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ON WEAK SOLUTIONS OF A SYSTEM OF ONE-DIMENSIONAL NONLINEAR THERMOELASTICITY

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ABSTRACT. In [7], we considered a model of nonlinear thermoelasticity and showed that classical solutions blow up in finite time. In this paper we study the same model, with a linear elastic response function, and establish a local existence result of weak solutions.

INTRODUCTION

In the absence of external forces and heat supply the evolution equations, govérning the motion of a homogeneous one-dimensional thermoelastic body occupying an interval I in a (fixed) reference configuration, take the form

$$
\begin{aligned}
u_{tt}(x,t) &= \sigma_x(x,t) \\
e_t(x,t) &= \sigma(x,t)u_{xt}(x,t) - q_x(x,t), \quad x \in I, \ t \geq 0,
\end{aligned}
\tag{1}
$$

where u is the displacement, σ is the stress, e is the internal energy, q is the heat flux, a subscript denotes a partial derivative with respect to the relevant variable, and a dot denotes a time derivative.

A thermoelastic material is characterized by constitutive relations of the form

$$
\begin{aligned}
\sigma(x,t) &= \hat{\sigma}(u_x(x,t), T(x,t)) \\
e(x,t) &= \hat{e}(u_x(x,t), T(x,t)) \\
q(x,t) &= \hat{q}(u_x(x,t), T(x,t), T_x(x,t)),
\end{aligned}
\tag{2}
$$

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where T, the absolute temperature, and the deformation gradient satisfy

$$u_x > -1,\ T > 0.$$

The second law of thermodynamics is used to obtain restrictions on the functions $\hat{\sigma}, \hat{e}, \hat{q}$. In particular we have the compatibility relation

$$\hat{e}_\varepsilon(\varepsilon,T) = \hat{\sigma}(\varepsilon,T) - T\,\sigma_T(\varepsilon,T),\ \forall\ \varepsilon > -1,\ \ T > 0 \tag{1}$$

and the heat conduction inequality

$$g \cdot \hat{q}(\varepsilon,T,g) \leq 0, \quad \forall \varepsilon\ T > 0,\ \ g\ \in\ \mathbb{R}. \tag{2}$$

The idea of using the second law of thermodynamics to restrict the constitutive relations is due to Coleman and Noll [1] then to Coleman and Mizel [2]. (see the book by Day [5] for more information).

Global existence and formation of singularities in classical, as well as, in weak solutions for nonlinear problems have attracted a broad interest of researchers. Slemrod [13], in 1981, established global existence and decay of solutions to certain initial-boundary value problems. His proof made a crucial use of Poincare's inequality; which makes his result unextendable to problems with unbounded intervals. To overcome this difficulty, Hrusa and Tarabek [6], in 1988, combined some of Slemrod's estimates, that remain valid for unbounded intervals, with additional ones that exploit relations associated with the second law of thermodynamics and established a global existence and a decay result to the Cauchy problem. Zheng and Shen [15], [16] discussed a class of a quasilinear hyperbolic-parabolic systems, of which the system of thermoelasticity is a special case. They used Fourier analysis together with some weighted estimates to establish global existence and to obtain a precise decay of the solution of the Cauchy problem.

In the case of large initial data, one should not expect global existence. In fact, the work by Coleman and Gurtin [3] on the growth and decay of acceleration waves in one-dimensional nonlinear thermoelasticity provides a strong indication that the damping effect of the heat diffusion restrains waves with small amplitude, however the nonlinear elastic response destabilizes waves with large amplitudes. For specialized constitutive relations, Defermos and Hsiao

[4] showed that classical solutions to the Cauchy problem blow up in finite time if the initial data are taken small enough in L^∞ norm with large enough gradients. As they pointed out, their relations are not compatible with thermodynamics. Also these relations are not fully coupled and they don't satisfy the global existence conditions. In 1989, Hrusa and Messaoudi [7] studied a special class of nonlinear thermoelastic materials and considered equations similar to those studied in [4] and proved a blow up result to the Cauchy problem. Their relations are fully coupled and totally compatible with thermdynamics; in particular they satisfy the assumptions of the global existence established by Hrusa and Tarabek [6] and Jiang [8]. Similar to [6], Hrusa and Messaoudi also exploited a relation associated with the second Law of thermodynamics to prove their blow up result. Furthermore in the one-dimensional case, Tarabek [14] considered a system of nonlinear thermoelasticity, where the heat conduction is given by Cattaneo's Law, and established a global existence theorem, for small and smooth initial data, to the Cauchy problem. His proof is based on the use of the classical energy method. In the other hand, Saouli [12] used the nonlinear semigroup theory to prove a local existence result to a system close to the one considered in [14]. Concerning weak solutions, Messaoudi [11] studied a semilinear system and showed that the solution blows up in finite time if the initial data satisfy certain assumptions. This result has been improved latey by Kirane and Tatar[9].

In this article, we are concerned with the system (1.1) together with the constitutive relations[1] of the form

$$\begin{aligned} \hat{\sigma}(u_x, T) &= au_x + d + \beta(T - \overline{T}) \\ \hat{e}(u_x, T) &= \frac{1}{2} au_x^2 + du_x + c(T - \overline{T}) - \beta\overline{T}u_x \\ \hat{q}(u_x, T, T_x) &= -\kappa T_x, \end{aligned} \tag{3}$$

$a, d, c, \beta, \kappa, \overline{T}$ are constant satisfying

$$a, c, \kappa, \overline{T} > 0. \tag{4}$$

By introducing the difference temperature

$$\theta := T - \overline{T},$$

[1]These relations are fully compatible with the second law of thermodynamics.

the system (1.1) takes the form

$$(5)\qquad u_{tt}(x,t) = au_{xx}(x,t) + \beta\theta_x(x,t)$$

$$(6)\qquad c\theta_t(x,t) = \kappa\theta_{xx}(x,t) + \beta\overline{T}u_{xt}(x,t) + \beta\theta(x,t)u_{xt}(x,t)$$

$$x \in I = (0,1),\ t \in [0,t_0),\ t_0 > 0.$$

We associate with the system (1.7), (1.8) the initial data and the boundary conditions

$$(7)\qquad u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x),\ x \in I$$

$$(8)\qquad u(0,t) = u(1,t) = 0,\ \theta(0,t) = \theta(1,t) = 0,\quad t \in [0,t_0).$$

In order to state our result, we shall assume that

$$(9)\qquad u_0, \theta_0 \in X_2,\ u_1 \in X_1,$$

where

$$(10)\qquad X_2 := H^2(I) \cap H_0^1(I),\ X_1 := H_0^1(I),\ X_0 := L^2(I).$$

Theorem 1. *Assume that $a, \kappa, c, \overline{T}$ satisfy (1.6) and let u_0, u_1, θ_0 be given and satisfying (1.11). Then the initial-boundary value problem (1.7) - (1.10) has a unique local 'weak' solution (u,θ) defined on a maximum time interval $[0,t'')$, $t'' \le t_0$,with*

$$u \in \bigcap_{\alpha=0}^{2} W^{\alpha,+\infty}([0,t''); \quad X_{2-\alpha})$$

$$(11)\qquad \theta \in L^\infty([0,t''); \quad X_2)$$

$$\dot\theta \in L^\infty([0,t''); \quad X_0) \cap L^2([0,t''); \quad X_1).$$

To prove our result, we need to solve a related linear problem, get estimates on the solution, and then use these estimates to solve the nonlinear problem.

2. LINEAR PROBLEM

We consider the following linear problem

$$
\begin{aligned}
(1)\qquad u_{tt}(x,t) &= au_{xx}(x,t) + \beta\theta_x(x,t), \quad x \in I,\ t \in [0,t_0) \\
(2)\qquad c\theta_t(x,t) &= \kappa\theta_{xx}(x,t) + \beta\overline{T}u_{xt}(x,t) + b(x,t)u_{xt}(x,t) \\
(3)\qquad u(x,0) &= u_0(x),\ u_t(x,0) = u_1(x),\ \theta(x,0) = \theta_0(x) \\
(4)\qquad u(0,t) &= u(1,t) = 0,\ \theta(0,t) = \theta(1,t) = 0,
\end{aligned}
$$

where $a, c, \beta, \kappa, \overline{T}$ are constants satisfying (1.6) and b such that

$$
\begin{aligned}
& b \in L^\infty([0,t_0);\quad H^2(I)) \\
(5)\qquad & \dot{b} \in L^\infty([0,t_0);\quad L^2(I)) \cap L^2([0,t_0);\quad H^1(I)).
\end{aligned}
$$

In addition to existence, uniqueness, and regularity, we need to establish an estimate on the solution. For this purpose we define

$$
\begin{aligned}
(6)\qquad \varepsilon(t) :=& \sum_{\alpha=0}^{2} || \overset{(\alpha)}{u} (.,t)||_{2-\alpha}^2 + ||\theta(.,t)||_2^2 + ||\dot{\theta}(.,t)||_0^2,\ t \in [0,t_0) \\
\varepsilon_0 :=& \sum_{\alpha=0}^{2} ||u_\alpha||_{2-\alpha}^2 + ||\theta_0||_2^2 + ||\theta_1||_0^2,
\end{aligned}
$$

where

$$
\begin{aligned}
(7)\qquad u_2(x) :=& \ au_0''(x) + \beta\theta_0'(x) \\
\theta_1(x) :=& \ \frac{\kappa}{c}\theta_0''(x) + \frac{\beta\overline{T} + b(x,0)}{c}u_1'(x), \qquad x \in I
\end{aligned}
$$

Remark 2.1 With such definitions, it is easy to verify that $\varepsilon_0 = \varepsilon(0)$.

We also set

$$
\begin{aligned}
(8)\qquad B\ :&= \max_{0\le t\le t_0} ||b(\cdot,t)||_2^2 + \max_{0\le t\le t_0} ||\dot{b}(\cdot,t)||_0^2 \\
&+ \int_0^{t_0} ||\dot{b}(\cdot,t)||_1^2 dt.
\end{aligned}
$$

Theorem 2. *Assume that (1.6), (2.5) hold and let* u_0, u_1, θ_0 *be given and satisfying (1.11) then the initial-boundary value problem (2.1) - (2.4) has a unique 'weak' solution* (u, θ) *satisfying (1.13). Moreover we have the estimate*

$$(9) \qquad \varepsilon(t) + \int_0^t \int_0^1 |\theta_{xt}(x,s)|^2 dx\, ds \le \Gamma\varepsilon_0 + k\int_0^t \varepsilon(s)ds, \qquad t \in [0, t_0),$$

where Γ *is a generic positive constant depending only and continuously on* $a, c, \kappa, \overline{T}$ *and* k *is a constant depending solely and continuously on* $a, c, \kappa, \overline{T}, B$.

To prove this theorem, we begin by establishing a lemma that gives an existence result under a stronger smoothness assumption on b. Precisely, we assume that

$$(10) \qquad b \in C^1([0, t_0); \quad H^2(I))$$

Lemma 2.1 *Assume that (1.6), (2.10) hold and let* u_0, u_1, θ_0 *be as in theorem 2. Then the initial-boundary value problem (2.1) - (2.4) has a unique solution* (u, θ) *satisfying*

$$u \in \bigcap_{\alpha=0}^{2} W^{\alpha,+\infty}([0,t_0); \quad X_{2-\alpha})$$

$$(11) \qquad \theta \in L^\infty([0,t_0); \quad X_2)$$

$$\dot\theta \in L^\infty([0,t_0), \quad L^2(I)) \cap L^2([0,t_0); \quad X_1).$$

Proof. We take a formal partial derivative with respect to t, of the equations (2.1), (2.2), to get

$$(12)\ \ddot u\,(x,t) = a\dot u_{xx}(x,t) + \beta\dot\theta_x(x,t)$$

$$(13)\ c\ddot\theta(x,t) = \kappa\dot\theta_{xx}(x,t) + \beta\overline{T}\ddot u_x(x,t) + \dot b(x,t)\dot u_x(x,t) + b(x,t)\ddot u_x(x,t)$$

and proceed to prove the existence of a 'weak' solution (u, θ) by applying Galerkin, method to the equations (2.12), (2.13) together with the boundary conditions (2.3) and the initial conditions

$$(14) \quad \overset{(\alpha)}{u}\,(x,0) = u_\alpha(x), \alpha = 0,1,2; \ \overset{(\beta)}{\theta}\,(x,0) = \theta_\beta(x); \ \beta = 0,1. \ x \in I,$$

where u_2 and θ_1 are defined in (2.7).

Let $\{v_m\}_{m=1}^{\infty}$ be a basis of H_0^1 and V_m be the finite dimensional subspace spaned by $\{v_1, v_2, \dots v_m\}$. It is clear that $V_m \subset V_{m+1}$, $\forall m = 1, 2, \dots$ We approximate the solution (u, θ) by functions lying in these subspaces and having the forms:

$$u^m(x,t) = \sum_{i=1}^{m} \xi_i(t) v_i, \quad \theta^m(x,t) = \sum_{j=1}^{m} \eta_j v_j \tag{15}$$

which satisfy

$$\langle \overset{(3)}{u^m}(.,t), v_k\rangle = a\langle \dot{u}^m_{xx}(.,t), v_k\rangle + \beta < \dot{\theta}^m_x(.,t), v_k\rangle \tag{16}$$

$$\begin{aligned} c\langle \overset{(2)}{\theta^m}(.,t), v_k\rangle &= \kappa\langle \dot{\theta}^m_{xx}(.,t), v_k\rangle + \beta\overline{T}\langle \ddot{u}^m_x(.,t), v_k\rangle \\ &\quad + \langle \dot{b}(.,t)\dot{u}_x(.,t), v_k\rangle + \langle b(.,t)\ddot{u}_x(.,t), v_k\rangle \end{aligned} \tag{17}$$

$$\begin{aligned} \langle \overset{(\alpha)}{u^m}(.,0), v_k\rangle &= \langle u_\alpha, v_k\rangle, \ \alpha = 0, 1, 2 \\ \langle \overset{(\beta)}{\theta^m}(.,0), v_k\rangle &= \langle \theta_\beta, v_k\rangle, \ \beta = 0, 1, \ k = 1, 2, \dots m \end{aligned} \tag{18}$$

where $\langle\ ,\ \rangle : H^{-1}(\Omega) \times H_0^1(\Omega) \longrightarrow \mathbb{R}$ is the duality parring. The standard theory of linear ordinary differential equations guarantees the existence of ξ_i, η_j such that (u^m, θ^m) given by (2.15) satisfies (2.16) - (2.18) for each $m = 1, 2, \dots$

We then substitute v_k by $\overset{(2)}{u^m}$ in (2.16) and by $\dot{\theta}^m$ in (2.17), integrate over $[0, t)$, using integration by parts, and add equalities to obtain

$$\begin{aligned} &\frac{1}{2}||\ddot{u}^m(\cdot t)||_0^2 + \frac{a}{2}||\dot{u}^m_x(\cdot,t)||_0^2 + \frac{c}{2\overline{T}}||\dot{\theta}^m(\cdot,t)||_0^2 + \frac{\kappa}{\overline{T}}\int_0^t\int_0^1 \left|\frac{\partial\dot{\theta}^m}{\partial x}(x,\tau)\right|^2 dx\, d\tau \\ &= \frac{1}{2}||\ddot{u}^m(\cdot,0)||_0^2 + \frac{a}{2}||\dot{u}^m_x(\cdot,0)||_0^2 + \frac{c}{2\overline{T}}||\dot{\theta}^m(\cdot,0)||_0^2 \\ &\quad + \frac{1}{\overline{T}}\int_0^t\int_0^1 \dot{b}(x,\tau)\dot{u}^m_x(x,\tau)\dot{\theta}^m(x,\tau) dx\, d\tau \\ &\quad - \frac{1}{\overline{T}}\int_0^t\int_0^1 b_x(x,\tau)\ddot{u}^m(x,\tau)\dot{\theta}^m(x,\tau) dx\, d\tau \\ &\quad - \frac{1}{\overline{T}}\int_0^t\int_0^1 b(x,\tau)\ddot{u}^m(x,\tau)\dot{\theta}^m_x(x,\tau) dx\, d\tau. \end{aligned} \tag{19}$$

By exploiting (2.10), young's inequality, Gronwall's inequality, and the relations

$$(20) \qquad \begin{aligned} u^m(x,t) &= u^m(x,0) + \int_0^t \dot{u}^m(x,\tau)d\tau \\ \theta^m(x,t) &= \theta^m(x,0) + \int_0^t \dot{\theta}^m(x,\tau)d\tau, \end{aligned}$$

we conclude that (u^m) is bounded in $W^{2,+\infty}([0,t_0];X_0)\cap W^{1,+\infty}([0,t_0];X_1)$ *and* (θ^m) is bounded in $W^{1,+\infty}([0,t_0];X_0)\cap W^{1,2}([0,t_0];X_1)$. Therefore we extract subsequences (still denoted u^m,θ^m) which converge weakly and weakly $*$ in these spaces to a limit (u,θ). By using the appropriate test functions, we easily conclude that (u,θ) is a weak solution of (2.12) - (2.14); hence a solution of (2.1) - (2.4), by virtue of (1.11) and (2.7). For the desired regularity of (u,θ), we use the system (2.1), (2.2). Thus (2.11) is established.

Lemma 2.2 *Assume that (1.6), (2.10) hold and let u_0, u_1, θ_0 be as in theorem 2. Then the solution (u,θ) of (2.1) - (2.4) is unique within this class of functions and has the additional estimate (2.9).*

Proof. To establish the estimate (2.9), we introduce the operator

$$(21) \qquad \Delta_h W(x,t) := W(x,t+h) - W(x,t),\ \forall t \in [0,t_0),\ \forall h \in (0,t_0-t),$$

and apply it to the equations (2.1), (2.2). We then multiply the resulting equalities by $\Delta_h u_t$ and $\Delta_h\theta$ respectively, integrate over I x $(0,t)$, use integration by parts, add equalities, divide by h^2, and finally let h go to zero. We thus obtain

$$\frac{1}{2}\int_0^1 u_{tt}^2(x,t)dx + \frac{a}{2}\int_0^1 u_{xt}^2(x,t)dx + \frac{c}{2\overline{T}}\int_0^1 \theta_t^2(x,t)dx$$

$$+\frac{\kappa}{\overline{T}}\int_0^t\int_0^1 \theta_{xt}^2(x,s)dx\,ds = \frac{1}{2}\int_0^1 u_2^2(x)dx + \frac{a}{2}\int_0^1 u_1'^2(x)dx$$

$$(22) \qquad +\frac{c}{2\overline{T}}\int_0^1 \theta_1^2(x)dx - \frac{1}{\overline{T}}\int_0^t\int_0^1 b(x,s)u_{tt}(x,s)\theta_{xt}(x,s)dx\,ds$$

$$-\frac{1}{\overline{T}}\int_0^t\int_0^1 b_t(x,s)\mu_{xt}(x,s)\theta_t(x,s)dx\,ds - \frac{1}{\overline{T}}\int_0^t\int_0^1 b_x(x,s)u_{tt}(x,s)\theta_t(x,s)dx\,ds.$$

By combining (2.22), young's inequality, the relations

$$
(23) \qquad \begin{aligned}
u_t(x,t) &= u_1(x) + \int_0^t u_{tt}(x,s)ds, \\
u(x,t) &= u_0(x) + \int_0^t u_t(x,s)ds, \\
\theta(x,t) &= \theta_0(x) + \int_0^t \theta_t(x,s)ds,
\end{aligned}
$$

and the equations (2.1), (2.2), we obtain bounds on (u,θ) in higher order spaces. The estimate (2.9) is then established. and the uniqueness follows easily from (2.9) and Gronwall's inequality.

Proof of theorem 2. We approximate[2] b by a sequence $\{b_n\}$ in $C^1([0,t_o);H^2(I))$ such that

$$
(24) \qquad \begin{aligned}
b_n &\longrightarrow b \quad \text{in} \quad L^\infty([0,t_0];H^2(I)) \\
\dot{b}_n &\longrightarrow \dot{b} \quad \text{in} \quad L^\infty([0,t_o];L^2(I)) \cap L^2([0,t_o],H^1(I))
\end{aligned}
$$

and consider the approximating problems

$$
\begin{aligned}
u^n_{tt}(x,t) &= a u^n_{xx}(x,t) + \beta\theta^n_x(x,t) \\
(25) \qquad c\theta^n_t(x,t) &= \kappa\theta^n_{xx}(x,t) + \beta\overline{T}u^n_{xt}(x,t) + b_n(x,t)u^n_{xt}(x,t) \\
u^n(0,t) &= u^n(1,t) = 0,\ \theta^n(0,t) = \theta^n(1,t) = 0 \\
(26) \qquad u^n(x,0) &= u_0(x),\ u^n_t(x,0) = u_1(x),\ \theta^n(x,0) = \theta_0(x).
\end{aligned}
$$

By applying Lemma 2.1 to the set of problems (2.25), (2.26), we obtain a sequence of solutions satisfying

$$
(27) \qquad \begin{aligned}
&\{u^n\} \text{ is bounded in } \bigcap_{\alpha=0}^{2} W^{\alpha_1+\infty}([0,t_0], \quad X_{2-\alpha}) \\
&\{\theta^n\} \text{ is bounded in } L^\infty([0,t_0]; \quad X_2) \\
&\{\dot{\theta}^n\} \text{ is bounded in } L^\infty([0,t_0]; \quad X_0) \cap L^2([0,t_0]; \quad X_1).
\end{aligned}
$$

We then extract a subsequence (still denoted $\{u^n,\theta^n\}$) which converges weakly and weakly $_*$, in these spaces, to a limit (u,θ). By using the right test functions we easily verify that the limit (u,θ) satisfies (2.1) - (2.4).

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[2]This kind of approximation can be easily done by extension over the domain $[-t_0,\ 2t_0]$ and regularization in the usual way (See Lions [10])

3. QUASILINEAR PROBLEM

In this section, we prove theorem 1 by using theorem 2 and an iteration scheme. To achieve this goal, we introduce the set $W(M,t_0)$ which consists of all (v,φ) satisfying

$$
\begin{aligned}
& v \in \bigcap_{\alpha=0}^{2} W^{\alpha,+\infty}([0,t_0]; \quad X_{2-\alpha}) \\
& \varphi \in L^\infty([0,t_0]; \quad X_2) \\
& \dot\varphi \in L^\infty([0,t_0]; X_0) \cap L^2([0,t_0]; \quad X_1) \\
(1) \qquad & v(.,0) = u_0, v_t(\cdot,0) = u_1, \ \varphi(\cdot,0) = \theta_0 \\
& \sup_{0 \le t \le t_0} \left\{ \sum_{\alpha=0}^{2} || \overset{(\alpha)}{v} (\cdot,t)||_{2-\alpha}^2 + ||\varphi(\cdot,t)||_2^2 \right\} \\
& + \int_0^{t_0} \int_0^1 | \dot\varphi_x (x,t)|^2 dx\, dt \le M^2,
\end{aligned}
$$

for M and t_0 to be chosen suitably.

Remark 3.1 $W(M,t_0)$ is nonempty, if M is sufficiently large. This follows from the trace theorem (see [9]).

We equip $W(M,t_0)$ by the complete[3] metric

$$
\begin{aligned}
\rho^2((u,\theta),(\overline{u},\overline{\theta})) = & \max_{0\le t\le t_0} \Big(||u(\cdot,t) - \overline{u}(\cdot,t)||_1^2 + ||u_t(\cdot,t) - \overline{u}_t(\cdot,t)||_0^2 \\
(2) \qquad & + ||\theta(\cdot,t) - \overline{\theta}(\cdot,t)||_0^2 \Big) + \int_0^{t_0} \int_0^1 \left(\theta_x - \overline{\theta}_x\right)^2 (x,t) dx\, dt,
\end{aligned}
$$

$\forall\ (u,\theta),(\overline{u},\overline{\theta}) \in W(M,t_0)$.

Proof of theorem 1. For any (v,φ) in $W(M,t_0)$, we consider the linear problem

$$u_{tt}(x,t) \quad = \quad a\, u_{xx}(x,t) + \beta\theta_x(x,t), \quad x \in I, \ t \in [0,t_0)$$

[3] The completeness of the metric ρ follows from weak and weak $*$ precompactness of bounded sets in $L^2([0,t_0];\ L^2(I))$ and $L^\infty([0,t_0];\ L^2(I))$ respectively and sequential weak lower semicontinuity of norms in these spaces.(See [13])

$$
\begin{aligned}
(3) \qquad c\theta_t(x,t) &= \kappa\theta_{xx}(x,t) + \beta(\overline{T} + \varphi(x,t))u_{xt}(x,t) \\
u(0,t) &= u(1,t) = 0, \;\; \theta(0,t) = \theta(1,t) = 0 \\
(4) \qquad u(x,0) &= u_0(x), \; u_t(x,0) = u_1(x), \; \theta(x,0) = \theta_0(x),
\end{aligned}
$$

By applying theorem 1, the problem (3.3); (3.4) has a unique solution satisfying the estimate (2.9). We then define the mapping f, which takes (v, φ) into the solution (u, θ) of (3.3), (3.4) and show that f maps $W(M, t')$ into itself, for suitably chosen M and t'. To this end we use the a priori estimate (2.9) to choose M and t' so that

$$M^2 \geq 2\Gamma\varepsilon_0 \,, \; k\, t' \leq \frac{1}{2};$$

hence (2.9) implies

$$(5) \qquad \frac{1}{2}\varepsilon\,(t) + \int_0^t \int_0^1 |\theta_{xt}(x,s)|^2 dx\, ds \leq \frac{M^2}{2}\,, \; \forall t \;\in\; [0,t'),$$

which gives in turn

$$(6) \qquad \varepsilon(t) + \int_0^t \int_0^1 |\theta_{xt}(x,s)|^2 dx\, ds \leq M^2, \; \forall t \;\in\; [0,t').$$

Therefore $(u, \theta) \in W(M, t')$.

Next we show, for t'' sufficiently small, that f is a contraction. For this prupose we set

$$U := u - \overline{u}, \;\; V := v - \overline{v}, \;\; \Theta := \theta - \overline{\theta}, \;\; \Phi := \varphi - \overline{\varphi},$$

where

$$(u, \theta) = f(v, \varphi), \quad (\overline{u}, \overline{\theta}) = f(\overline{v}, \overline{\varphi}).$$

A simple computation shows that (U, Θ) satisfies the following problem

$$(7) \qquad U_{tt}(x,t) = aU_{xx}(x,t) + \beta\Theta_x(x,t)$$

$$(8) \qquad c\Theta_t(x,t) = \kappa\Theta_{xx}(x,t) + \beta\overline{T}U_{xt}(x,t) + \beta(\phi u_{xt} - \overline{\phi}\overline{u}_{xt})(x,t)$$

$$(9) \qquad U(0,t) = U(1,t) = 0, \;\; \Theta(0,t) = \Theta(1,t) = 0$$

$$U(x,0)=0,\ U_t(x,0)=0,\ \Theta(x,0)=0 \tag{10}$$

By multiplying (3.7) by U_t and (3.8) by Θ, integrating over I x $(0,t)$, and adding equalities, we obtain

$$\int_0^1 U_t^2(x,t)dx + a\int_0^1 U_x^2(x,t)dx + \frac{c}{\overline{T}}\int_0^1 \Theta^2(x,t)dx +$$

$$+\frac{2\kappa}{\overline{T}}\int_0^t\int_0^1 \Theta_x^2(x,\tau)dx\,d\tau = \frac{2\beta}{\overline{T}}\int_0^t\int_0^1 (\phi u_{xt} - \overline{\phi}\overline{u}_{xt})\Theta(x,\tau)dx\,d\tau$$

$$= \frac{2\beta}{\overline{T}}\int_0^t\int_0^1 (\phi U_{xt}\Theta)(x,\tau)dx\,d\tau + \frac{2\beta}{\overline{T}}\int_0^t\int_0^1 (\overline{u}_{xt}\Phi\Theta)(x,\tau)dx\,d\tau$$

$$= -\frac{2\beta}{\overline{T}}\int_0^t\int_0^1 U_t(\phi_x\Theta + \phi\Theta_x)(x,\tau)dx\,d\tau$$

$$-\frac{2\beta}{\overline{T}}\int_0^t\int_0^1 \overline{u}_t(\Phi\Theta_x + \Phi_x\Theta)(x,\tau)dx\,d,\ \ \forall\, t\ \in\ [0,t'). \tag{11}$$

By using young's inequality, we arrive at

$$\int_0^1 U_t^2(x,t)dx + \int_0^1 U_x^2(x,t)dx + \int_0^1 \Theta^2(x,t)dx + \int_0^t\int_0^1 \Theta_x^2(x,\tau)dx\,d\tau$$

$$\leq \Gamma M\left\{\left(1+\frac{1}{2\varepsilon}\right)t' \max_{0\leq t\leq t'} ||U_t(\cdot,t)||_0^2 + \varepsilon\int_0^{t'}\int_0^1 \Theta_x^2(x,t)dx\,dt\right.$$

$$+t'\left(1+\frac{1}{2\varepsilon}\right)\max_{0\leq t\leq t'} ||\Theta(.,t)_0^2 + \frac{t'}{2\varepsilon}\max_{0\leq t\leq t'} ||\Phi(\cdot,t)||_0^2$$

$$\left.+\frac{\varepsilon}{2}\int_0^{t'}\int_0^1 \Phi_x^2(x,t)dx\,dt\right\}, \forall\, t\ \in\ [0,t'), \varepsilon > 0.$$

We then choose ε small enough so that $\Gamma M\varepsilon = 1/2$ and choose t" so that $\Gamma Mt"/2\varepsilon = 1/4$.

By carrrying all calculations, we get

$$\rho^2((u,\theta),(\overline{u},\overline{\theta})) \leq \frac{1}{2}\rho^2((v,\varphi),(\overline{v},\overline{\varphi})). \tag{12}$$

This shows that f is a strict constraction. Therefore by the contraction mapping therem f has a unique fixed point (u,θ) in $W(M,t")$. Obviously (u,θ) is the desired solution.

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