Arab J. Math. Sc.
Volume 7, Number 1, June 2001
pp. 47-53

LOGARITHMIC POTENTIALS

PREMALATHA

ABSTRACT. A necessary and sufficient condition is given for a subharmonic function u in $\mathbb{R}^2$ to be a logarithmic potential up to a harmonic function, the condition involving u only.

1. INTRODUCTION

With any subharmonic function u in $\mathbb{R}^2$ we can associate a measure $\mu \geq 0$ in $\mathbb{R}^2$ such that locally u equals $\log|x| * \mu$ up to a harmonic function. To obtain such a representation for u globally, we need some restrictions on u which are generally expressed in terms of both u and μ (M. Brelot [4]). In this note we give a condition for such a global representation in terms of u only. This leads to a characterisation involving the mean values of u, for u to have compact harmonic support.

2. LOGARITHMIC POTENTIALS

Let u be a subharmonic function in $\mathbb{R}^2$ with associated measure μ defined by $d\mu(x) = \frac{1}{2\pi}\Delta u dx$ in the sense of distributions. If H_u^D is the Dirichlet solution in the unit disc D with boundary values $u(x)$ on $|x| = 1$, let

$$\tilde{u}(x) = \begin{cases} u(x) - H_u^D(0) \text{ in } |x| \geq 1 \\ H_u^D(x) - H_u^D(0) \text{ in } |x| < 1. \end{cases}$$

Then $\tilde{u}$ is a subharmonic function in $\mathbb{R}^2$ which is harmonic in $|x| < 1$ with $\tilde{u}(0) = 0$. Let $\tilde{\mu}$ be the measure associated with $\tilde{u}$. For u subharmonic in $\mathbb{R}^2$, $M(r,u)$ stands for the mean value of $u(x)$ on $|x| = r$ and $\mu(B_0^r) = \mu(x : |x| < r)$. Note $\mu(B_0^r) = \tilde{\mu}(B_0^r)$ if $r > 1$.

Now, for $r > 1$,

$$M(r,\tilde{u}) = \int_{1\leq|y|\leq r} (\log\frac{1}{|y|} - \log\frac{1}{r})d\tilde{\mu}(y) \tag{1}$$

which, after an integration by parts, gives

$$M(r,\tilde{u}) = \int_1^r t^{-1}\tilde{\mu}(B_0^t)dt. \tag{2}$$

Equation (1) gives

$$M(r,u) - H_u^D(0) = M(r,\tilde{u}) = \mu(B_0^r)\log r - \int_{1\leq|y|\leq r} \log|y|d\mu(y), \tag{3}$$

while (2) gives

$$\begin{aligned} M(r,\tilde{u}) &= \int_1^r t^{-1}\tilde{\mu}(B_0^t)dt \\ &\leq \tilde{\mu}(B_0^r)\log r \\ &= \mu(B_0^r)\log r \\ &\leq \int_r^{r^2} \frac{\tilde{\mu}(B_0^t)}{t}dt \\ &= M(r^2,\tilde{u}) - M(r,\tilde{u}) \end{aligned} \tag{4}$$

Theorem 2.1. *For a Radon measure ϑ in $\mathbb{R}^2$, $v(x) = \int \log|x-y|d\vartheta(y)$ defines a subharmonic function in $\mathbb{R}^2$ if and only if $\int_{|y|>1} \log|y|d\vartheta(y)$ is finite.*

Proof. If $\int \log|x-y|d\vartheta(y)$ defines a subharmonic function in $\mathbb{R}^2$, so does the function

$$v_1(x) = \int_{|y|>1} \log|x-y|d\vartheta(y).$$

As v_1 is harmonic in $|x| < 1$, $v_1(0) = \int_{|y|>1} \log|y|d\vartheta(y)$ is finite.

Conversely, suppose $\int_{|y|>1} \log |y| d\vartheta(y)$ is finite. Then $\|\vartheta\| = \int_{\mathbb{R}^2} d\vartheta(y)$ is also finite. Let $x_0 \in \mathbb{R}^2$ and N be a small neighbourhood of x_0 such that $\frac{3}{2}|y| \geq |x-y| \geq \frac{1}{2}|y|$ if $x \in N$ and $|y|$ is sufficiently large, say $|y| > R$.

Then, by hypothesis, $\int_{|y|>R} \log |x-y| d\vartheta(y)$ is finite for $x \in N$ and hence it is harmonic in N. Since $\int_{|y|\leq R} \log |x-y| d\vartheta(y)$ is subharmonic in $\mathbb{R}^2$, we conclude that $\int \log |x-y| d\vartheta(y)$ is subharmonic in N. x_0 being arbitrary in $\mathbb{R}^2, v(x) = \int \log |x-y| d\vartheta(y)$ defines a subharmonic function in $\mathbb{R}^2$. This completes the proof of the theorem.

Corollary 2.2. *If a finite measure μ in $\mathbb{R}^2$ generates a logarithmic potential then*

$$\lim_{r\to\infty} (\|\mu\| - \mu(B_0^r)) \log r = 0.$$

Proof. By the above theorem, for any $\epsilon > 0$, there exists a positive R such that

$$\int_{|y|>R} \log |y| d\mu(y) < \epsilon.$$

Now

$$\int_{R\leq|y|\leq R'} \log |y| d\mu(y) \geq (\mu(B_0^{R'}) - \mu(B_0^R)) \log R$$

and hence

$$(\|\mu\| - \mu(B_0^R)) \log R \leq \int_{|y|\geq R} \log |y| d\mu(y) < \epsilon.$$

This gives $\lim_{r\to\infty}(\|\mu\| - \mu(B_0^r)) \log r = 0$.

Corollary 2.3. (Theorem 4, M.Brelot [4]). *Let u be a subharmonic function in $\mathbb{R}^2$ with associated measure μ. Then u is a logarithmic potential up to an additive harmonic function if and only if* $\lim_{r\to\infty}[M(r,u) - \mu(B_0^r) \log r]$ *exists and is finite.*

Proof. Follows from (3).

Remark. Recall that a subharmonic function u in $\mathbb{R}^2$ with associated measure μ is called an admissible subharmonic function if $\int d\mu(y)$ is finite [2]. Thus if the subharmonic function u in $\mathbb{R}^2$ is a logarithmic potential up to a harmonic function it is clear from Theorem 2.1 that it is admissible.

But the converse need not be true; this can be seen as follows:

Let μ be the measure defined as the uniform density on $|x| = n$ of total mass $\frac{1}{n(\log n)^2}$ for all $n \geq 2$. By M. Arsove [3], there exists a subharmonic function u in $\mathbb{R}^2$ with associated measure μ. This function u is admissible since $\|\mu\| = \sum_2^\infty \frac{1}{n(\log n)^2} < \infty$.

But

$$\int_{|y|>1} \log|y| d\mu(y) = \sum_2^\infty \frac{1}{n \log n} = \infty$$

and hence by Theorem 2.1, u is not a logarithmic potential even up to an additive harmonic function.

Theorem 2.4. *A subharmonic function u in $\mathbb{R}^2$ is a logarithmic potential up to an additive harmonic function if and only if* $\lim_{r\to\infty}[M(r^2,u) - 2M(r,u)]$ *exists and is finite.*

Proof. Let u be a logarithmic potential up to an additive harmonic function.

From Corollaries 2.2 and 2.3 we get

$$\lim_{r\to\infty}[M(r,u) - \|\mu\| \log r]$$

exists and is finite say α. Then

$$\begin{aligned}\lim_{r\to\infty}[M(r^2,u) - 2M(r,u)] &= \lim_{r\to\infty}[(M(r^2,u) - \|\mu\| \log r^2 \\ &\quad -2(M(r,u) - \|\mu\| \log r)] \\ &= -\alpha.\end{aligned}$$

Conversely, suppose that $\lim_{r\to\infty}[M(r^2,u) - 2M(r,u)]$ exists and is finite say β. Then for $r > 1$,

$$M(r^2,\tilde{u}) - M(r,\tilde{u}) = M(r^2,u) - M(r,u) \leq M(r,u) + \beta + \epsilon$$

if r is large.

Thus for large r, (4) gives

$$\begin{aligned}M(r,u) - H_u^D(0) &= M(r,\tilde{u}) \leq \mu(B_0^r)\log r \\ &\leq M(r^2,\tilde{u}) - M(r,\tilde{u}) \\ &\leq M(r,u) + \beta + \epsilon..\end{aligned}$$

That is, $0 \leq \mu(B_0^r)\log r - M(r,u) + H_u^D(0) \leq \beta + \epsilon + H_u^D(0)$ for large r.

But from (3) we have

$$\mu(B_0^r)\log r - M(r,u) + H_u^D(0) = \int_{1\leq|y|\leq r} \log|y|d\mu(y).$$

Therefore $\int_{|y|>1} \log|y|d\mu(y)$ is finite and hence by Theorem 2.1, u is a logarithmic potential up to an additive harmonic function. This proves the theorem.

Corollary 2.5. *Let u be a subharmonic function defined outside a compact set in $\mathbb{R}^2$ such that $\lim_{r\to\infty}[M(r^2,u) - 2M(r,u)]$ exists and is finite. Then outside a compact set, $u = p + h$ where h is harmonic in $\mathbb{R}^2$ and p is a logarithmic potential in $\mathbb{R}^2$ with respect to a finite signed measure.*

Proof. By Theorem 1 in [1] there exists a subharmonic function s in $\mathbb{R}^2$ and a constant α such that $u(x) = s(x) - \alpha\log|x|$ outside a compact set. From the hypothesis we conclude $\lim[M(r^2,s) - 2M(r,s)]$ exists and is finite. Hence by Theorem 2.4, s is a logarithmic potential up to an additive harmonic function.

Theorem 2.6. *Let u be a subharmonic function in $\mathbb{R}^2$. Then the following are equivalent:*

(i) $M(r^2,\tilde{u}) = 2M(r,\tilde{u}), r > 1$.

(ii) $M(r^2,\tilde{u}) \leq 2M(r,\tilde{u}), r > 1$.

(iii) supp $\tilde{\mu} \subset \{x : |x| = 1\}$; supp $\mu \subset \{x : |x| \leq 1\}$.

Proof. (ii) $\Rightarrow$ (iii). Let $M(r^2,\tilde{u}) \leq 2M(r,\tilde{u})$ for all $r > 1$. Then from (4) we obtain

$$M(r,\tilde{u}) \leq \tilde{\mu}(B_0^r)\log r \leq M(r^2,\tilde{u}) - M(r,\tilde{u}) \leq M(r,\tilde{u}) \text{ if } r > 1.$$

Hence $M(r^2,\tilde{u}) = 2M(r,\tilde{u})$ and $M(r,\tilde{u}) = \tilde{\mu}(B_0^r)\log r$.

This implies that

$$\tilde{\mu}(B_0^{r^2})\log r^2 = M(r^2,\tilde{u}) = 2M(r,\tilde{u}) = 2\tilde{\mu}(B_0^r)\log r.$$

Hence

$$\mu(B_0^{r^2}) = \mu(B_0^r)$$

and consequently

$$\mu(B_0^{r^{2n}}) = \mu(B_0^r) \text{ if } r > 1.$$

Thus, for $\epsilon > 0$,

$$\mu\{x : (1+\epsilon) < |x| < (1+\epsilon)^{2n}\} = 0.$$

Since ϵ and n are arbitrary, supp $\mu \subset \{x : |x| \leq 1\}$ and hence supp $\tilde{\mu} \subset \{x : |x| = 1\}$.

(iii) $\Rightarrow$ (i). If supp $\tilde{\mu} \subset \{x : |x| = 1\}$, then for $r > 1$,

$$M(r, \tilde{u}) = \int_1^r \frac{\tilde{\mu}(B_0^t)}{t} dt = \|\mu\| \log r.$$

Hence $M(r^2, \tilde{u}) = 2M(r, \tilde{u})$. This completes the proof of the theorem.

REFERENCES

1. V. Anandam, Subharmonic functions outside a compact set in $\mathbb{R}^n$, *Proc. Amer. Math. Soc.*, **84**(1982), 52-54.

2. V. Anandam and M. Brelot, On a generalised logarithmic kernel and its potentials, *Ann. Polon. Math.*, **39** (1981), 15-29.

3. M. Arsove, Functions representable as differences of subharmonic functions, *Trans. Amer. Math. Soc.*, **75**(1953),327-365.

4. M. Brelot, Sur le rôle du point à l'infini dans la théorie des fonctions harmoniques, *Ann. Ecole Norm. Sup.*, **61**(1944), 301-332.

THE RAMANUJAN INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS, UNIVERSITY OF MADRAS, MADRAS - 600 005, INDIA

Date received June 29, 1999.