nabla f;\nabla f,Q(e_i)) \le (n-1)k_0\delta\sum_i R(e_i,\nabla f;\nabla f,e_i).$$

Thus

$$\sum_i R(e,\nabla f;\nabla f,Q(e)) \le (n-1)k_0\delta Ric(\nabla f,\nabla f). \quad (5.7)$$

Now, using Lemma (3.3), we estimate

$$\int_M\{\sum_i Ric(Ae_i,Ae_i)\}dv \le (n-1)k_0\delta\int_M(\Delta f)^2dv - (n-1)k_0\delta\int_M Ric(\nabla f,\nabla f)dv. \quad (5.8)$$

In order to estimate the term $\|Q(\nabla f)\|^2$, we have, with $Q(e_i) = \mu_i e_i$ and $\mu_i = Ric(e_i,e_i)$,

$$\|Q(\nabla f)\|^2 = \sum_i g(Q(\nabla f),e_i)^2 = \sum_i \mu_i^2 g(\nabla f,e_i)^2$$

and as $0 < (n-1)k_0 \le \mu_i \le (n-1)k_0\delta$, the above equation takes the form of the inequality

$$\|Q(\nabla f)\|^2 \le (n-1)^2k_0^2\delta^2\|\nabla f\|^2. \quad (5.9)$$

Now, using Lemma (3.9) and the lower bound k_0 on the sectional curvatures, we see that

$$\int_M\{\sum_{i<j}(\lambda_i-\lambda_j)^2\}dv \ge (n-1)k_0\int_M(\Delta f)^2 - nk_0\int_M Ric(\nabla f,\nabla f)dv. \quad (5.10)$$

Using inequalities (5.7), (5.8), (5.9) and (5.10) in Equation (5.6), together with the bounds on the Ricci curvature, we get

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla \Delta f\|^2\}dv$$
$$+\int_M \{-\left(\frac{n-1}{n}\right)\lambda - 3(n-1)k_0\delta + 2(n-1)k_0\}(\Delta f)^2 dv$$
$$+\int_M \{4\lambda(n-1)k_0 - 2(n-1)^2k_0^2\delta^2 - 2(n-1)k_0^2\delta\}\|\nabla f\|^2 dv \le 0.$$

Finally, using Lemma (3.2) in the above inequality, we arrive at

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla \Delta f\|^2\}dv - \frac{(n-1)}{n}E(\lambda, k_0, \delta)\int_M \|\nabla f\|^2 dv \le 0, \tag{5.11}$$

where the constant $E(\lambda, k_0, \delta)$ is $E = \lambda^2 - 3nk_0(2-\delta)\lambda + 2nk_0^2\delta(1+(n-1)\delta)$. If $E \le 0$, then the inequality (5.11), together with Lemma (3.8), gives $A = -\frac{\lambda f}{n}I$ (as $k_0 > 0$). Thus $\nabla_X \nabla f = -\frac{\lambda f}{n}X$, $X \in \mathfrak{X}(M)$ holds, which is Obata's differential equation and it implies that M is isometric to an n-sphere $S^n(c)$, where $k_0 \le c \le k_0\delta$ and $\delta = 1$. Thus if $E \le 0$, M is isometric to $S^n(k_0)$, and the other option is $E > 0$. □

As a particular case of the above theorem we get Simon's theorem [14]. As for an Einstein manifold, $\delta = 1$ and in which case Theorem 5.3 implies that either M is isometric to $S^n(k_0)$ or else

$$\lambda^2 - 3nk_0\lambda + 2n^2k_0 > 0,$$

which implies that $\lambda > 2nk_0$ (as for M not isometric to a sphere, the Lichnerowicz theorem implies $\lambda > nk_0$).

Finally we have the following two characterizations for spheres as a by-product of the study of eigenvalues of the Laplace operator on compact manifolds in class $\mathcal{C}$.

Theorem 5.4 ([7]). *Let (M, g) by an n-dimensional compact Riemannian manifold of constant scalar curvature whose Ricci curvature satisfies Ric $\ge (n-1)k_0$ for a constant $k_0 > 0$. If there exists a nonzero eigenvalue λ of the Laplace operator on M satisfying*

$$\|R_X\|^2 \le \frac{(n-1)}{n}(\lambda k_0)[(n+2) - \frac{\lambda}{k_0}]\|X\|^2, \ X \in \mathfrak{X}(M),$$

then M *is isometric to* $S^n(k_0)$.

Proof. Let f be the eigenfunction corresponding to the nonzero eigenvalue λ. We subtract the integral in Theorem (5.1) from that in Lemma (3.6) to arrive at

$$\int_M \{\|\nabla A\|^2 - \|\nabla\Delta f\|^2 + 3\sum_i Ric(Ae_i, Ae_i) + \sum_i R(e_i, \nabla f; \nabla f, Q(e_i)) - \|R_{\nabla f}\|^2\}dv = 0. \quad (5.12)$$

The bound on the Ricci curvature together with Lemma (3.3) gives

$$\int_M \{\sum_i Ric(Ae_i, Ae_i)\}dv \geq (n-1)k_0 \int_M \{(\Delta f)^2 - Ric(\nabla f, \nabla f)\}dv. \quad (5.13)$$

Similarly we have for $Q(e_i) = \mu_i e_i$, $\mu_i = Ric(e_i, e_i) \geq (n-1)k_0$ that

$$\int_M \{\sum_i R(e_i, \nabla f; \nabla f, Q(e_i))\}dv \geq (n-1)k_0 \int_M Ric(\nabla f, \nabla f)dv. \quad (5.14)$$

Thus using inequalities (5.13) and (5.14) in (5.12), we arrive at

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla\Delta f\|^2\}dv + \int_M \{\frac{(n-1)}{n}(-\lambda + 3k_0 n)(\Delta f)^2 - 2(n-1)k_0 Ric(\nabla f, \nabla f) - \|R_{\nabla f}\|^2\}dv \leq 0.$$

Next we use Lemma (3.9) in the above inequality to get

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla\Delta f\|^2\}dv + \int_M \{\frac{(n-1)}{n}(-\lambda + (n+2)k_0)(\Delta f)^2 + \frac{2(n-1)}{n}k_0 \sum_{i<j}(\lambda_i - \lambda_j)^2 - \|R_{\nabla f}\|^2\}dv \leq 0$$

which, together with Lemma (3.2) in the form $\int_M (\Delta f)^2 dv = \lambda \int_M \|\nabla f\|^2 dv$, gives

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla\Delta f\|^2\}dv + \int_M \{\frac{(n-1)}{n}\lambda k_0((n+2) - \frac{\lambda}{k_0})\|\nabla f\|^2 - \|R_{\nabla f}\|^2\}dv + \frac{2(n-1)}{n}k_0 \int_M \{\sum_{i<j}(\lambda_i - \lambda_j)^2\}dv \leq 0.$$

This inequality, together with the hypothesis of the theorem and Lemma (3.8), gives $\lambda_1 = \cdots = \lambda_n = \mu$, say, and

$$\frac{(n-1)}{n}\lambda k_0((n+2) - \frac{\lambda}{k_0}) \|\nabla f\|^2 = \|R_{\nabla f}\|^2 . \tag{5.15}$$

Thus $A = -\frac{\lambda f}{n} I$, which by Obata's theorem implies M is isometric to a sphere $S^n(c)$. Since for $S^n(c)$, $\|R_{\nabla f}\|^2 = 2(n-1)c^2 \|\nabla f\|^2$ holds, we have from equation (5.15) that

$$-\lambda^2 + (n+2)k_0\lambda = 2\lambda_1 c,$$

where λ_1 is the first nonzero eigenvalue of the Laplace operator on $S^n(c)$ and $\lambda_1 = nc$. This gives

$$(nk_0 - \lambda) = 2[\frac{\lambda_1}{\lambda}c - k_0] \leq 2(c - k_0).$$

Since $\lambda \geq nk_0$, we get $c \leq k_0$. However, for $S^n(c)$, $Ric = (n-1)c$ and thus the statement of the Theorem implies $c \geq k_0$. This shows that $c = k_0$ and completes the proof. □

In [16, p.676] Yau has suggested a problem "to find constants α and β so that, if the Ricci curvature of a compact manifold satisfies $\alpha \leq Ric \leq \beta$, then the manifold admits an Einstein metric". We have the following Theorem which can be considered a result in this direction.

Theorem 5.5 ([7]). *Let (M, g) be an n-dimensional compact Riemannian manifold of constant scalar curvature whose sectional curvatures are bounded below by a constant $k_0 > 0$. If there exists a nonzero eigenvalue λ of the Laplace operator on M satisfying $\frac{2}{3}k_0\delta(n-1)(\delta-1) < \lambda < 2nk_0$ for a constant $\delta \geq 1$ and the Ricci curvature satisfies*

$$\frac{3(n-1)k_0\delta\lambda}{3\lambda - 2k_0(n-1)(\delta-1)} \leq Ric \leq (n-1)k_0\delta,$$

then M is isometric to $S^n(k_0)$.

Proof. Let f be the eigenfunction corresponding to the eigenvalue λ of the Laplace operator on M given in the statement. Then Equation (5.6) together with Lemma (3.9) gives

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla\Delta f\|^2\}dv + \int_M \{3\lambda Ric(\nabla f,\nabla f) - 2\|Q(\nabla f)\|^2$$
$$-3\sum_i Ric(Ae_i, Ae_i) - \sum_i R(e_i,\nabla f;\nabla f, Q(e_i)) + \sum_{i<j}(\lambda_i-\lambda_j)^2(2k_0-\frac{\lambda}{n})\}dv = 0.$$

Since $(n-1)k_0 \leq Ric \leq (n-1)k_0\delta$, $\delta \geq 1$, using equations (5.7), (5.9) and the estimate $\|Q(\nabla f)\|^2 \leq (n-1)k_0^2\delta^2\|\nabla f\|^2 \leq (n-1)k_0\delta^2 Ric(\nabla f,\nabla f)$ in the above equation, we arrive at

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla\Delta f\|^2\}dv + \int_M \{(3\lambda - 2(n-1)k_0\delta^2 + 2(n-1)k_0\delta)Ric(\nabla f,\nabla f)$$
$$- 3(n-1)k_0\delta(\Delta f)^2 + (2k_0 - \frac{\lambda}{n})\sum_{i<j}(\lambda_i-\lambda_j)^2\}dv \leq 0.$$

Using Lemma (3.2) in the above inequality, it takes the form

$$\int_M \{\|\nabla A\|^2 - \frac{1}{n}\|\nabla\Delta f\|^2\} + (3\lambda - 2(n-1)k_0\delta(\delta-1)\int_M \{Ric(\nabla f,\nabla f)$$
$$-\frac{3(n-1)k_0\delta\lambda}{3\lambda - 2(n-1)k_0\delta(\delta-1)}\|\nabla f\|^2\}dv + (2k_0 - \frac{\lambda}{n})\int_M \{\sum_{i<j}(\lambda_i-\lambda_j)^2\}dv \leq 0.$$

The inequality above together with the hypothesis of the theorem imply (as in the proof of Theorem 5.4) that M is isometric to $S^n(c)$ and that

$$Ric(\nabla f,\nabla f) = \frac{3(n-1)k_0\delta\lambda}{3\lambda - 2(n-1)k_0\delta(\delta-1)}\|\nabla f\|^2.$$

This equation and $Ric(\nabla f,\nabla f) = (n-1)c\|\nabla f\|^2$ for $S^n(c)$ imply

$$3\lambda c - 2(n-1)ck_0\delta(\delta-1) = 3k_0\delta\lambda. \tag{5.16}$$

Since $Ric \leq (n-1)k_0\delta$, for $S^n(c)$ we get $(n-1)c \leq (n-1)k_0\delta$ that is $c \leq k_0\delta$. Using $c \leq k_0\delta$ in (5.16) we get $\delta = 1$ and consequently equation (5.16) gives $c = k_0$ and hence M is isometric to $S^n(k_0)$. □

6. REMARKS

We observe that, due to the lack of tools on Riemannian manifolds in the classes $\mathcal{A}$ and $\mathcal{C}$, it is a difficult task to obtain sharp lower bounds on the eigenvalues of Laplace's operator, like Simon's result for the compact manifolds in class $\mathcal{E}$. Theorems (4.2) and (5.3), though they generalize the result of Simon to compact manifolds in $\mathcal{A}$ and $\mathcal{C}$ (as Simon's result is a particular case of Theorems 4.2 and 5.3), they are by no means the best possible results on the bounds of eigenvalues of Laplace's operator on compact manifolds in the classes $\mathcal{A}$ and $\mathcal{C}$. It will be an interesting problem to get simple inequalities expressing lower bounds on the eigenvalues of the Laplace operator on compact manifolds in $\mathcal{A}$ and $\mathcal{C}$ which improve the quadratic inequalities in Theorems (4.2) and (5.3). Apart from the class $\mathcal{A}$ of Riemannian manifolds between the classes $\mathcal{E}$ and $\mathcal{C}$, there are other important intermediate classes of Riemannian manifolds, denoted by $\mathcal{P}$ and $\mathcal{B}$, which consist of Riemannian manifolds with parallel Ricci tensor and the Ricci tensor being a Codazzi tensor, respectively, satisfying $\mathcal{E} \subset \mathcal{P} \subset \mathcal{C}$ and $\mathcal{E} \subset \mathcal{B} \subset \mathcal{C}$ with $\mathcal{P} = \mathcal{A} \cap \mathcal{B}$ (cf.[10]), and these inclusions are strict. Naturally if will be an interesting subject to obtain results for compact Riemannian manifolds in classes $\mathcal{P}$ and $\mathcal{B}$ which generalize Simon's result. However it is worth noting that a compact Riemannian manifold $\mathcal{M} \in \mathcal{P}$ or $\mathcal{M} \in \mathcal{B}$ having positive sectional curvature implies that $\mathcal{M} \in \mathcal{E}$. Consequently, Simon's result holds for positively curved compact Riemannian manifolds in $\mathcal{P}$ and $\mathcal{B}$. The special classes of Riemannian manifolds considered in [10] involve different constraints on the Ricci tensor. There are other two important classes of Riemannian manifolds, namely conformaly flat Riemannian manifolds and locally symmetric Riemannian manifolds. We cannot expect to generalize Simon's result to compact manifolds in these two classes, though we can expect to get lower bounds on the eigenvalues of Laplace's operator on compact conformaly flat Riemannian manifolds as well as compact locally symmetric Riemannian manifolds.

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