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THE PAGENUMBER OF N-FREE PLANAR ORDERED SETS THAT CONTAIN NEITHER $K_{2,3}$ NOR $K_{3,2}$ IS AT MOST THREE

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ABSTRACT. In this note we show that the pagenumber of an N-free planar ordered set which contains neither $K_{2,3}$ nor $K_{3,2}$ is at most three.

1. INTRODUCTION

A *book embedding* of a graph G consists of an embedding of its nodes along the spine of a book, and an embedding of its edges on pages so that edges embedded on the same page do not intersect. The *pagenumber* of $G, page(G)$, is the minimum number of pages needed, taken over all permutations of the vertices of G.

The *pagenumber* of an ordered set P, $page(P)$, is the pagenumber of the graph $cov(P)$ taken over only the permutations of the vertices of P which form a linear extension. The pagenumber for ordered sets was introduced by Nowakowski and Parker [6]; who derived bounds for special classes of ordered sets. Alzohairi and Rival gave a two-page algorithm for series-parallel planar ordered sets. Giacom et al. [4] presented a better algorithm (linear time) to embed series-parallel planar ordered sets into two pages. Zaguia et al. [5] gave a polynomial time algorithm to find the pagenumber of bipartite interval ordered sets. The main result of this paper is:

Theorem 1. *The pagenumber of an N-free planar ordered set which contains neither $K_{2,3}$ nor $K_{3,2}$ is at most three.*

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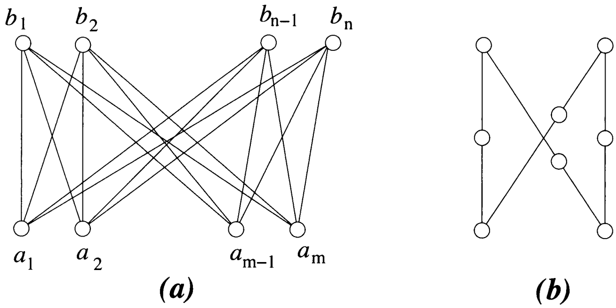

FIGURE 1

In fact, this a generalization of the following result in [3].

Theorem 2. *The pagenumber of an N-free planar ordered set which contains no covering four-cycle is at most two.*

2. N-FREE PLANAR ORDERED SETS.

We say the ordered set P *contains* $K_{m,n}, m, n \geq 2$ if it contains a subset

$$\{a_1, a_2, \cdots, a_m, b_1, , b_2, \cdots, b_n\}$$

satisfying $a_i \prec b_j$ for $1 \leq i \leq m, 1 \leq j \leq n$. Also, we write

$$K_{m,n} = \{a_1, a_2, \cdots, a_m, b_1, , b_2, \cdots, b_n\}$$

(see Figure 1(a)). Notice that $K_{2,2}$ is a covering four-cycle.

We obtain a *subdivision* of a $K_{m,n} = \{a_1, a_2, \cdots, a_m, b_1, , b_2, \cdots, b_n\}$ as follows. For each pair $(i, j), 1 \leq i \leq m, 1 \leq j \leq n$

(1) remove the edge (a_i, b_j).
(2) add the new element x_{ij} and the two edges $(a_i, x_{ij}), (x_{ij}, b_j)$.

Figure 1(b) illustrates a subdivision of the covering four-cycle $K_{2,2}$. As subdividing the edges does not affect planarity and does not create any N, then the N-free planar ordered set remains N-free planar after subdividing all $K_{m,n}, m \geq 2, n \geq 2$.

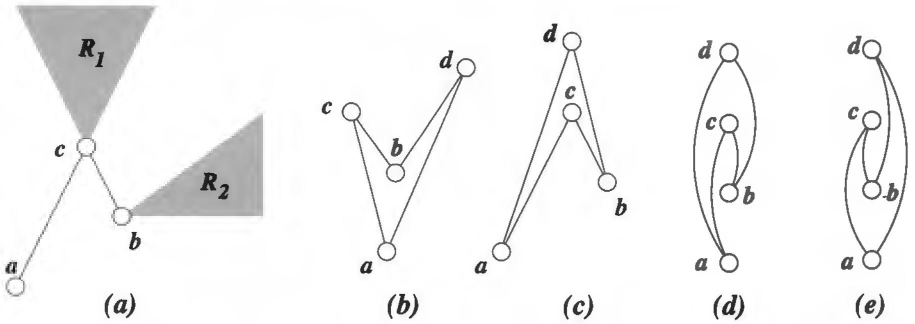

FIGURE 2

A *down set* of an element $x, D(x)$, in an ordered set P is defined by $D(x) = \{y \in P : y \leq x\}$. Dually, we define the *upper set* of $x, U(x) = \{y \in P : y \geq x\}$.

The following Lemma shows that there are only two essential planar upward drawings of any covering four-cycle $C = \{a, b, c, d\}$ illustrated in Figure 2(b) and Figure 2(c). We call the drawing of C in Figure 2(b) a V-shape drawing and that in Figure 2(c) an inverted V-shape drawing.

Lemma 3. *The only planar upward drawings of the covering four-cycle $C = \{a, b, c, d\}$ are the V-shape drawing and the inverted V-shape drawing.*

Proof. We may assume that the two edges (b, c) and (a, c) are drawn as illustrated in Figure 2(a). Let $R_1 = A_1 \bigcap B_1$ where A_1 is the half-plane which lies to the right of the line which contains the edge (b, c) and B_1 is the half-plane which lies to the left of the line which contains the edge (a, c) (see Figure 2(a)).

Let $R_2 = A_2 \bigcap B_2$ where A_2 is the half-plane which lies to the right of the line which contains the points a, b and B_2 is the upper half-plane given by equation $y > t$ where t is the maximum y-coordinate of a, b (see Figure 2(a)). In a planar upward drawing of C either d lies in R_1 or R_2. If d is in R_2, then C admits a V-shape drawing. If d is in R_1, then C admits an inverted V-shape drawing.

The next lemma is in [3].

Lemma 4. *If $a < b < c$ in a left greedy linear extension of a planar lattice P and $a < c$ in P,then either $a < b$ or $b < c$ in P.*

The proof of the next lemma is easy.

Lemma 5. *If P is an N-free ordered set which contains neither $K_{2,3}$ nor $K_{3,2}$ contains the covering four-cycle $C = \{a, b, c, d\}$, then the set of upper covers of a equals the set of upper covers of b equals $\{c, d\}$ and the set of lower covers of c equals the set of lower covers of d equals $\{a, b\}$.*

3. Proof of Theorem 1.

In this section we give the proof of Theorem 1.

For $x \in P$ and $x \neq y \in P$, we say $y \in B^-(x)$ if there are elements $x_1, x_2, \ldots, x_m$ of P such that $x > x_1 < x_2 > \ldots < x_m > y$ or $x > x_1 < x_2 > \ldots > x_m < y$. Also, let $x \in B^-(x)$. Dually, we define $B^+(x)$.

Theorem 6. *Let P be an N-free planar ordered set which contains neither $K_{2,3}$ nor $K_{3,2}$ such that:*

(a) for any covering four-cycle $C = \{a, b, c, d\}$ which admits a V-shape drawing in P, if there are $x, y \in P$ such that $x \prec b, x \prec y$, then there is $z \in P$ satisfying $z \prec b$ and $z \prec y$ (i.e $\{z, x, b, y\}$ is a covering four-cycle.).
(b) for any covering four-cycle $C = \{a, b, c, d\}$ which admits an inverted V-shape drawing in P, if there are $x, y \in P$ such that $c \prec x, y \prec x$, then there is $z \in P$ satisfying $c \prec z$ and $y \prec z$ (i.e $\{c, y, x, z\}$ is a covering four-cycle.).

Then pagenumber$(P) \leq 2$.

Proof. Fix a planar upward drawing of P such that for any covering four-cycle $C = \{a, b, c, d\}$ which admits a V-shape drawing in P, the y-coordinate of any maximal in $B^-(b)$ is less than the y-coordinate of b and for any covering four-cycle $C = \{a, b, c, d\}$ which admits an inverted V-shape drawing in P, the

y-coordinate of any minimal in $B^+(c)$ is greater than the y-coordinate of c. Obtain P' from P as follows:

(i) subdivide all covering four-cycles in P.
(ii) for each covering four-cycle $C = \{a, b, c, d\}$ which admits a V-shape drawing in P such that $B^-(b) = \{b\}$, add $b \succ a$.
(iii) starting with inner cycles, for each covering four-cycle $C = \{a, b, c, d\}$ which admits an inverted V-shape drawing in P such that $B^+(c) = \{c\}$, add $d \succ c$.
(iv) starting with inner cycles, for each covering four-cycle $C = \{a, b, c, d\}$ which admits a V-shape drawing in P and t is a maximal in $B^-(b)$ such that $b \not\succ t$ in P, add the relation $b \succ t$. Also add the relation $w \succ a$ for any minimal $w \in B^-(b)$.
(v) for each covering four-cycle $C = \{a, b, c, d\}$ which admits an inverted V-shape drawing in P and t is a minimal in $B^+(c)$ such that $t \not\succ c$ in P, add the relations $t \succ c$. Also add the relation $d \succ w$ for any maximal $w \in B^+(c)$.

Notice that each of the above steps does not create N. Thus P' is an N-free planar ordered set which contains no covering four-cycle. Therefore there is a two-page linear extension L' of P' obtained by Theorem 2.

Let $\overline{P}$ be the N-free planar lattice obtained from P' in the proof of Theorem 2 in [3].

Now we obtain the left greedy linear extension $\overline{L}$ of $\overline{P}$ and we distribute the edges as described in the two-page algorithm in [3].

Notice that if we restrict $\overline{L}$ to P, we obtain L which is a linear extension of P. Also all edges of P are edges in P' except the edges of covering four-cycles. To obtain a two-page linear extension L of P from $\overline{L}$ do the following:

(1) remove the set $\overline{P} - P$ from $\overline{L}$ and all edges connecting its vertices.
(2) for each covering four-cycle $C = \{a, b, c, d\}$ in P, draw its edges in L as illustrated in Figure 2(d) if C admits a V-shape drawing in P.

(3) for each covering four-cycle $C = \{a, b, c, d\}$ in P, draw its edges in L as illustrated in Figure 2(e) if C admits an inverted V-shape drawing in P.

It is enough to prove that adding the edges of covering four-cycles as described above will not create any edge intersection in the same page. We will show that for a covering four-cycle $C = \{a, b, c, d\}$ which admits a V-shape drawing in P and the proof will be similar if C admits an inverted V-shape drawing in P.

We have $a < b < c < d$ in $\overline{L}$. Therefore we have $a < b < c < d$ in L. In order to show that adding the edges of C in L will not create edge crossings in the same page we will need the following three facts:

Fact 1. *$b \prec c$ in L.*

Proof. Suppose there is $x \in P$ such that $b < x < c$ in L. Thus $b < x < c$ in $\overline{L}$. As $b < c$ in $\overline{P}$ and $\overline{L}$ is left greedy, by Lemma 4, either $b < x$ or $x < c$ in $\overline{P}$. If $b < x$ in $\overline{P}$, then $b < x$ in P. As b has only two upper covers c, d in P and $x < c < d$ in L, then $b \not< x$ in P. Hence $b \not< x$ in P. Thus $x < c$ in $\overline{P}$ which means $x < c$ in P. As $a < b < x$ in L, then $x \not< a$ and $x \not< b$ in P. That implies c has a lower cover in P different from a, b which contradicts the fact that the only lower covers of c are a, b.

Fact 2.*If $a < x < b$ in L, then $x < b$ in P.*

Proof. Since $a < x < b$ in L, then $a < x < b$ in $\overline{L}$. As $a < b$ in $\overline{P}$ and $\overline{L}$ is left greedy, by Lemma 4, either $a < x$ or $x < b$ in $\overline{P}$. As a has only two upper covers c, d in P and $x < c < d$ in L, then $a \not< x$ in P. Hence $x < b$ in $\overline{P}$. Therefore $x < b$ in P.

Fact 3. *If $c < x < d$ in L, then $c < x$ in P.*

Proof. Since $c < x < d$ in L, then $c < x < d$ in $\overline{L}$. As $c < d$ in $\overline{P}$ and $\overline{L}$ is left greedy, by Lemma 4, either $c < x$ or $x < d$ in $\overline{P}$. As d has only two lower covers a, b in P and $a < b < x$ in L, then $x \not< d$ in P. Hence $c < x$ in $\overline{P}$. Therefore $c < x$ in P.

Let (y, t) be an edge of P such that (y, t) is not an edge of C. According to Fact 1 we have six cases to consider.

Case 1. *$y < a < t < b$ in L.*

According to Fact 2, $b > t$ in P. Thus $y < b$ which means $y \in B^-(b)$. According to (iv) in this proof $a < y$ in $\overline{P}$ which implies that $a < y$ in $\overline{L}$. Hence $a < y$ in L which contradicts our assumption. Therefore this case can not happen.

Case 2. $y < a < c < t < d$ in L.

According to Fact 3, $c < t$ in P. Thus $c < t$ in $\overline{P}$. As $c < t$ in $\overline{L}$ and $y < a < b$ in $\overline{P}$, then t has at least two lower covers in $\overline{P}$. According to the two-page algorithm mentioned in [3], the edge (y, t) will be in the left page of $\overline{L}$. Hence the edge (y, t) will be in the left page of L. Therefore there is no edge intersection in this case.

Case 3. *$y < a < d < t$ in L.*

In this case the edge (y, t) will not intersect any edge of C.

Case 4. *$a < y < b < c < t < d$ in L.*

According to Fact 2, $y < b$ in P. Thus $t \in B^-(b)$. According to (iv) in this proof $t < b$ in $\overline{P}$ which implies $t < b$ in $\overline{L}$. Hence $t < b$ in L which contradicts our assumption. Therefore this case can not happen.

Case 5. *$a < y < b < d < t$ in L.*

The same proof of Case 4.

Case 6. *$c < y < d < t$ in L.*

Since $y < d < t$ in L, then $y < d < t$ in $\overline{L}$. As $y < t$ in $\overline{P}$ and $\overline{L}$ is left greedy, by Lemma 4, either $y < d$ or $d < t$ in $\overline{P}$. If $y < d$ in $\overline{P}$, then $y < d$ in P. As $a < b < y$ in L, then d has at least three lower covers in P which contradicts the fact that a, b are the only lower covers of d in P. Thus $d < t$ in $\overline{P}$ which means that t has at least two lower covers in $\overline{P}$. According to the two-page algorithm mentioned in [3], the edge (y, t) will be in the left page of $\overline{L}$. Hence the edge (y, t) will be in the left page of L. Therefore there is no edge intersection in this case.

Proof of Theorem 1. We obtain P' from P using the following procedures:

(a) if a covering four-cycle $C = \{a, b, c, d\}$ admits a V-shape drawing in P such that there are $x, y \in P$ satisfying $x \prec b, x \prec y$ and there is no $z \in P$ satisfying $z \prec b$ and $z \prec y$, then remove the edge (x, b) and add the two edges $(x, t), (t, b)$.

(b) if a covering four-cycle $C = \{a, b, c, d\}$ admits an inverted V-shape drawing in P such that there are $x, y \in P$ satisfying $c \prec x, y \prec x$ and there is no $z \in P$ satisfying $c \prec z$ and $y \prec z$, then remove the edge (c, x) and add the two edges $(c, t), (t, x)$.

Notice that P' is an N-free planar ordered set which contains neither $K_{2,3}$ nor $K_{3,2}$. It is clear P' satisfies the conditions of Theorem 6 which implies that there is a two-page linear extension L' of P'. Obtain L from L' by removing the vertices we added. As P' is an extension of P, then L is linear extension of P. Edges deleted from P are drawn in the third page of L. We will prove, by cases, that there are no edge intersections in the third page.

Case 1. There are two cycles $C = \{a, b, c, d\}, C' = \{a', b', c', d'\}$ which admit V-shape drawings in P such that there are $x, y, x', y' \in P$ satisfying the following conditions:

(i) $x \prec b, x \prec y$ and there is no $z \in P$ satisfying $z \prec b$ and $z \prec y$.

(ii) $x' \prec b', x' \prec y'$ and there is no $z' \in P$ satisfying $z' \prec b'$ and $z' \prec y'$.

Suppose that $x < x' < b < b'$ in L. Thus $x < x' < b < b'$ in $\overline{L}$, where $\overline{L}$ is the linear extension of the planar lattice $\overline{P}$ obtained in the proof of Theorem 6. As $x < x' < b$ in $\overline{L}$ and $x < b$ in $\overline{P}$, then according to Lemma 4 either $x < x'$ or $x' < b$ in $\overline{P}$.

If $x < x'$ in $\overline{P}$, then either $c < x'$ or $d < x'$ in $\overline{P}$. Thus $b < x'$ in $\overline{P}$. Hence $b < x'$ in L which contradicts our assumption.

If $x' < b$ in $\overline{P}$, then either $c' < b$ or $d' < b$ in $\overline{P}$. Thus $b' < b$ in $\overline{P}$. Hence $b' < b$ in L which contradicts our assumption.

Case 2. There are two cycles $C = \{a, b, c, d\}, C' = \{a', b', c', d'\}$ which admit inverted V-shape drawings in P such that there are $x, y, x', y' \in P$ satisfying the following conditions:

(i) $c \prec x, y \prec x$ and there is no $z \in P$ satisfying $c \prec z$ and $y \prec z$.
(ii) $c' \prec x', y' \prec x'$ and there is no $z' \in P$ satisfying $c' \prec z'$ and $y' \prec z'$.

Suppose that $c < c' < x < x'$ in L. Thus $c < c' < x < x'$ in $\overline{L}$. As $c < c' < x$ in $\overline{L}$ and $c < x$ in $\overline{P}$, then according to Lemma 4 either $c < c'$ or $c' < x$ in $\overline{P}$.

If $c < c'$ in $\overline{P}$, then either $d < a'$ or $d < b'$ in $\overline{P}$. Thus $x < c'$ in $\overline{P}$. Hence $x < c'$ in L which contradict our assumption.

If $c' < x$ in $\overline{P}$, then either $d' < a$ or $d' < b$ in $\overline{P}$. Thus $c' < c$ in $\overline{P}$. Hence $c' < c$ in L which contradicts our assumption.

Case 3. There are two cycles $C = \{a, b, c, d\}$ which admits a V-shape drawing in P and $C' = \{a', b', c', d'\}$ which admits an inverted V-shape drawing in P such that there are $x, y, x', y' \in P$ satisfying the following conditions:

(i) $x \prec b, x \prec y$ and there is no $z \in P$ satisfying $z \prec b$ and $z \prec y$.
(ii) $c' \prec x', y' \prec x'$ and there is no $z' \in P$ satisfying $c' \prec z'$ and $y' \prec z'$.

First suppose that $x < c' < b < x'$ in L. Thus $x < c' < b < x'$ in $\overline{L}$. As $x < c' < b$ in $\overline{L}$ and $x < b$ in $\overline{P}$, then according to Lemma 4 either $x < c'$ or $c' < b$ in $\overline{P}$.

If $x < c'$ in $\overline{P}$, then either $c < c'$ or $d < c'$ in $\overline{P}$. Thus $b < c'$ in $\overline{P}$. Hence $b < c'$ in L which contradicts our assumption.

If $c' < b$ in $\overline{P}$, then $d' < a$ in $\overline{P}$. Thus $x' < b$ in $\overline{P}$. Hence $x' < b$ in L which contradicts our assumption.

Finally we can show, in a similar way, that we can not have $c' < x < x' < b$ in L.

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