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SOME PROPERTIES CONCERNING THE INDICIAL ROOTS OF THE JACOBI OPERATOR ABOUT THE DELAUNAY HYPERSURFACE

MOHAMED JLELI [1]

ABSTRACT. In this paper, we prove a maximum principle of the Jacobi operator of the Delaunay hypersurfaces and we study the positivity of the indicial roots about these operators. We partially generalize, in any dimension, the result of R. Kusner, R. Mazzeo and D. Pollack.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

In $\mathbb{R}^3$, all constant mean curvature surfaces of revolutions are classified by Delaunay [1]. In particular, Delaunay discovered a beautiful one-parameter family of complete noncompact surfaces of constant mean curvature one (called Delaunay surfaces). After this classification the theory of constant mean curvature surfaces in $\mathbb{R}^3$ became the object of intensive study. In the case of complete noncompact constant mean curvature surfaces, the moduli space of such surfaces is now fairly well understood (in the genus 0 case). Then, many examples of such surfaces are produced [7],[13]. However, the set of compact constant mean curvature is not so well understood. The first examples of genus 1 constant mean curvature surfaces are constructed by H. Wente [19]. For the high genus case N. Kapouleas gives examples of genus 2 by fusing Wente tori in [9] and others examples of genus is greater than or equal to 3 are obtained by connecting together large number of mutually tangent unit spheres, using small catenoid necks [8].

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Recently, in [6] the authors gave a new idea for the construction of a constant mean curvature compact surfaces of arbitrary genus (≥ 3). This construction was based on three important tools which has been developed for the understanding of complete noncompact constant mean curvature surfaces. The first is the moduli space theory which is developed in [11], the second is the end-addition result which has been developed in [14] and [15] to produce complete noncompact constant mean curvature surfaces with prescribed ends. And finally the end-to-end construction which was developed in [17] to connect two constant mean curvature surfaces along their ends. In the most of the last constructions the use of the behavior of the Delaunay surfaces is crucial. In particular, the study of the Fredholm properties (the kernel and range) of the Jacobi operator about these surfaces which known as the "Linear Decomposition Lemma" (see [13] and [16]) on some weighted Lebesgue spaces and weighted Hölder spaces is based in the behavior of the indicial roots of the Delaunay surfaces [15].

In this paper, we generalize the result of R. Mazzeo, F. Pacard and D. Pollack [15] in $\mathbb{R}^{n+1}$, for $n > 3$. In, particular there exists a one parameter family of constant mean curvature hypersurfaces that will be denoted by $\mathcal{D}_\tau$, for $\tau \in (-\infty, 0) \cup (0, \tau_*)$. We give, in section 2, two different parameterizations of this one parameter family of hypersurfaces of revolution in $\mathbb{R}^{n+1}$, which are immersed or embedded and have constant mean curvature normalized to be equal to 1. These hypersurfaces, which were originally studied in [10], generalize the classical constant mean curvature surfaces in $\mathbb{R}^3$ which were discovered by Delaunay in [1] in the middle of the 19-th century.

In section 3, we define the Jacobi operator (the linearized mean curvature operator) $\mathcal{L}_{\mathcal{D}_\tau}$ about a n-Delaunay hypersurface. Then, we give the expression of the geometric Jacobi fields (some solutions of the homogeneous problem $\mathcal{L}_{\mathcal{D}_\tau} w = 0$). Next, for $\tau \in (-\infty, 0) \cup (0, \tau_*)$, we define the indicial roots associated to the Jacobi operator about a n-Delaunay hypersurface

$$\Gamma(\tau) := \{\pm\gamma_j(\tau) \, : \, j \in \mathbb{N}\}.$$

These reel numbers which characterize the rate of growth (or rate of decay) of the solutions of the homogeneous problem $\mathcal{L}_{\mathcal{D}_\tau} w = 0$ impose the choice of a weighted functional space to obtain a precise description of the mapping properties of the Jacobi operator. In section 4 by a maximum principle concerning $\mathcal{L}_{\mathcal{D}_\tau}$, however we need to impose a lower bound on the Delaunay parameter (τ belongs to $\in [\tau^*, 0) \cup (0, \tau_*]$) for the result to hold, since for τ tends to $-\infty$ there exists a bifurcation result concerning the hypersurface $\mathcal{D}_\tau$ (see [3]). The role played by this maximum principle will be central in the study of the positivity of the indicial roots of $\mathcal{D}_\tau$ when τ belongs to $\in [\tau^*, 0) \cup (0, \tau_*]$.

We need the following definition:

Definition 1.1. Let us denote by $\theta \mapsto e_j(\theta)$, for $j \in \mathbb{N}$ the eigenfunctions of the Laplace-Beltrami operator on S^{n-1}, which will be normalized to have L^2 norm equal to 1 and correspond to the eigenvalue λ_j. That is

$$-\Delta_{S^{n-1}} e_j = \lambda_j e_j,$$

and

$$\lambda_0 = 0, \quad \lambda_1 = \ldots = \lambda_n = n-1, \qquad \lambda_{n+1} = 2n, \ldots \qquad \text{and} \qquad \lambda_j \leq \lambda_{j+1}.$$

We also define

$$\delta_j := \left(\lambda_j + \left(\frac{n-2}{2} \right)^2 \right)^{\frac{1}{2}}. \tag{1.1}$$

Then, our main result reads

Theorem 1.1. *The indicial roots of the Jacobi operator about the Delauanay hypersurface enjoy the following properties:*
(1) For any $\tau \in (-\infty, 0) \cup (0, \tau_*)$,

$$\gamma_0(\tau) = \cdots = \gamma_n(\tau) = 0.$$

(2) There exists $\tau^* < 0$ such that for any $\tau \in (\tau^*, 0) \cup (0, \tau_*)$

$$\gamma_j(\tau) > 0 \qquad \text{for all} \quad j \geq n+1.$$

(3) For all $\eta > 0$ there exists $\tau_0 > 0$, such that for all $\tau \in (-\tau_0, 0) \cup (0, \tau_0)$, the numbers $\gamma_j(\tau)$ satisfy

$$\gamma_j(\tau) \geq \sqrt{\delta_j^2 - \eta}, \quad \text{for all} \quad j \geq n+1.$$

A similar results concerning the maximum principal hold for the Jacobi operator about the sphere S^n, the hyperplane $\mathbb{R}^n \times \{0\}$ and the catenoid.

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2. Constant mean curvature hypersurfaces of revolution

We are looking for constant mean curvature hypersurfaces of revolution (say around the x_{n+1} axis). Such an hypersurface can be locally parameterized by

$$\begin{array}{rcccc} X & : & (t_1, t_2) \times S^{n-1} & \longrightarrow & \mathbb{R}^{n+1} \\ & & (t, \theta) & \longmapsto & (\rho(t)\, \theta, t) \end{array}$$

where the function $t \longrightarrow \rho(t)$ is a smooth positive function which is defined over some interval (t_1, t_2).

The first fundamental form g of the hypersurface parameterized by X is given by

$$g = (1 + (\partial_t \rho)^2)\, dt \otimes dt + \rho^2\, d\theta_i \otimes d\theta_j$$

where $d\theta_i \otimes d\theta_j$ denotes the first fundamental form of S^{n-1}.

Let assume that the orientation of this hypersurface is chosen so that the unit inward normal vector field is given by

$$N := \frac{1}{\sqrt{1 + (\partial_t \rho)^2}}\, (-\theta, \partial_t \rho)\,.$$

With this chosen orientation, the second fundamental form b of the hypersurface parameterized by X is given by

$$b = \frac{1}{\sqrt{1 + (\partial_t \rho)^2}}\, \left(-\partial_t^2 \rho\, dt \otimes dt + \rho\, d\theta_i \otimes d\theta_j \right).$$

It follows at once from the above expressions that the mean curvature H of the hypersurface parameterized by X (which is the average of the trace of the shape form) is given by

$$H = \frac{n-1}{n}\, \frac{1}{\rho}\, \frac{1}{\sqrt{1 + (\partial_t \rho)^2}} - \frac{1}{n}\, \frac{1}{(1 + (\partial_t \rho)^2)^{\frac{3}{2}}}\, \partial_t^2 \rho. \tag{2.1}$$

Hence, the condition that the mean curvature of the hypersurface parameterized by X is equal to some given function H, is given by the equation

$$\partial_t^2 \rho - \frac{n-1}{\rho} \left(1 + (\partial_t \rho)^2\right) + n\, H \, \left(1 + (\partial_t \rho)^2)\right)^{\frac{3}{2}} = 0. \tag{2.2}$$

We introduce

$$\mathcal{H}(\rho, \partial_t \rho) := \frac{\rho^{n-1}}{\sqrt{1 + (\partial_t \rho)^2}} - H\, \rho^n. \tag{2.3}$$

In the case where the function H is constant, it follows from a simple computation that $\mathcal{H}(\rho, \partial_t \rho)$ is constant along solutions of (2.2). This property will be extensively used to derive *a priori* estimates for solutions of (2.2).

It will be more interesting to consider an isothermal type parameterization for which will be more convenient for analytical purposes. Hence, we looking for hypersurfaces of revolution which can be parameterized by

$$X(s, \theta) = (|\tau| \, e^{\sigma(s)} \, \theta, \kappa(s)), \tag{2.4}$$

for $(s, \theta) \in \mathbb{R} \times S^{n-1}$. The constant τ being fixed, the functions σ and κ are determined by asking that the hypersurface parameterized by X has constant mean curvature equal to H and also by asking that the metric associated to the parameterization is conformal to the product metric on $\mathbb{R} \times S^{n-1}$, namely

$$(\partial_s \kappa)^2 = \tau^2 e^{2\sigma} \left(1 - (\partial_s \sigma)^2\right). \tag{2.5}$$

We choose the orientation of the hypersurface parameterized by X so that, the unit normal vector field is given by

$$N := \left(- \frac{\partial_s \kappa}{|\tau| e^{\sigma}} \theta, \partial_s \sigma \right). \tag{2.6}$$

This time, using (2.5) the first fundamental form g of the hypersurface parameterized by X is given by

$$g = \tau^2 \, e^{2\sigma} \, (ds \otimes ds + d\theta_i \otimes d\theta_j),$$

and its second fundamental form b is given by

$$b = \left(\partial_s^2 \kappa \, \partial_s \sigma - \partial_s \kappa \, (\partial_s^2 \sigma + (\partial_s \sigma)^2)\right) \, ds \otimes ds + \partial_s \kappa \, d\theta_i \otimes d\theta_j.$$

Therefore, the mean curvature H of the hypersurface parameterized by X is given by

$$H = \frac{1}{n \tau^2 e^{2\sigma}} \left((n-1) \, \partial_s \kappa - \partial_s \kappa \, (\partial_s^2 \sigma + (\partial_s \sigma)^2) + \partial_s^2 \kappa \, \partial_s \sigma\right).$$

This is a rather intricate second order ordinary differential equation in the functions σ and τ which has to be complimented by the equation (2.5). In order to simplify our analysis, we use of (2.5) to get rid of the factor $\tau^2 e^{2\sigma}$ in the above equation. This yields

$$\partial_s\sigma\, \partial_s^2\kappa = \partial_s\kappa \left(1 - n + \partial_s^2\sigma + (\partial_s\sigma)^2 + n\, H\, \partial_s\kappa\, \left(1 - (\partial_s\sigma)^2\right)^{-1}\right).$$

Now, we can differentiate (2.5) with respect to s, and we obtain

$$\partial_s\kappa\, \partial_s^2\kappa = \tau^2 e^{2\sigma} \partial_s\sigma \left(1 - \partial_s^2\sigma - (\partial_s\sigma)^2\right).$$

The difference between the last equation, multiplied by $\partial_s\sigma$, and the former equation, multiplied by $\partial_s\kappa$, yields

$$\partial_s^2\sigma + (1-n)(1 - (\partial_s\sigma)^2) + n\, H\, \partial_s\kappa = 0. \tag{2.7}$$

Hence, in order to find constant mean curvature hypersurfaces of revolution, we have to solve (2.5) together with(2.7).

Let use define

$$\tau_* := \frac{1}{n}(1-n)^{\frac{n-1}{n}}.$$

For all $\tau \in (-\infty, 0) \cup (0, \tau_*]$, we define σ_τ to be the unique smooth nonconstant solution of

$$(\partial_s\sigma)^2 + \tau^2 \left(e^{\sigma} + \iota\, e^{(1-n)\sigma}\right)^2 = 1, \tag{2.8}$$

with initial condition $\partial_s\sigma(0) = 0$ and $\sigma(0) < 0$. Next, we define the function κ_τ to be the unique solution of

$$\partial_s\kappa = \tau^2 \left(e^{2\sigma} + \iota\, e^{(2-n)\sigma}\right), \qquad \text{with} \qquad \kappa(0) = 0. \tag{2.9}$$

Here, ι is the sign of τ.

In particular, the hypersurface parameterized by

$$X_\tau(s,\theta) := (|\tau| e^{\sigma_\tau(s)}\theta, \kappa_\tau(s)),$$

for $(s,\theta) \in \mathbb{R} \times S^{n-1}$, is an embedded constant mean curvature hypersurface of revolution when τ belongs $(0, \tau_*]$, this hypersurface will be referred to as the "n-unduloid" of parameter τ. In the other case, if $\tau < 0$, this hypersurface is only immersed and will be referred to as the "n-nodoid" of parameter τ.

Remark 2.1. Thanks to the Hamiltonian structure of (2.8), the function $s \mapsto \sigma(s)$ being periodic. Let denote by s_τ this period. Then, it is proved in [5]

$$s_\tau = -\frac{n}{n-1} \log \tau^2 + \mathcal{O}(1) \tag{2.10}$$

as τ tends to 0.

2.1. **Compactness results.** We begin with the study of the behavior of σ_τ as τ tends to 0. We define

$$\varphi_\tau := |\tau| \, e^{\sigma_\tau}, \qquad \text{and} \qquad \eta_\tau := |\tau| \, e^{(1-n)\sigma_\tau},$$

Using (2.5) and (2.7), one can check that, according to the sign of τ, the functions φ_τ and η_τ are nonconstant solutions of

$$(\partial_s \eta)^2 = (n-1)^2 \, \eta^2 \, (1 - (\varphi \pm \eta)^2), \tag{2.11}$$

and

$$(\partial_s \varphi)^2 = \varphi^2 \, (1 - (\varphi \pm \eta)^2), \tag{2.12}$$

with a $+$ when $\tau > 0$ and a $-$ when $\tau < 0$. In addition, we have

$$\varphi_\tau^{n-1} \, \eta_\tau = |\tau|^n. \tag{2.13}$$

Our first Lemma states that the functions φ_τ and η_τ and their derivatives, are uniformly bounded with respect to τ, provided that $|\tau|$ remains bounded.

Lemma 2.1. *Assume that $\tau_0 < 0$ is fixed. Then, for all $k \in \mathbb{N}$, there exists a constant $c_k > 0$ which only depends on τ_0 and k, such that*

$$\|\varphi_\tau\|_{\mathcal{C}^k} + \|\eta_\tau\|_{\mathcal{C}^k} \leq c_k$$

for all $\tau \in [\tau_0, 0) \cup (0, \tau_*]$.

Proof: When $\tau > 0$, observe that (2.12) already implies that the functions φ_τ and η_τ are uniformly bounded by 1. When $\tau < 0$ is bounded from below by τ_0, (2.12) together with (2.13) imply that the functions φ_τ and η_τ are uniformly bounded by a constant only depending on τ_0.

Now that we know that the functions φ_τ and η_τ are uniformly bounded. We use (2.11) and (2.12) inductively to show that the same property is also true for the derivatives of the functions φ_τ and η_τ. □

Assume that s_l is a sequence of real numbers, and that τ_l is a sequence which tends to 0. For all $l \in \mathbb{N}$, we define

$$\varphi_l := \varphi_{\tau_l}(\cdot - s_l) \qquad \text{and} \qquad \eta_l := \eta_{\tau_l}(\cdot - s_l).$$

The previous result together with Ascoli's theorem allows one to extract from the sequence $(\varphi_l, \eta_l)_l$, a subsequence which converges, as l tends to ∞, to $(\varphi_\infty, \eta_\infty)$ in $\mathcal{C}^k$ topology on any compact sets. The following Lemma classifies the possible limits $(\varphi_\infty, \eta_\infty)$.

Lemma 2.2. *Under the above hypothesis, the following holds :*

- *Either* $\varphi_\infty = \eta_\infty \equiv 0$,
- *or* $\varphi_\infty \equiv 0$ *and there exists* s_∞ *such that*
$$\eta_\infty = \frac{1}{\cosh((n-1)(\cdot - s_\infty))},$$
- *or* $\eta_\infty \equiv 0$ *and there exists* s_∞ *such that*
$$\varphi_\infty = \frac{1}{\cosh(\cdot - s_\infty)}.$$

Proof: Passing the limit in (2.13) we get

$$\varphi_\infty^{n-1}\, \eta_\infty \equiv 0.$$

This implies that, at least one of the functions η_∞ and φ_∞ has to be identically equal to 0. It only remains to identify the possible nontrivial limits.

If $\varphi_\infty \equiv 0$, we can pass to the limit in (2.11) and in the derivative of (2.11) with respect to s to get the equation satisfied by η_∞

$$\partial_s^2 \eta = (n-1)^2\, \eta\, \left(1 - 2\,\eta^2\right).$$

Furthermore, we have

$$(\partial_s \eta)^2 = (n-1)^2\, \eta^2\, \left(1 - \eta^2\right).$$

The nontrivial solutions of these equations are all of the form

$$s \longrightarrow \frac{1}{\cosh((n-1)(\cdot - s_0))}$$

for some $s_0 \in \mathbb{R}$.

Now, if $\eta_\infty \equiv 0$, we can pass to the limit in (2.12) and in the derivative of (2.12) to get equation satisfied by η_∞

$$\partial_s^2 \varphi = \varphi \left(1 - 2\,\varphi^2\right).$$

Furthermore, we have

$$(\partial_s \varphi)^2 = \varphi^2\,(1 - \varphi^2).$$

This time the only nontrivial solutions of this equation are of the form

$$s \longrightarrow \frac{1}{\cosh(\cdot - s_0)},$$

for some $s_0 \in \mathbb{R}$. This completes the proof of the result. □

3. The Jacobi operator about a n-Delaunay

Recall that the n-Delaunay hypersurface $\mathcal{D}_\tau$ can be parameterized as

$$X_\tau = (\iota\,\tau\,e^{\sigma_\tau}\,\theta, \kappa_\tau). \tag{3.1}$$

Assume that the orientation of $\mathcal{D}_\tau$ is chosen so that the unit normal vector field is given by

$$N_\tau := \left(-\iota\,\frac{\partial_s \kappa}{\tau\,e^{\sigma_\tau}}\,\theta, \partial_s \sigma_\tau\right). \tag{3.2}$$

Any hypersurface, close enough to $\mathcal{D}_\tau$, can be parameterized (at last locally) as a normal graph over $\mathcal{D}_\tau$. Namely, by

$$X_\omega = X_\tau + \omega\,N_\tau,$$

for some (small) smooth function ω. The hypersurface parameterized by X_ω will be denoted by $\mathcal{D}_\tau(\omega)$ and we define the mean curvature operator $H(\omega)$ to be the mean curvature of $\mathcal{D}_\tau(\omega)$.

It is well known [18] that the linearized mean curvature operator about $\mathcal{D}_\tau$, which is usually referred to as the Jacobi operator, is given by

$$\mathcal{L}_\tau := \Delta_\tau + |A_\tau|^2$$

where Δ_τ is the Laplace-Beltrami operator and $|A_\tau|^2$ is the square of the norm of the shape operator A_τ on $\mathcal{D}_\tau$.

Recall that we have defined in section 2 the function $\varphi_\tau := |\tau| \, e^{\sigma_\tau}$ and, in the above parameterization, the metric on $\mathcal{D}_\tau$ is given by

$$g = \varphi_\tau^2 \, \left(ds \otimes ds + d\theta_i \otimes d\theta_j\right),$$

and the second fundamental form is given by

$$b = \varphi_\tau^2 \, \left((1 \pm (1-n) \, |\tau|^n \, \varphi_\tau^{-n}) \, ds \otimes ds + (1 \pm |\tau^n| \, \varphi_\tau^{-n}) \, d\theta_i \otimes d\theta_j\right),$$

with a $+$ when $\tau > 0$ and a $-$ when $\tau < 0$. Using this, we find the expression of the Jacobi operator in term of the function φ_τ

$$\mathcal{L}_\tau := \varphi_\tau^{-n} \, \partial_s(\varphi_\tau^{n-2} \, \partial_s) + \varphi_\tau^{-2} \, \Delta_{S^{n-1}} + n + n\,(n-1) \, \tau^{2n} \, \varphi_\tau^{-2n}. \tag{3.3}$$

It will be convenient to define the conjugate operator

$$L_\tau := \varphi_\tau^{\frac{n+2}{2}} \;\; \mathcal{L}_\tau \, \varphi_\tau^{\frac{2-n}{2}}, \tag{3.4}$$

which is explicitly given in terms of the function φ_τ by

$$L_\tau = \partial_s^2 + \Delta_{S^{n-1}} - \left(\frac{n-2}{2}\right)^2 + \frac{n(n+2)}{4} \, \varphi_\tau^2 + \frac{n(3n-2)}{4} \, \tau^{2n} \, \varphi_\tau^{2-2n}. \tag{3.5}$$

Since the operators $\mathcal{L}_\tau$ and L_τ are conjugate, the mapping properties of one of them will easily translate for the other one. With slight abuse of terminology, we shall refer to any of them as the Jacobi operator about $\mathcal{D}_\tau$.

Now, we define the operator

$$\Delta_0 := \partial_s^2 + \Delta_{S^{n-1}} - \left(\frac{n-2}{2}\right)^2, \tag{3.6}$$

which appears in the expression of L_τ. This is conjugate to the Jacobi operator about the hyperplane $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$ in polar coordinates. Indeed if $r = e^{-s}$ and $\theta \in S^{n-1}$ then

$$\Delta_0 = e^{-\frac{n+2}{2}} \, \Delta_{\mathbb{R}^n} \, e^{\frac{n-2}{2}}.$$

Its easy to seen that we can parameterize the sphere $S^n \subset \mathbb{R}^{n+1}$ by

$$X_1 = \left(\frac{1}{\cosh s}\theta, \tanh s\right),$$

where $(s, \theta) \in \mathbb{R} \times S^{n-1}$ and we assume that the orientation of S^n is chosen so that the unit normal vector field is given by

$$N_1 = -\left(\frac{1}{\cosh s}\theta, \tanh s\right).$$

With these definitions, the Jacobi operator about the sphere is given by

$$\begin{aligned} \mathcal{L}_1 &:= \Delta_{S^n} + n \\ &= \psi_1^{-n}\, \partial_s(\psi_1^{n-2}\, \partial_s) + \psi_1^{-2}\, \Delta_{S^{n-1}} + n, \end{aligned}$$

where we have defined

$$\psi_1(s) := \frac{1}{\cosh s}.$$

Again, we consider the conjugate operator

$$L_1 := \psi_1^{\frac{n+2}{2}}\; \mathcal{L}_1\, \psi_1^{\frac{2-n}{2}}$$

which takes the simple form

$$L_1 = \Delta_0 + \frac{n(n+2)}{4}\frac{1}{(\cosh s)^2}. \tag{3.7}$$

Finally, the n-catenoid is defined to be the unique (up to rigid motion and dilation) minimal hypersurface of revolution in $\mathbb{R}^{n+1}$. It can be parameterized by

$$X_2(s, \theta) := (\psi_2(s)\, \theta, \kappa_2(s)),$$

for $(s, \theta) \in \mathbb{R} \times S^{n-1}$, where the functions ψ_2 and κ_2 are defined by

$$\psi_2(s) := (\cosh((n-1)s))^{\frac{1}{n-1}} \quad \text{and} \quad \kappa_2(s) := \int_0^s (\cosh((n-1)t))^{\frac{2-n}{n-1}}\, dt.$$

Observe that the function ψ_2 is a solution of

$$(\partial_s \psi_2)^2 = \psi_2^2 - \psi_2^{4-2n} \qquad \text{and} \qquad (\partial_s \kappa_2)^2 = \kappa_2^{2-n}.$$

We assume the orientation chosen so that the unit normal vector field is given by

$$N_2 := \left(- \frac{1}{\cosh((n-1)s)} \theta, \tanh((n-1)s) \right) = \left(- \frac{\partial_s \kappa_2}{\psi_2} \theta, \frac{\partial_s \psi_2}{\psi_2} \right).$$

So, the first and the second fundamental form are given respectively by

$$I = \psi_2^2 \, (ds \otimes ds + d\theta_i \otimes d\theta_j) \,,$$

and

$$II = \psi_2^{2-n} \, ((1-n) \, ds \otimes ds + d\theta \otimes d\theta) \,.$$

It is easy to check that X_2 parameterizes a minimal hypersurface. We refer to [2] for further details and references. Now, the Jacobi operator about the n-catenoid is given by

$$\mathcal{L}_2 := \psi_2^{-n} \, \partial_s (\psi_2^{n-2} \, \partial_s) + \psi_2^{-2} \, \Delta_{S^{n-1}} + n \, (n-1) \, \psi_2^{-2n},$$

Again, we define the conjugate operator

$$L_2 := \psi_2^{\frac{n+2}{2}} \, \mathcal{L}_2 \, \psi_2^{\frac{2-n}{2}}$$

which is explicitly given by

$$L_2 = \Delta_0 + \frac{n \, (3n-2)}{4} \, \frac{1}{(\cosh((n-1)s))^2}. \tag{3.8}$$

Remark

- The expansions of the Jacobi operator L_1 and L_2 can be obtained as a limit of the Jacobi fields L_τ when τ tends to 0 using Lemma 2.2.
- When $n = 2$, $L_1 = L_2$.

3.1. **Geometric Jacobi fields.** We ended this section by giving only the expression of some Jacobi fields, i.e., solution of the homogeneous problem

$$\mathcal{L}_\tau \, \omega = 0$$

since these Jacobi fields follow from a rigid motion or by changing the Delaunay parameter τ. More details are given in [4].

- For $\tau \in (-\infty, 0) \cup (0, \tau_*)$, we define $\Phi_\tau^{0,+}$ to be the Jacobi field corresponding to the translation of $\mathcal{D}_\tau$ along the x_{n+1} axis
$$\Phi_\tau^{0,+} := \varphi^{\frac{n-4}{2}} \, \partial_s \varphi.$$
It is easy to check that $\Phi_\tau^{0,+}$ only depends on s and is periodic. Then, this Jacobi field is bounded.

- Since we have n directions orthogonal to x_{n+1}, there are n linearly independent Jacobi fields which are obtained by translating $\mathcal{D}_\tau$ in a direction orthogonal to its axis. We get for $j = 1, \ldots, n$
$$\Phi_\tau^{j,+} := \left(\varphi^{\frac{n}{2}} \pm |\tau|^n \varphi^{-\frac{n}{2}} \right) e_j.$$
Again, we see that $\Phi_\tau^{j,+}$ is periodic (hence bounded) for all $j = 1, \ldots, n$.

- For $j = 1, \ldots, n$, we define
$$\Phi_\tau^{j,-}(s, \theta) := \varphi^{\frac{n-4}{2}} \left(\varphi \, \partial_s \varphi + \kappa \, \partial_s \kappa \right) e_j$$
to be the Jacobi field corresponding to the rotation of the axis of $\mathcal{D}_\tau$. Observe that $\Phi_\tau^{j,-}$ is not bounded, but grows linearly.
- Finally, the Jacobi field corresponding to a change of parameter $\tau \in (-\infty, 0) \cup (0, \tau_*)$ will be denoted by $\Phi_\tau^{0,-}$. It can be obtained by writing, for η small enough, the constant mean curvature hypersurface $\mathcal{D}_{\tau+\eta}$ as a normal graph over $\mathcal{D}_\tau$ for some function ω_η and differentiating ω_η with respect to η at $\eta = 0$. The corresponding Jacobi field takes the form
$$\Phi_\tau^{0,-} := \varphi^{\frac{n-2}{2}} \, \partial_\tau X_\tau . N_\tau = \varphi^{\frac{n-4}{2}} \left(\partial_\tau \kappa \, \partial_s \varphi - \partial_\tau \varphi \, \partial_s \kappa \right).$$
This Jacobi field is again linearly growing in s as $|s|$ tends to $+\infty$. Indeed, let T_τ to be the physical period of the Delaunay hypersurface which can be written as
$$T_\tau = \kappa_\tau(s_\tau).$$
Then, we have the identity
$$X_\tau(. + s_\tau, .) = X_\tau + T_\tau \, (0_{\mathbb{R}^n}, 1).$$
Differentiation with respect to τ yields
$$\partial_\tau X_\tau(. + s_\tau, .) + \partial_\tau s_\tau \partial_s X_\tau(. + s_\tau, .) = \partial_\tau X_\tau + \partial_\tau T_\tau \, (0_{\mathbb{R}^n}, 1).$$

Taking the scalar product with N_τ and using the definition of $\Phi_\tau^{0,+}$, we obtain

$$\Phi_\tau^{0,-}(.+s_\tau) = \Phi_\tau^{0,-} + \partial_\tau T_\tau\, \Phi_\tau^{0,-}.$$

which clearly shows that $\Phi_\tau^{0,-}$ grows linearly in s.

3.2. **Indicial roots associated to a n-Delaunay.** Because of the rotational invariance of the operator L_τ we can introduce the eigenfunction decomposition with respect to the cross-sectional Laplace-Beltrami operator $\Delta_{S^{n-1}}$. In this way we obtain the sequence of operators

$$L_{\tau,j} = \partial_s^2 - \lambda_j - \left(\frac{n-2}{2}\right)^2 + \frac{n(n+2)}{4}\varphi^2 + \frac{n(3n-2)}{4}\tau^{2n}\varphi^{2-2n} \tag{3.9}$$

for $j \in \mathbb{N}$. By definition, the *indicial roots* of the operator $L_{\tau,j}$ characterize the rate of growth (or rate of decay) of the solutions of the homogeneous equation

$$L_{\tau,j}\,\omega = 0$$

at infinity (see [11]). To explain this, observe that the potential in the expression of $L_{\tau,j}$ is given by

$$-\delta_j^2 + \frac{n(n+2)}{4}\varphi^2 + \frac{n(3n-2)}{4}\tau^{2n}\varphi^{2-2n}$$

where δ_j has been defined in (1.1), and hence this potential is periodic of period s_τ. We define a 2 by 2 matrix $T_j(\tau) \in M_2(\mathbb{R})$ such that, for all ω solution of $L_{\tau,j}\,\omega = 0$ in $\mathbb{R}$, we have

$$\begin{pmatrix}\omega\\ \partial_s\omega\end{pmatrix}(s_\tau) = T_j(\tau)\begin{pmatrix}\omega\\ \partial_s\omega\end{pmatrix}(0).$$

Let $\lambda_-(\tau,j)$ and $\lambda_+(\tau,j)$ denote the roots of the characteristic polynomial of $T_j(\tau)$.

Assume that v_1 and v_2 are the solutions of $L_{\tau,j}\,v_i = 0$ with $v_1(0) = \partial_s v_2(0) = 1$ and $v_2(0) = \partial_s v_1(0) = 0$. We denote by W the Wronskian of v_1, v_2

$$W := v_1\,\partial_s v_2 - v_2\,\partial_s v_1$$

The Wronskian of v_1 and v_2 being constant, we evaluate it at $s = 0$ and $s = s_\tau$, and we obtain

$$\det(T_j(\tau)) = \lambda_-(\tau,j)\lambda_+(\tau,j) = 1.$$

Observe that the matrix $T_j(\tau)$ has real entries, hence

$$\mathrm{Tr}\,(T_j(\tau)) = \lambda_-(\tau, j) + \lambda_+(\tau, j) \in \mathbb{R}$$

This being understood, we can write

$$\lambda_+(\tau, j) = \mu\, e^{i\xi} \qquad \text{and} \qquad \lambda_-(\tau, j) = \frac{1}{\mu}\, e^{-i\xi} \tag{3.10}$$

where $\xi := \xi(\tau, j) \in [0, 2\pi)$ and $\mu := \mu(\tau, j) \geq 1$ satisfy

$$(\mu^2 - 1)\, \sin \xi = 0.$$

The above analysis allows one to state the :

Definition 3.1. The indicial roots of $L_{\tau,j}$ are defined by $\pm\, \gamma_j(\tau)$ where

$$\gamma_j(\tau) := \frac{1}{s_\tau} \log\, \mu(\tau, j),$$

and where $\mu(\tau, j) \geq 1$ is defined as in (3.10). The set of all indicial roots of L_τ is the union of the $\pm\, \gamma_j(\tau)$, namely

$$\Gamma(\tau) := \{\pm \gamma_j(\tau)\ :\ j \in \mathbb{N}\}.$$

4. Maximum principle

This section is devoted to the proof of various forms of the maximum principle for the Jacobi operators which have been defined in the previous section.

4.1. Maximum principle for the hyperplane $\mathbb{R}^n$.

Our first result is simply the :

Proposition 4.1. *Assume that ω is a solution of $\Delta_0\, \omega = 0$ which is defined in $(s_1, s_2) \times S^{n-1}$, with boundary data $\omega = 0$ on $\{s_i\} \times S^{n-1}$ if any of the s_i is finite. Further assume that*

$$|\omega| \leq (\cosh s)^\delta$$

for some $\delta < \frac{n-2}{2}$. Then $\omega = 0$.

Proof: Since the potential in Δ_0 is negative, the result is straightforward when both of the s_i are finite. In the general case, we consider the eigenfunction decomposition of ω as

$$\omega = \sum_{j \in \mathbb{N}} \omega_j \, e_j .$$

We see that $(\partial_s^2 - \delta_j^2)\, \omega_j = 0$, hence ω_j is a linear combination of $s \longrightarrow e^{\pm \delta_j s}$ and has to be bounded by $(\cosh s)^\delta$. The result follows from the fact that $\delta_j \geq \frac{n-2}{2}$ for all $j \geq 0$. □

4.2. **Maximum principle for the Jacobi operator about S^n.** Recall that the conjugate of the Jacobi operator about S^n is defined by

$$L_1 = \Delta_0 + \frac{n\,(n+1)}{4} \frac{1}{(\cosh s)^2}.$$

We prove a maximum principle for the operator L_1, when it is restricted to the set of functions whose eigenfunction decomposition does not involve $e_0, \ldots, e_n$.

Proposition 4.2. *Assume that ω is a solution of $L_1 \omega = 0$ in $(s_1, s_2) \times S^{n-1}$, with boundary data $\omega = 0$ on $\{s_i\} \times S^{n-1}$ if any of the s_i is finite. Further assume that, for all $s \in (s_1, s_2)$, the function $\omega(s, \cdot)$ is L^2-orthogonal to Span$\{e_0, \ldots, e_n\}$ and that*

$$|\omega| \leq (\cosh s)^\delta$$

for some $\delta < \frac{n+2}{2}$. Then $\omega = 0$.

Proof: We consider the eigenfunction decomposition of the function ω, which reads

$$\omega(s, \theta) = \sum_{j \geq n+1} \omega_j(s)\, e_j(\theta).$$

since we have assumed that the function $\omega(s, \cdot)$ is L^2-orthogonal to Span $\{e_0, \ldots, e_n\}$ for all $s \in (s_1, s_2)$.

Then, the function ω_j solves

$$\partial_s^2 \omega_j - \delta_j^2\, \omega_j + \frac{n\,(n+1)}{4} \frac{1}{(\cosh s)^2}\, \omega_j = 0$$

in (s_1, s_2). Since $j \geq n+1$, we have

$$\delta_j \geq \frac{n+2}{2}.$$

Hence the potential in the above ordinary differential equation is negative. The maximum principle already implies that $\omega_j = 0$ if both s_1 and s_2 are finite. In the case where one of the s_i is not finite, say $s_2 = +\infty$, we observe that ω_j either blows up at $+\infty$ like $s \to e^{\delta_j s}$ or decays exponentially at $+\infty$ like $s \longrightarrow e^{-\delta_j s}$. Since we have assumed that $|\omega| \leq (\cosh s)^\delta$ for some $\delta < \frac{n+2}{2}$, we conclude that ω_j decays exponentially like $s \longrightarrow e^{-\delta_j s}$ at $+\infty$. The maximum principle again implies that $\omega_j = 0$. This completes the proof of the result. □

4.3. **Maximum principle for the Jacobi operator about the n-catenoid.** Recall that the conjugate of the Jacobi operator about the n-catenoid is defined by

$$L_2 = \Delta_0 + \frac{n\,(3n-2)}{4} \frac{1}{(\cosh((n-1)s)^2}.$$

The following result can be found in [2] and is the counterpart of the result of Proposition 4.2 for the operator L_2. However, this time the proof is slightly more involved. We give a simpler proof than the one in [2].

Proposition 4.3. *Assume that ω is a solution of $L_2\,\omega = 0$ in $(s_1, s_2) \times S^{n-1}$, with boundary data $\omega = 0$ on $\{s_i\} \times S^{n-1}$ if any of the s_i is finite. Further assume that, for all $s \in (s_1, s_2)$, the function $\omega(s, \cdot)$ is L^2-orthogonal to Span$\{e_0, \ldots, e_n\}$ and that*

$$|\omega| \leq (\cosh s)^\delta$$

for some $\delta < \frac{n+2}{2}$. Then $\omega = 0$.

Proof: For the time being, let us assume that both s_1 and s_2 are finite. Observe that, if ω solves $L_2\,\omega = 0$, then

$$v := \psi_2^{\frac{2-n}{2}}\,\omega$$

solves $\mathcal{L}_2\,v = 0$. We consider the eigenfunction decomposition of the function v, namely

$$v(s, \theta) = \sum_{j \geq n+1} v_j(s)\, e_j(\theta).$$

Step 1 We multiply the equation $\mathcal{L}_2 v = 0$ by $\psi_2^n v_j e_j$ and integrate the result over (s_1, s_2). We obtain, after an integration by parts

$$\int_{s_1}^{s_2} \psi_2^{n-2}(\partial_s v_j)^2 + \lambda_j \int_{s_1}^{s_2} \psi_2^{n-2} v_j^2 - n(n-1) \int_{s_1}^{s_2} \psi_2^{-n} v_j^2 = 0 \tag{4.1}$$

Step 2 Recall that the function ψ_2 satisfies

$$(\partial_s \psi_2)^2 = \psi_2^2 - \psi_2^{4-2n} \tag{4.2}$$

Multiplication by $\psi_2^{n-4} v_j^2$ and integrating the result over (s_1, s_2) we get

$$\int_{s_1}^{s_2} \psi_2^{n-4} (\partial_s \psi_2)^2 v_j^2 - \int_{s_1}^{s_2} \psi_2^{n-2} v_j^2 + \int_{s_1}^{s_2} \psi_2^{-n} v_j^2 = 0 \tag{4.3}$$

Step 3 Differentiation of (4.2) yields

$$\partial_s (\psi_2^{n-3} \partial_s \psi_2) = (n-2) \psi_2^{n-2} + \psi_2^{-n}.$$

We multiply this equality by v_j^2 and integrate the result over (s_1, s_2). Using Cauchy-Schwarz inequality we get

$$(n-2) \int_{s_1}^{s_2} \psi_2^{n-2} v_j^2 + \int_{s_1}^{s_2} \psi_2^{-n} v_j^2$$
$$- 2 \left(\int_{s_1}^{s_2} \psi_2^{n-2} (\partial_s v_j)^2 \right)^{\frac{1}{2}} \left(\int_{s_1}^{s_2} \psi_2^{n-4} (\partial_s \psi_2)^2 v_j^2 \right)^{\frac{1}{2}} \leq 0.$$

The sum of the last inequality multiplied by $n-2$, (4.3) multiplied by $n^2 - 2n + 2$ and (4.1), yields

$$(\lambda_j - 2n + 2) \int \psi_2^{n-2} v_j^2 + 2n \int_{s_1}^{s_2} \psi_2^{n-4} (\partial_s \psi_2)^2 v_j^2$$
$$+ \left(\left(\int_{s_1}^{s_2} \psi_2^{n-2} (\partial_s v_j)^2 \right)^{\frac{1}{2}} - (n-2) \left(\int_{s_1}^{s_2} \psi_2^{n-4} (\partial_s \psi_2)^2 v_j^2 \right)^{\frac{1}{2}} \right)^2 \leq 0,$$

which readily implies that $v_j = 0$ since $\lambda_j \geq 2n$ when $j \geq n+1$.

In the case where s_1 or s_2 is not finite, we argue exactly as above, being understood that we have to justify all integrations. Assume for example that $s_2 = +\infty$. Arguing as in the proof of Proposition 4.2, we can prove that

$\psi_2^{\frac{n-2}{2}} v_j$ decays exponentially like $s \longrightarrow e^{-\delta_j s}$ at $+\infty$ and this is enough to justify all the integrations by parts. □

4.4. Maximum principle for the n-Delaunay hypersurface. We now prove a similar maximum principle for the Jacobi operator about a n-Delaunay hypersurface. The statement and the proof are very close to the statement and proof of the corresponding result for the n-catenoid. However we need to impose a lower bound on the Delaunay parameter for the result to hold.

Proposition 4.4. *There exists $\tau^* < 0$ such that for all $\tau \in [\tau^*, 0) \cup (0, \tau_*]$, if ω is a bounded solution of $L_\tau \omega = 0$ in $(s_1, s_2) \times S^{n-1}$, with boundary data $\omega = 0$ on $\{s_i\} \times S^{n-1}$ and if, for all $s \in (s_1, s_2)$, the function $\omega(s, \cdot)$ is L^2-orthogonal to Span$\{e_0, \dots, e_n\}$ on S^{n-1}, then $\omega = 0$.*

Proof: Again, if ω is a solution of $L_\tau \omega = 0$, then

$$v := \varphi^{\frac{2-n}{2}} \omega$$

solves $\mathcal{L}_\tau v = 0$. We consider the eigenfunction decomposition of v

$$v = \sum_{j \geq n+1} v_j \, e_j.$$

Step 1 Multiplying the equation $\mathcal{L}_\tau v = 0$ by $\varphi_\tau^n v_j e_j$ and integrating by parts over $(s_1, s_2) \times S^{n-1}$, we obtain the identity

$$\int \varphi_\tau^{n-2} (\partial_s v_j)^2 + \lambda_j \int \varphi_\tau^{n-2} v_j^2 = n \int \varphi_\tau^n v_j^2 + n(n-1)\, \tau^{2n} \int \varphi_\tau^{-n} v_j^2 \tag{4.4}$$

where all integrals are understood over (s_1, s_2).

Step 2 Recall that

$$(\partial_s \varphi_\tau)^2 = \varphi_\tau^2 - (\varphi_\tau^2 + \iota\, |\tau|^n \, \varphi_\tau^{2-n})^2, \tag{4.5}$$

where $\iota = 1$ if $\tau > 0$ and $\iota = -1$ if $\tau < 0$. Multiplying this identity by $\varphi_\tau^{n-4} v_j^2$ we get

$$\int \varphi_\tau^{n-4} (\partial_s \varphi_\tau)^2 \, v_j^2 = \int \varphi_\tau^{n-2} \, v_j^2 - \int \varphi_\tau^n \, v_j^2 - \tau^{2n} \int \varphi_\tau^{-n} \, v_j^2 - 2\, \iota\, |\tau|^n \int v_j^2. \tag{4.6}$$

Step 3 Differentiation of (4.5) yields

$$\partial_s\left(\varphi_\tau^{n-3}\,\partial_s\varphi_\tau\right) = (n-2)\,\varphi^{n-2} + (1-n)\,\varphi_\tau^n + \tau^{2n}\,\varphi_\tau^{-n} - (n-2)\,\iota\,|\tau|^n$$

We multiply this equality by v_j^2 and integrate the result over (s_1, s_2). Using Cauchy-Schwartz inequality, we get

$$(n-2)\int \varphi_\tau^{n-2}\,v_j^2 + (1-n)\int \varphi_\tau^n\,v_j^2 + \tau^{2n}\int \varphi_\tau^{-n}\,v_j^2 - (n-2)\,\iota\,|\tau|^n\int v_j^2$$
$$\leq 2\left(\int \varphi_\tau^{n-2}\,(\partial_s v_j)^2\right)^{\frac{1}{2}}\left(\int \varphi_\tau^{n-4}\,(\partial_s\varphi_\tau)^2\,v_j^2\right)^{\frac{1}{2}}.$$

The sum of the last equation multiplied by $n-2$, the former equation multiplied by n^2-2n+2 and (4.4), yields

$$(\lambda_j - 2n + 2)\int \varphi_\tau^{n-2}\,v_j^2 + n^2\,\iota\,|\tau|^n\int v_j^2 + 2n\int \varphi_\tau^{n-4}\,(\partial_s\varphi_\tau)^2\,v_j^2$$
$$+\left(\left(\int \varphi_\tau^{n-2}\,(\partial_s v_j)^2\right)^{\frac{1}{2}} - (n-2)\left(\int \varphi_\tau^{n-4}\,(\partial_s\varphi_\tau)^2\,v_j^2\right)^{\frac{1}{2}}\right)^2 \leq 0.$$

When $\tau > 0$, it is enough to use the fact that $\lambda_j \geq 2n$ for all $j \geq n+1$, to conclude that $v_j = 0$.

Step 4 In the case where $\tau < 0$, the last inequality together with the fact that $\lambda_j \geq 2n$, imply that

$$2\int \varphi_\tau^{n-2}\,v_j^2 \leq n^2\,|\tau|^n\int v_j^2.$$

If v_j were to be non zero, this would imply that

$$A := \inf \varphi_\tau \leq \left(\frac{n^2|\tau|^n}{2}\right)^{\frac{1}{n-2}}.$$

However, it follows from (4.5) that the minimum of φ_τ is the unique positive solution of the equation

$$|\tau|^n\,A^{2-n} = A + A^2$$

Therefore, this would imply that

$$\frac{2}{n^2} \leq \left(\frac{n^2|\tau|^n}{2}\right)^{\frac{1}{n-2}} + \left(\frac{n^2|\tau|^n}{2}\right)^{\frac{2}{n-2}},$$

which is clearly not true if $|\tau|$ is chosen small enough. This completes the proof of the result. □

Remark 4.1. In the forthcoming analysis, we agree, reducing $|\tau^*|$ if necessary, the constant $\tau^* < 0$ is chosen so that

$$2 \inf \varphi_\tau^{n-2} - n^2 \, |\tau|^n > 0$$

for all $\tau \in [\tau^*, 0)$.

A closer look at the proof of the previous result shows that we have also proven the :

Proposition 4.5. *Assume that $j \geq n+1$ and $\tau \in [\tau^*, 0) \cup (0, \tau_*]$ are fixed. Let ω be a solution of*

$$L_{\tau,j} \omega \geq 0$$

in (s_1, s_2), with

$$\omega(s_1) \leq 0 \quad \textit{and} \quad \omega(s_2) \leq 0$$

Then, $\omega \leq 0$ in (s_1, s_2).

Proof: We argue by contradiction and assume that the result is not true. There would exists $\bar{s}_1$ and $\bar{s}_2 \in (s_1, s_2)$ such that

$$L_{\tau,j} \, \omega \geq 0$$

and $\omega(s) > 0$ in $(\bar{s}_1, \bar{s}_2)$, and

$$\omega(\bar{s}_1) = \omega(\bar{s}_2) = 0.$$

We now apply the strategy of the proof of the previous result. The only difference being that the equality (4.4) now becomes the inequality

$$\int \varphi_\tau^{n-2} (\partial_s \omega)^2 + \lambda_j \int \varphi_\tau^{n-2} \omega^2 \leq n \int \varphi_\tau^n \omega^2 + n(n-1) \, \tau^{2n} \int \varphi_\tau^{-n} \omega^2$$

In any case, this is enough to conclude that $\omega \leq 0$ in (s_1, s_2). □

The next result mainly states that an exponentially decaying solution of the homogeneous problem $L_{\tau,j} \, \omega = 0$ does not change sign, provided $j \geq n+1$.

Proposition 4.6. *Assume that $\tau \in [\tau^*, 0) \cup (0, \tau_*]$ and $S \in \mathbb{R}$ are fixed. Let ω be a solution of $L_\tau \omega = 0$ in $(S, +\infty) \times S^{n-1}$ with $\omega = 0$ on $\{S\} \times S^{n-1}$. Further assume that ω decays exponentially at $+\infty$ and that, for each $s \geq S$, the function $\omega(s, \cdot)$ is L^2-orthogonal to Span$\{e_0, \ldots, e_n\}$ on S^{n-1}. Then $\omega = 0$.*

Proof: The proof of this result is identical to the proof of Proposition 4.4, though we now have to justify all the integrations by parts. But ω is assumed to decay exponentially at $+\infty$ and, by standard elliptic estimates, this is also the case for $\partial_s\omega$ and this is enough to justify all the integrations by part. □

As a by product of the proof of Proposition 4.4 Step 3, we have the

Lemma 4.1. *If v is defined in (s_1, s_2) and is a solution of*

$$\mathcal{L}_\tau(v\, e_j) = 0$$

in $(s_1, s_2) \times S^{n-1}$, then

$$(\lambda_j - 2n + 2) \int_{s_1}^{s_2} \varphi_\tau^{n-2}\, v^2 + n^2\, \iota\, |\tau|^n \int_{s_1}^{s_2} v^2 \leq h(s_2) - h(s_1),$$

where

$$h := \varphi_\tau^{n-2}\, v\, \partial_s v + (n-2)\, \varphi_\tau^{n-3} v^2\, \partial_s \varphi_\tau,$$

and where $\iota = 1$ when $\tau > 0$ and $\iota = -1$ when $\tau < 0$.

We end this section with the :

Proposition 4.7. *Assume that $\tau \in [\tau^*, 0) \cup (0, \tau_*]$ and that ω is a bounded solution of $L_\tau\, \omega = 0$ in $\mathbb{R} \times S^{n-1}$. Further assume that, for all $s \in \mathbb{R}$, the function $\omega(s, \cdot)$ is L^2-orthogonal to Span$\{e_0, \ldots, e_n\}$ on S^{n-1}. Then $\omega = 0$.*

Proof: As usual, we consider the eigenfunction decomposition of $v := \varphi_\tau^{\frac{2-n}{2}}\ \omega$ as

$$v(s, \theta) = \sum_{j \geq n+1} v_j(s)\, e_j(\theta),$$

Let us first show that

$$\int_{\mathbb{R}} v_j^2(s)\, ds < +\infty$$

for $j \geq n+1$. Indeed, we apply the result of Lemma 4.1 to get

$$(\lambda_j - 2n + 2) \int_{-S}^{S} \varphi_\tau^{n-2}\, v_j^2 + n^2\, \iota\, |\tau|^n \int_{-S}^{S} v_j^2 \leq h(S) - h(-S),$$

where

$$h := \varphi_\tau^{n-2}\, v_j\, \partial_s v_j + (n-2)\, \varphi_\tau^{n-3} v_j^2\, \partial_s \varphi_\tau.$$

Thanks to the choice of τ and j, we have

$$(\lambda_j - 2n + 2)\, \inf \varphi_\tau^{n-2} + n^2\, \iota\, |\tau|^n > 0$$

Hence we conclude that

$$\int_{-S}^{S} v_j^2 \leq c\,(h(S) - h(-S)).$$

for some constant c which only depends on τ and j.

Since v_j is bounded, we also get that $\partial_s v_j$ is a bounded function. Therefore, the right hand side of the last inequality is bounded independently of S. Passing the limit as $S \to +\infty$, we conclude that $v_j \in L^2(\mathbb{R})$. In particular, there exists a sequences of real numbers $(\alpha_i)_i$ (resp.$(\beta_i)_i$) which tends to $-\infty$ (resp. $+\infty$) and for which

$$\lim_{i \to +\infty} v_j(\alpha_i) = \lim_{i \to +\infty} v_j(\beta_i) = 0$$

Again, we use the result of Lemma 4.1 to get

$$\int_{\alpha_i}^{\beta_i} v_j^2 \leq c\,(h(\beta_i) - h(\alpha_i)),$$

and we pass to the limit as $i \to +\infty$ to conclude that $v_j = 0$. This completes the proof of the result. □

Observe that the explicit knowledge of some Jacobi fields yields some information about the indicial roots of the operator L_τ. Indeed, since the Jacobi fields $\Phi_\tau^{j,\pm}$, described in subsection 3.2, are at most linearly growing, the associated indicial roots are all equal to 0. Hence we conclude that

$$\gamma_j(\tau) = 0,$$

for $j = 0, \ldots, n$ and for all $\tau \in (-\infty, 0) \cup (0, \tau_*]$. The situation is completely different when $j \geq n+1$ since if $\tau \in [\tau^*, 0) \cup (0, \tau_*]$ then,

$$\gamma_j(\tau) > 0.$$

To prove these, we argue by contradiction and assume that $\gamma_j(\tau) = 0$ for some $j \geq n+1$ and some $\tau \in [\tau^*, 0) \cup (0, \tau_*]$. The discussion of section 3 shows that the homogeneous problem $L_{\tau,j}\, \omega = 0$ admits at least one nontrivial periodic solution. However, since we have assumed that $j \geq n+1$, this would contradict the result of Proposition 4.7. These prove the two first part of the Theorem.

5. The limit of the indicial roots as τ tends to 0

We show that, as τ tends to 0, the indicial roots $\gamma_j(\tau)$ remain bounded from below by any constant less than δ_j. This result relies on a precise estimate of the potential which appears in the expression of L_τ.

Lemma 5.1. *Given $\eta > 0$, there exists $\tau_0 > 0$ and $s_0 > 0$ such that for all $\tau \in (-\tau_0, 0) \cup (0, \tau_0)$, we have*

$$\frac{n(n+2)}{4}\varphi_\tau^2 + \frac{n(3n-2)}{4}\tau^{2n}\varphi_\tau^{2-2n} \leq \eta$$

in $[s_0, \frac{s_\tau}{2} - s_0] \cup [\frac{s_\tau}{2} + s_0, s_\tau - s_0]$.

Proof: It follows from the result of Lemma 2.2 that, as τ tends to 0 :

- the sequence of functions $\tau^{2n}\,\varphi_\tau^{2-2n}$ converges, uniformly on compact sets, to the function $s \longrightarrow \frac{1}{\cosh(n-1)s}$,
- the sequence of functions φ_τ converges, uniformly on compact sets, to 0,
- the sequence of functions $\tau^{2n}\,\varphi_\tau^{2-2n}(\cdot + \frac{s_\tau}{2})$ converges, uniformly on compact sets, to 0,
- the sequence of functions $\varphi_\tau(\cdot + \frac{s_\tau}{2})$ converges, uniformly on compact sets, to the function $s \longrightarrow \frac{1}{\cosh s}$.

Given $\eta > 0$, we fix $s_0 > 0$ so that

$$\frac{n(n+2)}{4}\frac{1}{\cosh s_0} \leq \eta/4 \quad \text{and} \quad \frac{n(3n-2)}{4}\frac{1}{\cosh((n-1)\,s_0)} \leq \eta/4.$$

Next, we use the converge properties stated above, to fix $\tau_0 > 0$ such that

$$\frac{n(n+2)}{4}\,\varphi_\tau^2\left(\frac{s_\tau}{2} - s_0\right) \leq \frac{\eta}{2} \quad \text{and} \quad \frac{n(3n-2)}{4}\,\tau^{2n}\,\varphi_\tau^{2-2n}(s_0) \leq \frac{\eta}{2}$$

for all $\tau \in (-\tau_0, 0) \cup (0, \tau_0)$. Recall that the function $s \to \sigma_\tau(s)$ is monotone increasing in $[0, \frac{s_\tau}{2}]$. Hence, we conclude that

$$\frac{n(n+2)}{4}\varphi^2(s) \leq \eta/2 \qquad \text{for all} \quad s \in [0, \frac{s_\tau}{2} - s_0]$$

and

$$\frac{n(3n-2)}{4}\tau^{2n}\varphi^{2-2n}(s) \leq \eta/2 \qquad \text{for all} \quad s \in [s_0, \frac{s_\tau}{2}].$$

This yields the estimate in the set $[s_0, \frac{s_\tau}{2} - s_0]$. Finally, a similar analysis gives the estimate in $[\frac{s_\tau}{2} + s_0, s_\tau - s_0]$. □

We now prove that, provided $j \geq n+1$, the indicial roots $\gamma_j(\tau)$ remain bounded from below by any constant less than δ_j as τ tends to 0.

Proof: The fact that $\gamma_j(\tau) > 0$ for $j \geq n+1$ and $\tau \in [\tau^*, 0) \cup (0, \tau_*]$already ensure that there exists a function v solution of

$$\mathcal{L}_\tau (v\, e_j) = 0,$$

which decays exponentially at $+\infty$. Applying the result of Proposition 4.6, we see that v does not vanish, hence we may well assumed that v is normalized so that

$$v(0) = 1.$$

Step 1 Using the result of Lemma 4.1, we see that

$$(\lambda_j - 2n + 2) \int_s^{+\infty} \varphi_\tau^{n-2}\, v^2 + n^2\, \iota\, |\tau|^n \int_s^{+\infty} v^2 \leq -h(s),$$

for all s, where we recall that

$$h := \varphi_\tau^{n-2}\, v\, \partial_s v + (n-2)\, \varphi_\tau^{n-3}\, v^2\, \partial_s \varphi_\tau.$$

Hence

$$h(s) < 0.$$

And this implies that

$$\partial_s \left(\varphi_\tau^{n-2}\, v \right) < 0.$$

In particular, using the fact that the function σ_τ and hence the function φ_τ are increasing on $[0, s_\tau/2]$, we conclude that the function

$$\omega := \varphi_\tau^{\frac{n-2}{2}}\, v$$

is decreasing in $[0, s_\tau/2]$. We even have

$$\partial_s \omega < 0$$

in this set. Recall that the function ω is a solution of $L_{\tau,j}\, \omega = 0$.

Step 2 Assume that $\eta > 0$ is fixed. We fix s_0 and τ_0 as in Lemma 5.1, so that

$$\frac{n(n+2)}{4} \varphi_\tau^2 + \frac{n(3n-2)}{4} \tau^{2n} \varphi_\tau^{2-2n} \leq \eta/4$$

on $[s_0, \frac{s_\tau}{2} - s_0]$, for all $\tau \in (-\tau_0, 0) \cup (0, \tau_0)$. To simplify the notations, we set

$$S_0 := \frac{s_\tau}{2} - s_0 \qquad \text{and} \qquad \beta_j := \sqrt{\delta_j^2 - \eta/4}.$$

We denote

$$Q_{j,\tau} := \delta_j^2 - \frac{n(n+2)}{4}\varphi_\tau^2 - \frac{n(3n-2)}{4}\tau^{2n}\varphi_\tau^{2-2n},$$

By assumption,

$$Q_{j,\tau} > \beta_j^2$$

over $[s_0, S_0]$. We choose $a, b \in \mathbb{R}$ such that the function

$$\bar{\omega} := a\, e^{\beta_j s} + b\, e^{-\beta_j s}$$

agrees with ω at $s = s_0$ and $s = S_0$. Namely

$$\bar{\omega}(S_0) = \omega(S_0) \qquad \text{and} \qquad \bar{\omega}(s_0) = \omega(s_0).$$

We find explicitly

$$a = \frac{\omega(S_0)e^{\beta_j S_0} - \omega(s_0)e^{\beta_j s_0}}{e^{2\beta_j S_0} - e^{2\beta_j s_0}} \quad \text{and} \quad b = \frac{\omega(s_0)e^{-\beta_j s_0} - \omega(S_0)e^{-\beta_j S_0}}{e^{-2\beta_j s_0} - e^{-2\beta_j S_0}}$$

Since $Q_{j,\tau} > \beta_j^2$ over $[s_0, S_0]$, we see that

$$\partial_s^2(\omega - \bar{\omega}) - \beta_j^2\,(\omega - \bar{\omega}) \leq 0$$

in $[s_0, S_0]$ and $(\omega - \bar{\omega})(s_0) = (\omega - \bar{\omega})(S_0) = 0$. The maximum principle then implies that $\omega \leq \bar{\omega}$ on $[s_0, S_0]$.

We claim that

$$\omega(S_0) \leq 2\omega(s_0)e^{-\beta_j(S_0 - s_0)}.$$

Indeed, we have shown that the function ω is strictly decreasing over $[0, s_\tau/2]$, hence $\omega(S_0) < \omega(s_0)$ and this implies that $b > 0$.

Without loss of generality, we may as well assume that

$$\omega(S_0) > \omega(s_0)\, e^{-\beta_j(S_0 - s_0)}.$$

Otherwise there is nothing to prove. Under this assumption, we have $a > 0$ and also $b \in (0, \omega(s_0)e^{\beta_j s_0})$. Still using the fact that the function ω is strictly decreasing on $[0, s_\tau/2]$, we find that

$$\omega(S_0) = \inf_{s \in [s_0, S_0]} \omega \leq \inf_{s \in [s_0, S_0]} \bar{\omega}$$

But, a and b being positive, the infinium of $\bar{\omega}$ over $\mathbb{R}$ is achieved at the point $s_m > s_0$ which satisfies $e^{2\beta_j s_m} = b/a$ (observe that $s_m > s_0$ since $\bar{\omega}(S_0) < \bar{\omega}(s_0)$). First we rule out the case $s_m < S_0$. Indeed, if this were the case then we would have $\omega(S_0) < \bar{\omega}(s_m) = 2\sqrt{ab}$. Introducing in this inequality the expression for both a and b we find that

$$(\omega(S_0)\cosh(\beta_j(S_0 - s_0)) - \omega(s_0))^2 < 0,$$

which is not possible. Therefore we always have $s_m \geq S_0$ and this implies that

$$a = b\, e^{-2\beta_j s_m} \leq b\, e^{-2\beta_j S_0}.$$

In particular we obtain

$$\omega(S_0) \leq \bar{\omega}(S_0) = a\, e^{\beta_j S_0} + b\, e^{-\beta_j S_0} \leq 2\omega(s_0)\, e^{-\beta_j(S_0 - s_0)}.$$

This completes the proof of the claim.

The function ω being decreasing on $[0, s_\tau/2]$, we conclude that

$$\omega(s_\tau/2) \leq 2\,\omega(s_0)\, e^{-\beta_j(S_0 - s_0)} \leq 2\,\omega(0)\, e^{-\beta_j(S_0 - s_0)}.$$

Since, s_τ tends to $+\infty$ as τ tends to 0, reducing $|\tau|$ if this is necessary, we can assume that

$$2\, e^{-\beta_j\,(S_0 - s_0)} \leq e^{-\tilde{\beta}_j \frac{s_\tau}{2}}$$

where

$$\tilde{\beta}_j = \sqrt{\delta_j^2 - \eta/2}.$$

Hence, we conclude that

$$\omega(s_\tau/2) \leq \omega(0)\, e^{-\tilde{\beta}_j \frac{s_\tau}{2}}. \tag{5.1}$$

provided τ is chosen small enough.

Step 3 We claim that, provided τ is chosen small enough, the function ω is decreasing in $[s_\tau/2 + s_0, s_\tau - \bar{s}_0]$, for some suitable choice of $\bar{s}_0 \geq s_0$. Recall that ω and $Q_{j,\tau}$ are strictly positive in $[s_\tau/2 + s_0, s_\tau - s_0]$. Hence

$$\partial_s^2\, \omega > 0$$

i.e. ω is strictly convex in this set. Observe that, for all $s \in \mathbb{R}$

$$\omega(s + s_\tau) = \omega(s_\tau)\, \omega(s)$$

since the difference between these two functions vanishes at $s = 0$ and is exponentially decaying, hence this difference is identically equal to 0 by Proposition 4.6. This implies that ω is strictly decreasing over $[s_\tau, 3s_\tau/2]$.

Case 1 We now rule out the case where, for some sequence of τ tending to 0, the function ω is strictly increasing in $[s_\tau/2 + s_0, s_\tau - s_0]$. Assume that this is the case. Then the function ω is increasing over $[s_\tau/2 + s_0, s_\tau - s_0]$ and is decreasing on $[s_\tau, 3s_\tau/2]$. Hence the maximum of ω on $[s_\tau/2 + s_0, 3s_\tau/2]$ is achieved at a point $s_\tau^* \in [s_\tau - s_0, s_\tau]$. We define

$$\hat{\omega}(s) := \frac{\omega(s + s_\tau^*)}{\omega(s_\tau^*)}.$$

The sequence of functions $\hat{\omega}$ is bounded on compact sets and solves an homogeneous ordinary differential equation. Hence the derivative of $\hat{\omega}$ is also bounded on compacts and by Ascoli's theorem, we can extract a subsequence which converges, as τ tends to 0, to ω_*, a nontrivial bounded solution of

$$L_1 (\omega_* \, e_j) = 0,$$

in $\mathbb{R}$. However, the result of Proposition 4.2 shows that $\omega_* \equiv 0$, which is a contradiction.

Case 2 It remains to consider the case where, for some sequence of τ tending to 0, the function ω has a local minimum in $[s_\tau/2 + s_0, s_\tau - s_0]$, say at a point s_τ^*. If $s_\tau - s_\tau^*$ tends to $+\infty$, as τ tends to 0, then, the function ω being convex on $[s_\tau/2 + s_0, s_\tau - s_0]$, this implies that ω is increasing over $[s_\tau^*, s_\tau - s_0]$ and we can argue as in Step 1 to rule out this possibility.
Therefore, the sequence $s_\tau - s_\tau^*$ tends to some positive constant, taking $\bar{s}_0$ to be the supremum of all such constants, we have shown that ω is decreasing in $[s_\tau/2 + s_0, s_\tau - \bar{s}_0]$. Which completes the proof of the claim.

Arguing as in the last Step 2, we get

$$\omega(s_\tau - s_0) \leq \omega(s_\tau/2 + s_0) \, e^{-\tilde{\beta}_j(\frac{s_\tau}{2} + s_0)}. \tag{5.2}$$

It is an easy exercise to show, using the positivity of ω, that there exists a constant $c > 0$ independent of τ such that

(5.3) $$\omega(s_\tau/2 + s_0) \leq c\,\omega(s_\tau/2) \quad \text{and} \quad \omega(s_\tau) \leq c\,\omega(s_\tau - s_0).$$

Collecting (5.2), (5.3) and reducing the range of τ if this is necessary, we conclude that

$$\omega(s_\tau) \leq \omega(0)\, e^{-\hat{\beta}_j s_\tau},$$

where we have set $\hat{\beta}_j = \sqrt{\delta_j^2 - \eta}$. This inequality implies that

$$\gamma_j(\tau) \geq \hat{\beta}_j$$

for all τ small enough. □

As a by product of the previous result, we get the :

Corollary 5.1. *Assume that $j \geq n+1$ and $\tau \in (\tau^*, 0) \cup (0, \tau_*)$. There exists a unique solution of*

$$L_{\tau,j}\omega = 0$$

in $(0, +\infty)$ with $\omega(0) = 1$, which satisfies

(5.4) $$\frac{1}{c} e^{-\gamma_j s} \leq \omega(s) \leq c\, e^{-\gamma_j s}$$

for some constant $c > 1$ depending on τ.

We end this chapter by proving that the indicial roots are increasing as a function of λ_j.

Corollary 5.2. *There exists $\tau_0 > 0$ such that, for all $\tau \in (-\tau_0, 0) \cup (0, \tau_0)$, we have*

$$\gamma_{j+1}(\tau) > \gamma_j(\tau).$$

if $\lambda_{j+1} > \lambda_j$ and $j \geq n+1$.

Proof: Assume that $j \geq n+1$ and that ω_j is the exponentially decreasing function defined in Corollary 5.1 for $L_{\tau,j}$. Given $\delta \in \mathbb{R}$, we compute

$$L_{\tau,j+1}\, \omega_j^{1+\delta} = (\lambda_j - \lambda_{j+1})\, \omega_j^{1+\delta} + \delta\,(1+\delta)\,(\partial_s \omega_j)^2\, \omega_j^{\delta-1} + \delta\, \omega_j^{\delta}\, \partial_s^2 \omega_j.$$

The function ω_j being positive, and a solution of $L_{\tau,j}\,\omega_j = 0$, there exists a constant $c > 0$ (independent of $|\tau|$ small enough) such that

$$|\partial_s^2 \omega_j| + |\partial_s \omega_j| \leq c\, \omega_j.$$

in $(0, +\infty)$. Choosing $\delta > 0$ small enough, we conclude that

$$L_{\tau,j+1}\,\omega_j^{1+\delta} < 0$$

in $(0, +\infty)$.

In particular, the function $\omega_j^{1+\delta}$ is a supersolution and this, together with the result of Proposition 4.5, implies that ω_{j+1} the exponentially decaying function defined in Corollary 5.1 for $L_{\tau,j+1}$, satisfies

$$\omega_{j+1} \leq \omega_j^{1+\delta}.$$

Hence

$$\gamma_{j+1}(\tau) \geq (1+\delta)\,\gamma_j(\tau),$$

and the proof of the result is complete. □

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Department of Mathematics, College of Science, King Saud University.

P.O. Box 2455, Riyadh 11451, Saudia Arabia.

E-mail: jleli@ksu.edu.sa

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