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BREAKDOWN OF SOLUTIONS OF A SYSTEM DESCRIBING HEAT PROPAGATION WITH SECOND SOUND

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ABSTRACT. In this work we consider a hyperbolic quasilinear system describing the propagation of heat waves for rigid solids at very low temperature. We establish a blow-up result for classical solutions.

1. INTRODUCTION

In the classical theory of thermodynamics, heat conduction is viewed as a purely diffusive process, typically described using Fourier's Law. As a result we get the usual heat equation. This equation provides a useful description of heat conduction under a large range of conditions and predicts an infinite speed of propagation, that is any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox (infinite speed propagation) and disturbances which are almost entirely thermal, may propagate in a finite speed. This wave-like of heat propagation is known as second sound. It was first detected in the **He**, and then in high purity dielectric crystals of sodium fluoride, **NaF**, and bismuth **Bi**. The range of temperature, for which the second sound is detectable, is in fact quite small and normal diffusive propagation takes place above it.

In this theory it is assumed that the heat flux satisfies Cattaneo's law

$$\tau(\theta)q_t + q = -k(\theta)\,\theta_x, \tag{1.1}$$

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where θ is the absolute temperature, q is the heat flux, and τ and k are strictly positive functions depending on the absolute temperature. With this relation, the internal energy, given by

$$e = \hat{e}(\theta), \tag{1.2}$$

is no longer compatible with the second Law of thermodynamics. Coleman, Fabrizio and Owen [1] showed in 1982 that, if (1.1) is adopted then compatibility with thermodynamics requires that (1.2) should be replaced by

$$e = \tilde{e}(\theta, q) = a(\theta) + b(\theta)q^2,$$

where b is a function determined by τ and k. In particular $b(\theta) > 0$. As a consequence, we obtain the system governing the evolution of θ and q

$$\begin{gathered} q_x - (a'(\theta) + b'(\theta)q^2)\theta_t + 2b(\theta)q \;\; q_t = 0 \\ \tau(\theta)q_t + q + k(\theta)q_x = 0. \end{gathered} \tag{1.3}$$

Global existence and decay of classical solutions to the Cauchy problem, as well as to some initial boundary value problems have been established by Coleman, Hrusa, and Owen [2]. In their paper, the authors used a classical energy argument to prove their result. Concerning the formation of singularities, Messaoudi [4] studied the following system

$$\begin{gathered} \tau(\theta)q_t = q + k(\theta)\,\theta_x = 0 \\ c(\theta)\,\theta_t + q_x = 0 \end{gathered}$$

and showed, under the same restrictions on τ, c and k, that classical solutions to the Cauchy problem break down in finite time if the initial data are chosen small in L^∞ norm with large enough derivatives. This result has been generalized by Messaoudi [5], [6] to the system (1.3).

Another approach to second sound is the one presented in [3], [8] and [9], where the authors introduced an internal parameter which accounts for the history memory effects of the heat flux. This approach gave rise to a new theory of heat conduction which we discuss in the next section.

2. DERIVATION OF EQUATIONS

In this section, we investigate the breakdown of classical solutions for the following quasilinear system:

$$v_t - p_x = 0 \tag{2.1}$$

$$p_t - (\sigma(v))_x = f(v)p \tag{2.2}$$

This system describes the propagation of heat wave for rigid solids at very low temperature, below about 20^0K.

Equation (2.1) comes from the balance of energy which in the one-dimensional case takes the form

$$(\varepsilon(v))_t + q_x = 0, \tag{2.3}$$

where $v > 0$ is the absolute temperature, ε is the internal energy, and q is the one-dimensional heat flux. Equation (2.2) is the evolution equation for an internal parameter p, which is introduced to account for memory effects the heat flux. The effect of memory may be considered, for example, as a functional of a history of temperature gradient,

$$q = -\alpha(v) \int_{-\infty}^{t} e^{-\sigma(t-s)} v_x(x,s)ds, \quad \alpha(v) > 0, \quad \sigma > 0. \tag{2.4}$$

By setting

$$p = \int_{-\infty}^{t} e^{-\sigma(t-s)}\, v_x(x,s)\, ds \tag{2.5}$$

equation (2.4) can be equivalently replaced by

$$q = -\alpha(v)p,$$

and a simple derivation shows that

$$p_t = -\sigma p + v_x \tag{2.6}$$

Equation (2.6) is linear and does not fully describe the properties of heat propagation in solids. To improve the model one may generalize the history dependence of q by modifying equation (2.4) as in [7] or, by introducing a suitable nonlinear dependence in (2.6) as in [8]. Namely,

$$p_t = g_1(v)\, v_x + g_2(v)p, \tag{2.7}$$

The functions α, g_1, and g_2 are material dependent. The second law of thermodynamics imposes the restrictions that $\alpha(v) = \kappa v^2 g_1(v)$ and $g_2(v) < 0$, where the constant $\kappa > 0$ comes from the Helmoltz free energy ψ which has the form $\psi = \psi_1(v) + \frac{1}{2}\kappa v p^2$. We also make an assumption that $g_1(v) > 0$. Combining (2.3), (2.4), and (2.7) we get the following system.

$$(\varepsilon(v))_t - (\alpha(v)p)_x = 0$$
$$p_t + (G_1(v))_x = g_2(v)p, \quad G_1'(v) = -g_1(v).$$

If we set $\varepsilon(v) = v$, $\alpha(v) = 1$, $\sigma = -G_1$, and $f = g_2$, system (2.1), (2.2) follows.

3. FORMATION OF SINGULARITIES

This section is devoted to the statement and the proof of the blow-up of solutions for the problem

$$\text{(3.1)} \qquad \begin{cases} v_t - p_x = 0, & x \in \mathbb{R}, \quad t \geq 0 \\ p_t - (\sigma(v))_x = f(v)p, & x \in \mathbb{R}, \quad t \geq 0 \\ v(x,0) = v_0(x), \quad p(x,0) = p_0(x), & x \in \mathbb{R}. \end{cases}$$

We first begin with a theorem which gives a pointwise upper bound on the solution in terms of the initial data.

Theorem 2.1 Assume that σ' and f are C^1 functions, with

$$\sigma'(y) > 0, \qquad |f(y)| \leq c_0, \, \forall y \in \mathbb{R}.$$

Let $v_0, p_0 \in H^2(\mathbb{R})$ be given. Then any solution (v, p) of problem (3.1) satisfies

$$\text{(3.2)} \qquad \max_{x\in\mathbb{R},\ 0\leq t\leq T} \{|v(x,t)| + |p(x,t)|\} \leq \gamma \max_{x\in\mathbb{R}} \{|v_0(x)| + |p_0(x)|\},$$

where γ is a positive constant which depends only on c_0 and T.

Proof. We introduce the quantities

$$\text{(3.3)} \qquad r(x,t) = p(x,t) - A(v(x,t)), \qquad s(x,t) = p(x,t) + A(v(x,t)),$$

where

$$A(v) = \int_0^v \sqrt{\sigma'(\xi)}\ d\xi$$

and the differential operators

$$\partial_t^+ = \frac{\partial}{\partial t} + \sqrt{\sigma'(v)}\,\frac{\partial}{\partial x}, \qquad \partial_t^- = \frac{\partial}{\partial t} - \sqrt{\sigma'(v)}\,\frac{\partial}{\partial x}. \tag{3.4}$$

We then compute

$$\begin{aligned} \partial_t^+ r &= r_t + \sqrt{\sigma'(v)}\, r_x \\ &= \left[-\sqrt{\sigma'(v)}\, v_t + p_t\right] + \sqrt{\sigma'(v)}\left[-\sqrt{\sigma'(v)}\, v_x + p_x\right] \\ &= \left[-\sqrt{\sigma'(v)}\, v_t + \sqrt{\sigma'(v)}\, p_x\right] + \left[p_t - \sigma'(v)\, v_x\right] \\ &= \sqrt{\sigma'(v)}\left[-v_t + p_x\right] + \left[p_t - \sigma'(v)\, v_x\right]. \end{aligned} \tag{3.5}$$

By using (2.1), (2.2), estimate (3.5) becomes

$$\partial_t^+ r = f(v)p. \tag{3.6}$$

Similar computations also lead to

$$\partial_t^- s = f(v)p. \tag{3.7}$$

We then define

$$R(t) := \max_x |r(x,t)|, \qquad S(t) := \max_x |s(x,t)| \tag{3.8}$$

The maxima in (3.8) are attained because r and s go to 0 as $x \to \pm\infty$ since they are H^1-functions. In this case, for any $t \in (0,T)$, we can choose x_1 and x_2 so that

$$R(t) = |r(x_1,t)|, \qquad S(t) = |s(x_2,t)|.$$

Therefore, for any $h \in (0,t)$, we have

$$R(t-h) \geq |r(x_1 - h\sqrt{\sigma'(v)}, t-h)|$$

Consequently

$$\begin{aligned} R(t) - R(t-h) &\leq |r(x_1,t)| - |r(x_1 - h\sqrt{\sigma'(v)},\, t-h)| \\ &\leq |r(x_1,t) - r(x_1 - h\sqrt{\sigma'(v)},\, t-h)|. \end{aligned}$$

By dividing by h and letting h go to 0, we obtain, for almost each $t \in (0,T)$,

$$R'(t) \leq \left|\partial_t^+ r\right| = |f(v)\,p|. \tag{3.9}$$

Similarly, we can show that

$$S'(t) \leq \left|\partial_t^- s\right| = |f(v)\,p| \tag{3.10}$$

We then add (3.9) to (3.10) and use (3.3), (3.8), and the boundedness of f, to get

$$\begin{aligned} \frac{d}{dt}[R(t) + S(t)] &\leq 2|f(v)p| = c_0[r(x,t) + s(x,t)] \\ &\leq c_0\,[R(t) + S(t)], \end{aligned} \tag{3.11}$$

for almost every $t \in [0, T)$. Integration of both sides of (3.11) gives

$$R(t) + S(t) \leq (R(0) + S(0)) + c\int_0^t (R(\eta) + s(\eta))\,d\eta$$

and Gronwall's inequality leads to

$$(R(t) + S(t)) \leq (R(0) + S(0))\,e^{c_0 t}, \quad \forall\, t \in [0, T]. \tag{3.12}$$

Therefore, (3.2) follows by using (3.3) and (3.8).

Theorem 2.2 Let σ' and f be as in Theorem 2.1. Assume further that $\sigma'' > 0$. Then given any $L > 0$, there exist initial data $v_0, p_0 \in H^2(\mathbb{R})$, for which the solution (v,p) blows up in finite time $T^* < L$.

Proof. We take an $x-$partial derivative of (3.6) to get

$$\begin{aligned} \left(\partial_t^+ r\right)_x &= \left(r_t + \sqrt{\sigma'(v)}\,r_x\right)_x \\ &= r_{tx} + \frac{1}{2}\,\sigma''(v)\,\sigma'^{-\frac{1}{2}}(v)\,v_x\,r_x + \sqrt{\sigma'(v)}\,r_{xx} \\ &= \left(r_{tx} + \sqrt{\sigma'(v)}\,r_{xx}\right) + \frac{1}{2}\,\sigma''(v)\,\sigma'^{-\frac{1}{2}}\,v_x\,r_x \end{aligned} \tag{3.13}$$

$$= \partial_t^+ r_x + \frac{1}{2}\,\sigma''(v)\,\sigma'^{-\frac{1}{2}}(v)\,v_x\,r_x = (f(v)p)_x.$$

From (3.3), we have

$$v_x = \frac{s_x - r_x}{2\sqrt{\sigma'(v)}}, \qquad p_x = \frac{s_x + r_x}{2}. \tag{3.14}$$

We substitute (3.14) in (3.13) to obtain

$$\begin{aligned}\partial_t^+ r_x &= \frac{-1}{2}\,\sigma''(v)\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\,\sigma'^{-\frac{1}{2}}(v)\,r_x + f'(v)\left(\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\right)\left(\frac{r+s}{2}\right)\\ &\quad + f(v)\left(\frac{r_x + s_x}{2}\right).\end{aligned} \tag{3.15}$$

Now, we introduce $w = \alpha(v) r_x$ and compute

$$\begin{aligned}\partial_t^+ w &= \partial_t^+(\alpha(v) r_x) = \left[\partial_t^+ \alpha(v)\right] r_x + \alpha(v)\partial_t^+ r_x\\ &= \left[\alpha_t(v) + \sqrt{\sigma'(v)}\,\alpha_x(v)\right] r_x + \alpha(v)\left[\frac{-1}{2}\,\sigma''(v)\,\sigma'^{-\frac{1}{2}}(v)\left(\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\right) r_x\right.\\ &\quad \left. + f'(v)\left(\frac{s_x - r_x}{2\sqrt{\sigma'(v)}}\right)\left(\frac{r+s}{2}\right) + f(v)\left(\frac{r_x + s_x}{2}\right)\right] \qquad (3.16)\\ &= \alpha'(v)\left[v_t + \sqrt{\sigma'(v)}\,v_x\right] r_x + \alpha(v)\left(\left[\frac{-\sigma''(v)(s_x - r_x)}{4\sigma'(v)}\,r_x\right]\right.\\ &\quad + \left.\frac{f'(v)(s_r - r_x)}{4\sqrt{\sigma'(v)}}\,(r+s) + f(v)\left(\frac{r_x + s_x}{2}\right)\right).\end{aligned}$$

At this point we choose $\alpha(v)$ so that

$$\alpha'(v)\left[v_t + \sqrt{\sigma'(v)}\,v_x\right] r_x - \frac{\sigma''(v)\,s_x r_x\,\alpha(v)}{4\sigma'(v)} = 0$$

By using (2.1), we arrive at

$$\alpha'(v)\left[p_x + \sqrt{\sigma'(v)}\,v_x\right]\,r_x - \frac{\sigma''(v) s_x\,r_x\,\alpha(v)}{4\sigma'(v)} = 0$$

and exploiting (3.14) we get

$$\alpha'(v)\left[p_x+\sqrt{\sigma'(v)}\,v_x\right]r_x=\frac{\sigma''(v)\alpha(v)[p_x+\sqrt{\sigma'(v)}\,v_x]}{4\sigma'(v)}r_x.$$

Therefore α satisfies

$$\alpha'(v)-\frac{\sigma''(v)\alpha(v)}{4\sigma'(v)}=0;$$

hence

$$\alpha(v)=[\sigma'(v)]^{\frac{1}{4}}.$$

By substituting in (3.16) we obtain

$$\begin{aligned}(3.17)\quad \partial_t^+ w &= \sigma'(v)^{\frac{1}{4}}\left[\frac{\sigma''(v)}{4\sigma'(v)}r_x^2+\frac{f'(v)(s_x-r_x)}{4\sqrt{\sigma'(v)}}(r+s)\right.\\ &\qquad\left.+f(v)\left(\frac{r_x+s_x}{2}\right)\right]\\ &=\frac{[\sigma'(v)]^{\frac{-3}{4}}\sigma''(v)}{4}r_x^2-\sigma'(v)^{\frac{1}{4}}\left(\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s)-\frac{f(v)}{2}\right)r_x\\ &\qquad+\sigma'(v)^{\frac{1}{4}}\left(\frac{f'(v)(r+s)}{4\sqrt{\sigma'[(v)}}+\frac{f}{2}\right)s_x\\ &=\frac{\sigma''(v)}{4\sigma'(v)^{\frac{5}{4}}}w^2-\left(\frac{f(v)}{4\sqrt{\sigma'(v)}}(r+s)-\frac{f(v)}{2}\right)w\\ &\qquad+\sigma'(v)^{\frac{1}{4}}\left[\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s)+\frac{f}{2}\right]s_x\end{aligned}$$

The last terms in (3.17) can be handled as follows:

$$(3.18)\qquad [\sigma'(v)]^{\frac{1}{4}}\left[\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s)\right]s_x=\sigma'(v)^{\frac{1}{4}}\left[\frac{f'(v)}{4\sqrt{\sigma'(v)}}(2r+2A(v))\right]s_x$$

By using

$$s_x = \frac{-1}{2\sqrt{\sigma'(\imath)}}\,\partial_t^+(r-s)$$

we obtain from (3.18),

$$\sigma'(v)^{\frac14}\left[\frac{f'(v)}{4\sqrt{\sigma'(v)}}(r+s)\right] s_x = \sigma'^{\frac14}(v)\frac{f'(v)}{4\sqrt{\sigma'(v)}}(2r+2A(v))\left(\frac{-\partial_t^+(r-s)}{2\sqrt{\sigma'(v)}}\right)$$

$$(3.19) \qquad = \frac{-1}{4}(\sigma'(v))^{\frac{-3}{4}} f'(v)(r+A(v)\partial_t^+(r-s).$$

By recalling (3.3), direct calculations yield

$$\partial_t^+(r-s) = \partial_t^+(-2A(v)) = \partial_t^+\left(-2\int_0^v \sqrt{\sigma'(\xi)}\,d\xi\right) = -2\sqrt{\sigma'(v)}\;\;\partial_t^+ v$$

Thus (3.19) becomes

$$\frac{-1}{4}(\sigma'(v))^{-\frac34}[f'(v)(r+A(v)]\,\partial_t^+(r-s)$$

$$(3.20) \quad = \frac12(\sigma'(v))^{\frac{-1}{4}} f'(v) r\,\partial_t^+ v + \frac{\sigma'(v)^{-\frac14}}{2} f'(v)A(v)\partial_t^+ v$$

$$= \frac12\left[\partial_t^+\left(r\int_0^v \sigma'(\xi)^{\frac{-1}{4}} f'(\xi)d\xi\right) - \left(\partial_t^+ r\right)\int_0^v \sigma'(\xi)^{\frac{-1}{4}}\,f'(\xi)\,d\xi\right]$$

$$+\partial_t^+\left[\frac12\int_0^v \sigma'(\xi)^{\frac{-1}{4}} f'(\xi)A(\xi)\,d\xi\right]$$

By substituting (3.20) in (3.17) we deduce

$$(3.21) \quad \partial_t w = \frac{\sigma'' w^2}{4\sigma^{\frac54}} - \left(\frac{f'}{4\sqrt{\sigma'}}(r+s) - \frac{f}{2}\right) w$$

$$+\frac12\partial_t^+\left(r\int_0^v \sigma'(\xi)^{\frac{-1}{4}} f'(\xi)d\xi\right) - \frac12\left(\partial_t^+ r\right)\int_0^v \sigma'(\xi)^{\frac{-1}{4}}\,f'(\xi)\,d\xi$$

$$+\,\partial_t^+\left[\frac12\int_0^v \sigma'(\xi)^{\frac{-1}{4}} f'(\xi)A(\xi)\,d\xi\right]$$

$$= \frac{\sigma'' w^2}{4\sigma^{\frac54}} - \left(\frac{f'}{4\sqrt{\sigma'}}(r+s) - \frac{f}{2}\right) w$$

$$- f(v)\left(\frac{r+s}{2}\right)\int_0^v \sigma'(\xi)^{\frac{-1}{4}}\,f'(\xi)\,d\xi + \partial_t^+ g,$$

where

$$g = \frac{1}{2}\left[r\int_0^v \sigma'(\xi)^{\frac{-1}{4}} f'(\xi)d\xi + \int_0^v \sigma'(\xi)^{\frac{-1}{4}} f'(\xi)A(\xi)\,d\xi\right]$$

By setting $k = w - g$, (3.21) takes on the form

$$(3.22) \qquad \partial_t^+ k = \frac{\sigma''(v)(k+g)^2}{4\sigma^{\frac{5}{4}}(v)} - \left(\frac{f'(v)}{4\sqrt{\sigma}}(r+s) - \frac{f(v)}{2}\right)(k+g)$$

$$- \frac{f(v)(r+s)}{2}\int_0^v \left(\sigma'^{\frac{-1}{4}} f'\right)(\xi)d\xi$$

$$= \frac{\sigma''(v)}{4\sigma'(v)^{\frac{5}{4}}}k^2 + \left(\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}}}g - \frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}} + \frac{f(v)}{2}\right)k$$

$$+\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}}}g^2 - \left(\frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}} - \frac{f(v)}{2}\right)g$$

$$- \frac{f(v)(r+s)}{2}\int_0^v (\sigma' f')(\xi)\,d\xi.$$

By choosing initial data small enough (in L^∞ norm), we are guaranteed to have

$$a := inf\left(\frac{\sigma''(v)\,\sigma'^{\frac{-5}{4}}(v)}{4}\right) > 0.$$

and by setting

$$m := \max\left|\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}}}g^2 - \left(\frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}} - \frac{f(v)}{2}\right)g - \frac{f(v)(r+s)}{2}\int_0^v (\sigma' f')(\xi)\,d\xi\right|$$

$$M = \max\left|\frac{\sigma''(v)}{2\sigma'^{\frac{5}{4}}}g - \frac{f'(v)(r+s)}{2\sqrt{\sigma'(v)}} + \frac{f}{2}\right|$$

we get, from (3.22),

$$\partial_t^+ k \geq ak^2 - Mk - m.$$

We then exploit Young's inequality

$$Mk \leq \frac{a}{2}k^2 + \frac{1}{2a}M^2$$

to arrive at

$$\begin{aligned} \partial_t^+ k(t) &\geq ak^2 - \frac{a}{2}k^2 - \frac{1}{2a}M^2 - m \\ &\geq \frac{a}{2}k^2(t) - (\frac{1}{2a}M^2 + m). \end{aligned} \tag{3.23}$$

It suffices to choose v_0 and p_0 small enough in L^∞ norm with derivatives large enough so that

$$\frac{a}{4}k^2(0) > \frac{1}{2a}M^2 + m, \qquad \frac{4}{ak(0)} < L$$

consequently, (3.23) reduces to:

$$\partial_t^+ k(t) \geq \frac{a}{4}k^2(t). \tag{3.24}$$

Integration along the forward characteristics then yields

$$k(t) \geq \frac{1}{\frac{1}{k(0)} - \frac{a}{4}t}$$

Therefore, $k(t) \longrightarrow \infty$, as $t \to T^* \leq 4/ak(0) < L$. Hence, r_x blows up in finite time.

Remark 2.1. The blow up of r_x implies that either u_x or v_x (hence v_t or u_t) blows up in finite time. However, the solution (u, v) remains bounded in the L^∞ norm.

Remark 2.2. A similar result can also be obtained for certain initial boundary value problem.

Remark 2.3. We obtain the same blow-up result, if $\sigma < 0$. In this case we study the evolution of s_x over the backward characteristic.

Remark 2.4. The blow-up result also holds for $\sigma(0) \neq 0$. In this case a slight modification in the proof is needed.

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