Arab J. Math. Sc.
Volume 8, Number 2, December 2002
pp. 41-52

A CHARACTERIZATION OF EINSTEIN MANIFOLDS

SHARIEF DESHMUKH

ABSTRACT. Let (M, g) be an n-dimensional compact Riemannian manifold, $n \geq 3$, with sectional curvatures bounded below by a constant $k_0 > 0$. If Q is the Ricci operator, S the scalar curvature and F the tensor field of type (1,2) which is the divergence of the curvature tensor of M, then it is shown that the inequality

$$n\|Q\|^2 - S^2 \geq \frac{n-4}{4nk_0}\|grad\ S\|^2 + \frac{1}{2k_0}\|F\|^2$$

implies that (M, g) is an Einstein manifold.

1. INTRODUCTION

The divergence of the curvature tensor R of an n-dimensional Riemannian manifold (M, g) is a tensor field F of type (1,2), defined by

$$F(X,Y) = \sum_i (\nabla e_i R)(X,Y)e_i, \quad X, Y \in \mathfrak{X}(M),$$

where ∇ is the covariant derivative operator with respect to the Riemannian connection, $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields and $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. If $F = 0$, then the curvature tensor is said be divergence free. This operator is important in the theory of general relativity; moreover, it is also important in the characterization of an Einstein manifold. For instance Gray [3] has shown that if M is compact and all its sectional curvatures are positive, then $F = 0$ implies that M is an Einstein manifold (cf. equation (2.4) and Theorem 1.1 in [3]). The Ricci operator $Q : \mathfrak{X}(M) \to \mathfrak{X}(M)$ of a Riemannian manifold (M, g) is defined by $g(Q(X), Y) = Ric(X, Y), X, Y \in \mathfrak{X}(M)$, where Ric is the Ricci tensor of M.

Mathematics Subject Classification: 53C20, 53C25
This research is supported by the Research Center, College of Sciences, KSU, under project no. (math /1419/22)

For any Riemannian manifold (M, g) the Ricci operator Q and the scalar curvature $S = trQ$ are related by the Schwarz's inequality $n\|Q\|^2 - S^2 \geq 0$, and for $\dim M \geq 3$ equality holds if and only if (M, g) is an Einstein manifold. For an Einstein manifold (M, g), $\dim M \geq 3$, the scalar curvature S is constant and $F = 0$, though the converse of this statement is not true (see example given in §4). However, the converse is true for compact Riemannian manifolds of positive sectional curvature. In this paper we consider n-dimensional compact Riemannian manifolds, $n \geq 3$, with sectional curvatures bounded below by a positive constant k_0 and consider the non-negative quantity $\frac{(n-4)}{4nk_0}\|grad\ S\|^2 + \frac{1}{2k_0}\|F\|^2$, the vanishing of which becomes a premise for the above converse. Therefore a natural question arises: under what condition does the above quantity vanish? In this paper we answer this question by proving the following:

Theorem 1.1. *Let (M, g) be an n-dimensional compact and connected Riemannian manifold, $n \geq 3$, with sectional curvatures bounded below by a constant $k_0 > 0$. If the Ricci operator Q, the scalar curvature S, and the divergence F of the curvature tensor field of M satisfy*

$$n\|Q\|^2 - S^2 \geq \frac{n-4}{4nk_0}\|grad\ S\|^2 + \frac{1}{2k_0}\|F\|^2,$$

then (M, g) is an Einstein manifold.

2. PRELIMINARIES

Let (M, g) be an n-dimensional Riemannian manifold. For a local orthonormal frame $\{e_1, \ldots, e_n\}$ on M, we have

$$g(Q(X), Y) = Ric(X, Y) = \sum_i R(e_i, X; Y, e_i) = \sum_i R(X, e_i; e_i, Y), \tag{2.1}$$

where $R(X, Y; Z, W) = g(R(X, Y)Z, W)$, $\quad X, Y, Z, W \in \mathfrak{X}(M)$, R being the curvature tensor field of M. Then, from (2.1), we have the following expression for the Ricci operator

$$Q(X) = \sum_i R(X, e_i)e_i, \quad X \in \mathfrak{X}(M). \tag{2.2}$$

The divergence F of the curvature tensor field is defined by

$$F(X, Y) = \sum_i (\nabla e_i R)(X, Y)e_i, \quad X, Y \in \mathfrak{X}(M), \tag{2.3}$$

where ∇ is the covariant derivative operator with respect to the Riemannian connection on M. Using curvature properties and the Bianchi identities, it is easy to verify that F satisfies $F(X,Y) = -F(Y,X)$, $F(fX,Y) = fF(X,Y)$, $F(X,fY) = fF(X,Y)$, for a smooth function $f : M \to R$, and that

$$g(F(X,Y),Z)+g(F(Y,Z),X)+g(F(Z,X),Y) = 0, \quad X,Y,Z \in \mathfrak{X}(M). \tag{2.4}$$

Also, using equation (2.2) together with second Bianchi identity, we immediately obtain

Lemma 2.1. *The Ricci operator Q satisfies*

$$(\nabla Q)(X,Y) - (\nabla Q)(Y,X) = -F(X,Y), \quad X,Y \in \mathfrak{X}(M),$$

where $(\nabla Q)(X,Y) = \nabla_X Q(Y) - Q(\nabla_X Y)$.

Lemma 2.2. *The gradient of the scalar curvature S of an n- dimensional Riemannian manifold (M,g) satisfies*

$$\frac{1}{2} grad\, S = \sum_j (\nabla Q)(e_j, e_j),$$

where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M.

Proof. We have $S = \sum_j g(Q(e_j), e_j)$ and consequently, for $X \in \mathfrak{X}(M)$, using Lemma 1, we get

$$\begin{aligned} X(S) &= \sum_j g((\nabla Q)(X,e_j),e_j) \\ &= \sum_j g((\nabla Q)(e_j,X),e_j) - \sum_{ji} g((\nabla e_i R)(X,e_j)e_i,e_j) \\ &= \sum_j g(X,(\nabla Q)(e_j,e_j)) - \sum_{ij} g((\nabla e_i R)(X,e_j)e_i,e_j), \end{aligned} \tag{2.5}$$

where we have used $g((\nabla Q)(X,Y),Z) = g(Y,(\nabla Q)(X,Z))$, which is an outcome of the symmetry of the operator Q.

Next, for a local orthonarmal frame $\{e_1, ..., e_n\}$ on a neighborhood of any point $p \in M$ with $\nabla_{e_i} e_j = 0$ at p, we have at p

$$\begin{aligned}
\sum_{ij} g((\nabla e_i R)(X,e_j)e_i,e_j) &= \sum_{ij} g(\nabla e_i R(X,e_j)e_i,e_j) \\
&\quad -\sum_{ij} g(R(\nabla e_i X,e_j)e_i,e_j) \\
&= \sum_{ij} e_i g(R(X,e_j)e_i,e_j) \\
&\quad -\sum_{ij} g(R(\nabla e_i X,e_j)e_i,e_j) \\
&= \sum_{ij} e_i R(X,e_j;e_i,e_j) - \sum_{ij} R(\nabla e_i X,e_j,e_i,e_j) \\
&= -\sum_i e_i Ric(X,e_i) + \sum_i Ric(\nabla e_i X,e_i) \\
&= -\sum_i (\nabla e_i Ric)(X,e_i) \\
&= -\sum_i g(X,(\nabla Q)(e_i,e_i)).
\end{aligned}$$

This last equation, together with (2.5), proves the Lemma. □

Lemma 2.3.

$$(\nabla^2 Q)(X,Y,Z) - (\nabla^2 Q)(X,Z,Y) = -(\nabla F)(X,Y,Z),\ X,Y,Z \in \mathfrak{X}(M),$$

where $(\nabla^2 Q)(X,Y,Z) = \nabla_X(\nabla Q)(Y,Z) - (\nabla Q)(\nabla_X Y,Z) - (\nabla Q)(Y,\nabla_X Z)$.

Proof. The proof is a straightforward calculation using Lemma 1.

Lemma 2.4. *Let* $\{e_1,\ldots,e_n\}$ *be a local orthonormal frame on a compact Riemannian manifold* M. *Then*

$$\int_M \{\sum_i g(\nabla_{e_i} grad\, S, Q(e_i))\} dv = -\frac{1}{2}\int_M \|grad\, S\|^2$$

Proof. We compute

$$\begin{aligned}
div(Q(grad\, S)) &= \sum_i g(\nabla_{e_i} Q(grad\, S),e_i) = \sum_i g((\nabla Q)(e_i,grad\, S),e_i) \\
&\quad +\sum_i g(\nabla_{e_i} grad\, S, Q(e_i))
\end{aligned}$$

$$= \sum_i g(grad\, S, (\nabla Q)(e_i, e_i)) + \sum_i g(\nabla_{e_i} grad\, S, Q(e_i))$$
$$= \frac{1}{2}\|grad\, S\|^2 + \sum_i g(\nabla_{e_i} grad\, S, Q(e_i)),$$

where we have used Lemma 2. Integrating this last equation over M, and using Stokes theorem, we get the result. □

Lemma 2.5. *Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame on a neighborhood of any point of a compact Riemannian manifold (M, g). Then*

$$\int_M \{\sum_{i\alpha} g((\nabla F)(e_\alpha, e_\alpha, e_i), Q(e_i))\} dv = \frac{1}{2}\int_M \|F\|^2,$$

where $\|F\|^2 = \sum_{ij} \|F(e_i, e_j)\|^2$.

Proof. Define a function $f : M \to R$ by $f = \frac{1}{2}\|Q\|^2 = \frac{1}{2}\sum_i g(Q(e_i), Q(e_i))$, then, choosing a local orthonormal frame $\{e_1, \dots, e_n\}$ on a neighborhood of any point $p \in M$ with $\nabla_{e_i} e_j = 0$ at p, we have

$$\Delta f = \sum_\alpha e_\alpha e_\alpha f = \sum_\alpha e_\alpha g(\nabla_{e_\alpha} Q(e_i), Q(e_i)) = \sum_\alpha e_\alpha g((\nabla Q)(e_\alpha, e_i), Q(e_i)) \tag{2.6}$$

at p, where Δ is the Laplacian operator, and

$$div \sum_i ((\nabla Q)(e_i, Q(e_i))) = \sum_{i\alpha} g(\nabla_{e_\alpha}(\nabla Q)(e_i, Q(e_i))), e_\alpha) = \sum_{i\alpha} e_\alpha g(Q(e_i), (\nabla Q)(e_i, e_\alpha)). \tag{2.7}$$

Using Lemma 1, we also have

$$\|F\|^2 = \sum_{ij} g((\nabla Q)(e_i, e_j) - (\nabla Q)(e_j, e_i), (\nabla Q)(e_i, e_j) - (\nabla Q)(e_j, e_i)) = 2[\|\nabla Q\|^2 - \sum_{ij} g((\nabla Q)(e_i, e_j), (\nabla Q)(e_j, e_i))]. \tag{2.8}$$

Thus

$$\sum_{i\alpha} g((\nabla F)(e_\alpha, e_\alpha, e_i), Q(e_i)) = \sum_{i\alpha} g(\nabla_{e_\alpha} F(e_\alpha, e_i), Q(e_i))$$
$$= \sum_{i\alpha} e_\alpha g(F(e_\alpha, e_i), Q(e_i)) - g(F(e_\alpha, e_i), (\nabla Q)(e_\alpha, e_i))$$

$$
\begin{aligned}
&= -\sum_{i\alpha} e_\alpha g((\nabla Q)(e_\alpha, e_i), Q(e_i)) + \sum_{i\alpha} e_\alpha g((\nabla Q)(e_i, e_\alpha), Q(e_i)) \\
&\quad -\sum_{i\alpha} g(F(e_\alpha, e_i), (\nabla Q)(e_\alpha, e_i)),
\end{aligned}
$$

where we have again used Lemma 1. Consequently, using (2.6), (2.7) and Lemma 1, we get

$$
\begin{aligned}
\sum_{i\alpha} g((\nabla F)(e_\alpha, e_\alpha, e_i), Q(e_i)) &= -\Delta f + div(\sum_i (\nabla Q)(e_i, Q(e_i))) \\
&\quad + \sum_{i\alpha} g((\nabla Q)(e_\alpha, e_i) \\
&\quad -(\nabla Q)(e_i, e_\alpha), (\nabla Q)(e_\alpha, e_i)) \\
&= -\Delta f + div(\sum_i ((\nabla Q)(e_i, Q(e_i))) + \|\nabla Q\|^2 \\
&\quad -\sum_{i\alpha} g((\nabla Q)(e_i, e_\alpha), (\nabla Q)(e_\alpha, e_i)).
\end{aligned}
$$

Finally, we can use equation (2.8) and integrate over M to get the desired result. □

Lemma 2.6. *Let (M, g) be an n-dimensional connected Riemannian manifold, $n \geq 3$. Then*

$$
\|\nabla Q\|^2 \geq \frac{1}{n}\|grad\ S\|^2,
$$

and for a positively curved M the equality hold if and only if (M, g) is an Einstein manifold.

Proof. Define $B : \mathfrak{X}(M) \to \mathfrak{X}(M)$ by $B(X) = Q(X) - \frac{S}{n}X$. Then we find

$$
(\nabla B)(X, Y) = (\nabla Q)(X, Y) - \frac{1}{n}X(S)Y,
$$

from which it follows that

$$
\begin{aligned}
\|\nabla B\|^2 &= \|\nabla Q\|^2 + \frac{1}{n}\|grad\ S\|^2 - \frac{2}{n}\sum_{ij} g((\nabla Q)(e_i, e_j), e_j) g(grad\ S, e_i) \\
&= \|\nabla Q\|^2 + \frac{1}{n}\|grad\ S\|^2 \\
&\quad -\frac{2}{n}[\sum_{ij} g((\nabla Q)(e_j, e_i) - F(e_i, e_j), e_j) g(grad\ S, e_i)]
\end{aligned}
$$

$$
\begin{aligned}
&= \|\nabla Q\|^2 + \frac{1}{n}\|grad\ S\|^2 - \frac{2}{n}\sum_{ij} g(e_i, (\nabla Q)(e_j, e_j))g(grad\ S, e_i) \\
&\quad + \frac{2}{n}\sum_{ij} g(F(e_i, e_j), e_j)g(grad\ S, e_i) \\
&= \|\nabla Q\|^2 + \frac{1}{n}\|grad\ S\|^2 - \frac{2}{n}g(grad\ S, \sum_j (\nabla Q)(e_j, e_j)) \\
&\quad + \frac{2}{n}\sum_j g(F(grad\ S, e_j), e_j).
\end{aligned}
$$

Using Lemma 2, we get

$$
\|\nabla B\|^2 = \|\nabla Q\|^2 + \frac{2}{n}\sum_j g(F(grad\ S, e_j), e_j) \tag{2.9}
$$

Now we compute

$$
\begin{aligned}
\sum_j g(F(grad\ S, e_j), e_j) &= \sum_{j\alpha} g((\nabla_{e_\alpha} R)(grad\ S, e_j)e_\alpha, e_j) \\
&= \sum_{j\alpha} [g((\nabla_{e_\alpha} R(grad\ S, e_j)e_\alpha, e_j) - g(R(\nabla_{e_\alpha} grad\ S, e_j)e_\alpha, e_j)] \\
&= \sum_{j\alpha} [e_\alpha R(grad\ S, e_j; e_\alpha, e_j) - R(\nabla_{e_\alpha} grad\ S, e_j; e_\alpha, e_j)] \\
&= \sum_\alpha [-e_\alpha Ric(grad\ S, e_\alpha) + Ric(\nabla_{e_\alpha} grad\ S, e_\alpha)] \\
&= -\sum_\alpha [\nabla_{e_\alpha} Ric)(grad\ S, e_\alpha) \\
&= -\sum_\alpha g((\nabla Q)(e_\alpha grad\ S), e_\alpha) \\
&= -g(grad\ S, \sum_\alpha (\nabla Q)(e_\alpha, e_\alpha)) = -\frac{1}{2}\|grad\ S\|^2.
\end{aligned}
$$

Thus (2.9) becomes $\|\nabla B\|^2 = \|\nabla Q\|^2 - \frac{1}{n}\|grad\ S\|^2$, which proves the inequality. If the equality holds with M positively curved, then $\nabla B = 0$ together with M being irreducible (the curvature of M being positive), so we shall have $B = \mu I$ for some constant $\mu \in R$. Thus in this case we have

$$
Q(X) = \left(\frac{S}{n} + \mu\right) X,
$$

or $Ric(X,Y) = \left(\frac{S}{n} + \mu\right) g(X,Y)$. Since $\dim M \geq 3$, we must have $\frac{S}{n} + \mu =$ constant and $S = \left(\frac{S}{n} + \mu\right) n$, which gives $\mu = 0$. Hence $Ric(X,Y) = \frac{S}{n} g(X,Y)$, that is, M is an Einstein manifold. □

Lemma 2.7. *If $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M which diagonalizes Q, with $Q(e_i) = \lambda_i e_i$, then*

$$\int_M \{\sum_{i<\alpha} (\lambda_i - \lambda_\alpha)^2\} dv = \int_M \{n \|Q\|^2 - S^2\} dv$$

Proof. We have

$$\begin{aligned} \sum_{i\alpha} (\lambda_i - \lambda_\alpha)^2 &= \sum_{i\alpha} \lambda_i^2 + \sum_{i\alpha} \lambda_\alpha^2 - 2 \sum_{i\alpha} \lambda_i \lambda_\alpha \\ &= n\|Q\|^2 + n\|Q\|^2 - 2 \sum_\alpha (\sum_i \lambda_i) \lambda_\alpha \\ &= 2n\|Q\|^2 - 2 \sum_\alpha S\lambda_\alpha = 2(n\|Q\|^2 - S^2) \end{aligned}$$

and $\sum_{i\alpha} (\lambda_i - \lambda_\alpha)^2 = 2 \sum_{i<\alpha} (\lambda_i - \lambda_\alpha)^2$. Integrating the resulting equation over M we get the Lemma. □

3. PROOF OF THE THEOREM

Consider the function $f : M \to R$ defined by $f = \frac{1}{2} \sum_{ij} g(Q(e_i), e_j)^2$, where $\{e_1, \ldots, e_n\}$ is a local orthonormal frame on M. Then it is easy to show that the Hessian H_f of f is given by

$$\begin{aligned} H_f(X,Y) = &\sum_{ij} g((\nabla Q)(X, e_i), e_j) g((\nabla Q)(Y, e_i), e_j) \\ &+ \sum_i g((\nabla^2 Q)(X, Y, e_i), Q(e_i)). \end{aligned}$$

Using Lemma 3, we get

$$\begin{aligned} H_f(X,X) = &\sum_{ij} g((\nabla Q)(X, e_i), e_j)^2 + \sum_i g((\nabla^2 Q)(X, e_i, X) \\ &- (\nabla F)(X, X, e_i), Q(e_i)). \end{aligned}$$

Using the Ricci identity in the above equation we get

$$\begin{aligned} H_f(X,X) &= \sum_{ij} g((\nabla Q)(X,e_i),e_j)^2 + \sum_i g((\nabla^2 Q)(e_i,X,X),Q(e_i)) \\ &\quad + \sum_i R(X,e_i;Q(X),Q(e_i)) - \sum_i R(X,e_i;X,Q^2(e_i)) \\ &\quad - \sum_i g((\nabla F)(X,X,e_i),Q(e_i)). \end{aligned} \tag{3.1}$$

Using Lemma 2, we have $\frac{1}{2}\nabla_X grad\ S = \sum_j (\nabla^2 Q)(X,e_j,e_j)$. This, together with equation (3.1), gives

$$\begin{aligned} \Delta f = \sum_\alpha H_f(e_\alpha,e_\alpha) &= \|\nabla Q\|^2 + \frac{1}{2}\sum_i g((\nabla_{e_i} grad\ S, Q(e_i)) \\ &\quad + \sum_{i\alpha}[R(e_\alpha,e_i,Q(e_\alpha),Q(e_i)) - R(e_\alpha,e_i,e_\alpha,Q^2(e_i))] \\ &\quad - \sum_{i\alpha} g((\nabla F)(e_\alpha,e_\alpha,e_i),Q(e_i)). \end{aligned}$$

Integrating this equation over M, and using Lemmas 4 and 5, we arrive at

$$\begin{aligned} \int_M \{\|\nabla Q\|^2 - \tfrac{1}{4}\|grad\ S\|^2 - \tfrac{1}{2}\|F\|^2 + \textstyle\sum_{i\alpha}[R(e_\alpha,e_i;Q(e_\alpha),Q(e_i)) \\ -R(e_\alpha,e_i;e_\alpha,Q^2(e_i))]\}dv = 0. \end{aligned} \tag{3.2}$$

To evaluate the term $\sum_{i\alpha}[R(e_\alpha,e_i;Q(e_\alpha),Q(e_i)) - R(e_\alpha,e_i;e_\alpha,Q^2(e_i))]$, we choose a local orthonormal frame $\{e_1,\ldots,e_n\}$ which diagonalizes Q with $Q(e_i) = \lambda_i e_i$, and obtain

$$\begin{aligned} &\sum_{i\alpha}[R(e_\alpha,e_i;Q(e_\alpha),Q(e_i)) - R(e_\alpha,e_i;e_\alpha,Q^2(e_i))] \\ &\qquad = \sum_{i\alpha} -\lambda_i\lambda_\alpha K_{i\alpha} + \lambda_i^2 K_{i\alpha} \\ &\qquad = \frac{1}{2}\{\sum_{i\alpha} 2\lambda_i^2 K_{i\alpha} - \sum_{i\alpha} 2\lambda_i\lambda_\alpha K_{i\alpha}\} \\ &\qquad = \frac{1}{2}[\sum_{i\alpha}\lambda_i^2 K_{i\alpha} + \sum_{i\alpha}\lambda_\alpha^2 K_{i\alpha} - \sum_{i\alpha} 2\lambda_i\lambda_\alpha K_{i\alpha}] \\ &\qquad = \frac{1}{2}\sum_{i\alpha}(\lambda_i - \lambda_\alpha)^2 K_{i\alpha} = \sum_{i<\alpha}(\lambda_i - \lambda_\alpha)^2 K_{i\alpha}, \end{aligned}$$

where $K_{i\alpha}$ is the sectional curvature of the plane section spanned by $\{e_i, e_\alpha\}$. Thus (3.2) takes the form

$$\int_M \{(\|\nabla Q\|^2 - \frac{1}{n}\|grad\ S\|^2) - \left(\frac{n-4}{4n}\right)\|grad\ S\|^2 - \frac{1}{2}\|F\|^2 + \sum_{i<\alpha}(\lambda_i - \lambda_\alpha)^2 K_{i\alpha}\}dv = 0.$$

Now, using Lemma 7 and $K_{i\alpha} \geq k_0 > 0$ in the above equation, we get

$$\int_M \{\|\nabla Q\|^2 - \frac{1}{n}\|grad\ S\|^2\}dv + k_0 \int_M \{n\|Q\|^2 - S^2 - \frac{n-4}{4nk_0}\|grad\ S\|^2 - \frac{1}{2k_0}\|F\|^2\}dv \leq 0.$$

Thus, if $n\|Q\|^2 - S^2 \geq \frac{n-4}{4nk_0}\|grad\ S\|^2 + \frac{1}{2k_0}\|F\|^2$ holds, then in (3.3) we must have

$$\int_M \{\|\nabla Q\|^2 - \frac{1}{n}\|grad\ S\|^2\}dv \leq 0,$$

which, together with Lemma 6 gives

$$\|\nabla Q\|^2 = \frac{1}{n}\|grad\ S\|^2.$$

Since M is positively curved (as $k_0 > 0$), this equality, again by Lemma 6, implies that (M, g) is an Einstein manifold. □

4. REMARK

We note that if, for a Riemannian manifold (M, g), the scalar curvature S is a constant and $F = 0$, then we have the Schwarz's inequality $n\|Q\|^2 - S^2 \geq 0$ which satisfies the inequality required by the theorem. However there are Riemannian manifolds having constant scalar curvature and $F = 0$, which are not Einstein manifolds. Here we give two examples.

(1) Consider R^n with the Euclidean metric and the open set

$$H^n = \{(x_1, \ldots, x_n) \in R^n / x_n > 0\}.$$

On H^n we define the metric $\bar{g} = x_n^p g$, where $p = 4/(n-2)$. Then the curvature tensor $\bar{R}$ with respect to $\bar{g}$ of H^n, after a long but simple calculation, can be shown to be

$$\begin{aligned}\bar{R}(X,Y)Z &= \frac{p}{(n-2)x_n^2}\{g(X,Z)Y - g(Y,Z)X \\ &-\frac{n}{2}[X(x_n)Z(x_n)Y - Y(x_n)Z(x_n)X \\ &+g(X,Z)Y(x_n)\frac{\partial}{\partial x^n} - g(Y,Z)X(x_n)\frac{\partial}{\partial x^n}]\},\end{aligned}$$

from which we find the Ricci curvatures

$$\bar{Ric}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^i}\right) = \frac{-2}{(n-2)x_n^2}, \ 1 \le i < n, \tag{4.1}$$

and

$$\bar{Ric}\left(\frac{\partial}{\partial x^n}, \frac{\partial}{\partial x^n}\right) = \frac{2(n-1)}{(n-2)x_n^2},$$

which gives that the scalar curvature $\bar{S} = 0$. After another long but easy calculation, we arrive at

$$F\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{p(n-2)-4}{(n-2)x_n^{p+3}}\left[g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^n}\right)\frac{\partial}{\partial x^j} - g\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^n}\right)\frac{\partial}{\partial x^i}\right] = 0,$$

as $p(n-2) = 4$. Thus for $(H^n, \bar{g})$ both *grad* $\bar{S} = 0$ and $\|F\| = 0$, yet it is not an Einstein manifold as shown by (4.1).

(2) Consider the standard metric on $S^2(c)$, the 2-sphere of constant curvature c, and the product metric on the product $M = S^2(c_1) \times S^2(c_2)$ of two 2-spheres of unequal radii. Then M is a symmetric space and so the RHS in the inequality of the Theorem vanishes, but it is not an Einstein manifold, because the Ricci operator has two different eigenvalues (corresponding to the two unequal radii). Here the sectional curvatures are non-negative but not strictly positive.

REFERENCES

1. B.Y. Chen and L. Vanhecke, Differential Geometry of Geodesic spheres, J. reine angew. Math. 325(1981), 28-67.

2. I. Chavel, Riemannian Geometry: A modern introduction, Cambridge Univ. Press no. 108(1993).

3. A. Gray, Einstein-like manifolds which are not Einstein, Geom. Ded. 7(1978), 259-280.

Department of Mathematics, King Saud University
P.O. Box 2455 Riyadh 11451, Saudi Arabia.
E-mail address: Shariefd@ksu.edu.sa

Date received October 10, 2002.