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AN r-QUORUM QUEUEING SYSTEM WITH RANDOM SERVER CAPACITY AND IMPATIENT CUSTOMERS UNDER N-POLICY

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ABSTRACT. The authors study a queueing system under N-policy characterized by a Poisson input, a delayed service, a random server capacity following an idle period, and two scenarios of impatient customers. An ergodicity condition and steady-state probabilities are derived for the discrete-time process using the embedded Markov chain technique, and for the continuous-time parameter process using semi-regenerative techniques. Various system characteristics are computed and illustrative examples are provided.

1. INTRODUCTION

An r-quorum queueing system is a queueing system where the service is delayed until at least r customers are present in the queue. N-policy discipline means that each time the system becomes empty the server waits until exactly N customers are available, then works until the system is again empty. Muh [5] combined for the first time N-policy and r-quorum models in a single model as follows: if at least r units are waiting in the queue at a service completion, then the server picks a batch of exactly r units and starts processing them immediately; otherwise, and if less than r units are in the queue at a service completion, then the server starts an idle period and resumes work taking a batch of r only when $N(\geq r)$ have been accumulated in the queue. Muh used the embedded processes technique to obtain the system size at a departure epoch.

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In the present paper we extend Muh's model by incorporating the following two features:

(i) Random server capacity: We assume that following an idle period the server capacity becomes random and instead of serving exactly r customers, the server may pick a batch of any size randomly chosen between 1 and r. More generality is obtained by allowing the service time to depend on the size of the batch taken for service.

(ii) Customer impatience: We take into account two scenarios of impatience. The first scenario (scenario 1) happens when the queue length Q_n is above r at a service completion. In this case and as we know, the server picks a batch of r customers, then some of the $Q_n - r$ unserved customers leave the system . The second scenario (scenario 2) occurs following an idle period. In this case the server picks a random number of j customers $(1 \leq j \leq r)$ and again some of the $N - j$ unserved customers leave the system.

Random server capacity following an idle period in a bulk service queueing system was for the first time studied by Dshalalow and Tadj [3]. The delayed service feature obviously reduces start-up costs and finds applications in transportation systems. The random server capacity is encountered in computer networks for instance. Customer impatience may represent items deterioration in an inventory system. Our model presents characteristics that need not be all present in the same real-world situation, however, it generalizes many special cases widely considered in the literature.

Section 2 describes the model and the notation. Sections 3 and 4 consider the discrete and continuous time versions of the queueing process, respectively. In each case a necessary and sufficient condition for ergodicity is established, steady-state probabilities are derived and illustrative examples are provided.

2. MODEL AND NOTATION

We study a queueing model with a single server and an infinite waiting room. The service discipline is FIFO. The arrival process is an orderly stationary Poisson process with fixed rate $\lambda > 0$. We are interested in the system size $\{Q(t)\}$ at an arbitrary instant of time $t \geq 0$. Defining $Q(t)$ as a right continuous process, let $Q_n = Q(T_n^+), n = 0, 1, ...,$ where $\{T_n\}$ represents the completion time of the nth service.

If $Q_n < r$, the server waits for the queue to accumulate to N, then takes $c_{n+1} = j (j = 1, ..., r)$ customers for service. We assume that $\{c_n\}$ forms a sequence of iid random variables with $\gamma_j = P\{c_1 = j\}$. The service time in this case is distributed according to a general distribution function B_j depending on the batch taken for service and having a finite mean b_j. Some of the unserved $N - c_{n+1}$ customers may leave the system (scenario 1). Let $\Delta_1(N - c_{n+1})$ be the number of customers who stay in the system and wait for the next service and denote by

$$(2.1) \qquad f_s^{(N-j)} = P\{s \text{ out of } (N-j) \text{ customers stay in the system} | Q_n = i\}, \quad s = 0, 1, ..., N - j,$$

$$(2.2) \qquad \bar{f}_{N-j} = E[\Delta_1(N-j) | Q_n = i].$$

Define the probability generating function of $\Delta_1(N - j)$ by

$$(2.3) \qquad f_{N-j}(z) = \sum_{s=0}^{N-j} f_s^{(N-j)} z^s.$$

If $Q_n \geq r$, the server takes r customers for service. The service time is general with distribution function B having finite mean b. Some of the unserved $Q_n - r$ customers may leave the system (scenario 2). Let $\Delta_2(Q_n - r)$ be the number of those customers who stay in the system and wait for the next service and denote by

$$(2.4) \qquad g_s^{(i-r)} = P\{s \text{ out of } (i-r) \text{ customers stay in the system} | Q_n = i\}, \quad s = 0, 1, ..., i - r,$$

$$(2.5) \qquad \bar{g}_{i-r} = E[\Delta_2(i-r) | Q_n = i].$$

For the probability generating function of $\Delta_2(i-r)$, let $M(\geq N)$ such that:

$$g_{i-r}(z) = \sum_{s=0}^{i-r} g_s^{(i-r)} z^s, r \leq i \leq M, \tag{2.6}$$

and

$$g_{i-r}(z) = z^{i-r}, i > M. \tag{2.7}$$

In other words, we assume a "stable impatience" of the customers beginning from the level M. This assumption is needed for analytical convenience. We may think of M as being so large that it will never be reached by the queue length.

We recall that given a probability distribution function B, its Laplace-Stieltjes transform B^* is given by:

$$B^*(\theta) = \int_0^\infty e^{-\theta x} B(dx), Re(\theta) \geq 0, \tag{2.8}$$

where $Re(\theta)$ stands for the real part of θ.

3. THE PROCESS $\{Q_n\}$

3.1 Probability generating function of the system

The process $\{Q_n\}$ is a Markov chain since

$$Q_{n+1} = \begin{cases} \Delta_1(N - c_{n+1}) + V_{n+1} & \text{if } Q_n < r \\ \Delta_2(Q_n - r) + V_{n+1} & \text{if } Q_n \geq r \end{cases} \tag{3.1}$$

where V_n is the number of arrivals during nth service. Denote by $P = (p_i, i \geq 0)$ its steady-state probability vector, if it exists, by A its transition probability matrix, and by $A_i(z)$ the probability generating function of the ith row of $A, i \geq 0$. Then

$$A_i(z) = \begin{cases} \sum_{k=1}^{r} f_{N-k}(z) F_k(z) \gamma_k & \text{if } i < r \\ g_{i-r}(z) F(z) & \text{if } i \geq r \end{cases} \tag{3.2}$$

where $f_{N-k}(z)$ is defined by (2.3), $g_{i-r}(z)$ by (2.6),

$$F_k(z) = B_k^*(\lambda - \lambda z), k = 1, ..., r, \tag{3.3}$$

$$F(z) = B^*(\lambda - \lambda z), \tag{3.4}$$

and $B_k^*(\theta), B^*(\theta)$ are the Laplace-Stieltjes transforms of B_k and B respectively. Note that in the case $i < r, A_i(z)$ is independent of i and consequently we suppress the subscript i and write $A(z)$.

It follows from (3.1) that A is a delta $\Delta_{r,M}$-matrix. Delta matrices were introduced by Abolnikov and Dukhovny [1].

Definition 1 (Abolnikov and Dukhovny [1]). A finite or an infinite stochastic matrix $A = (a_{ij}; i,j \geq 0)$ is called a $\Delta_{m,n}$ matrix, $n \geq m \geq 1$, if $a_{ij} = k_{j-i+m}$ for $i > n, j \geq i - m; a_{ij} = 0$ for $i > n, j < i - m$ where $\{k_i, i \geq 0\}$ is a given probability mass function.

Proposition 1 (Abolnikov and Dukhovny [1]). *Let $\{Q_n\}$ be an irreducible aperiodic Markov chain with transition probability matrix A in the form of a $\Delta_{m,n}$-matrix and let $A_i(z)$ be the generating function of the ith row of A and $K(z) = \sum_{j=0}^{\infty} k_j z^j$. Then $\{Q_n\}$ is recurrent-positive if and only if*

$$\left.\frac{d}{dz} A_i(z)\right|_{z=1} < \infty, i = 0, 1, ..., n, \tag{3.5}$$

and

$$\left.\frac{d}{dz} K(z)\right|_{z=1} < m. \tag{3.6}$$

Proposition 2 (Abolnikov and Dukhovny [1]). *Under condition* (3.6) *the function $z^r - K(z)$ has exactly r roots that belong to the closed unit ball $\bar{B}(0,1) = \{z \in C : \|z\| \leq 1\}$. Those of the roots lying on the boundary $\partial B(0,1)$ are simple.*

Now we formulate the main result of this section.

Proposition 3. *The Markov chain $\{Q_n\}$ is ergodic if and only if*

$$\rho = \lambda b < r, \tag{3.7}$$

where λ is the rate of the Poisson process and b is the mean service time.

The generating function $P(z)$ of the components of vector P is given by:

$$P(z) = \frac{\sum_{i=0}^{M}\{z^r A_i(z) - z^i F(z)\}p_i}{z^r - F(z)}, \tag{3.8}$$

and the $(M+1)$ unknown probabilities $p_0, p_1, ..., p_M$ form the unique solution of the following system of $(M+1)$ equation:

$$\sum_{i=0}^{M} \frac{d^k}{dz^k}\{A_i(z) - z^i\}p_i z = z_s = 0, k = 0, ..., k_s - 1; s = 1, ..., S+1, \tag{3.9}$$

$$\sum_{i=0}^{r-1}\{r + \hat{f} - i + \lambda(\bar{b} - b)\}p_i + \sum_{i=r}^{M}\{r + \bar{g}_{i-r} - i\}p_i = r - \rho, \tag{3.10}$$

where

$$\hat{b} = \sum_{k=1}^{r} b_k \gamma_k, \tag{3.11}$$

$$\hat{f} = \sum_{k=1}^{r} \bar{f}_{N-k}\gamma_k, \tag{3.12}$$

z_s are the roots of $z^r - F(z) = 0$ in the region $\bar{B}(0,1) \setminus \{1\}$ with their multiplicities k_s such that $\sum_{s=1}^{S} k_s = r - 1$, and $z_{S+1} = 0$ is of multiplicity $M + 1 - r$.

Proof. Ergodicity follows from Proposition 1. We easily check that the first condition is satisfied. Since $F(z) = B^*(\lambda - \lambda z)$, the second condition is equivalent to (3.7). Relation (3.8) is proved using $P(z) = \sum_{i=0}^{\infty} A_i(z)p_i$. To derive the system of equations (3.9)-(3.10), we rewrite relation (3.8) as:

$$\frac{P(z) - \sum_{i=0}^{M} p_i z^i}{z^{M+1}} = \frac{\sum_{i=0}^{M}\{A_i(z) - z^i\}p_i}{z^{M+1-r}\{z^r - F(z)\}}. \tag{3.13}$$

The left-hand-side of the above relation is analytic in $\bar{B}(0,1) \setminus \{1\}$ and therefore so is the right-hand-side. Now the denominator of the right-hand-side has $r - 1$ simple roots in $\bar{B}(0,1) \setminus \{1\}$ (Proposition 2) and therefore these roots are also roots of the numerator. Call them $z_s, s = 1, ..., S$ and let $k_s, s = 1, ..., S$, be their respective multiplicities such that $\sum_{s=1}^{S} k_s = r - 1$. Also let $z_{S+1} = 0$ be the $(S+1)$th root, of multiplicity $M + 1 - r$. These considerations yield system (3.9) of M equations with

$M+1$ unknowns $p_0, ..., p_M$. The last equation is equivalent to the identity $P(1) = 1$. Finally, uniqueness of $\{p_0, ..., p_M\}$ is proved as in Dshalalow and Tadj [3].

3.2 Examples

1) Assume that B is exponentially distributed with parameter $1/b$. Then $F(z) = [1+\rho(1-z)]^{-1}$. Take for example $r = 3$. Equation $z^3 - F(z) = 0$ is equivalent to

$$\rho z^3 - z^2 - z - 1 = 0, \tag{3.14}$$

which has two simple roots, say z_1 and z_2, that belong to the unit ball $\bar{B}(0,1) \setminus \{1\}$. The denominator in the right-hand-side of (3.13) is $z^{M-2}(z^3 - F(z)$. $z_3 = 0$ is a root of multiplicity $k_3 = M-2$. To obtain $P(z)$ explicitly, we need to derive $p_0, ..., p_M$, which is done by solving the simple system of $M+1$ equations:

$$\sum_{i=0}^{M} \{A_i(z_1) - z_1^i\} p_i = 0,$$

$$\sum_{i=0}^{M} \{A_i(z_2) - z_2^i\} p_i = 0,$$

$$\sum_{i=0}^{M} \frac{d^k}{dz^k} \{A_i(z) - z^i\} p_i \bigg|_{z=0} = 0, k = 0, ..., M-2,$$

$$\sum_{i=0}^{2} \{3 + \hat{f} - i + \lambda(\hat{b} - b)\} p_i + \sum_{i=3}^{M} \{3 - i + \bar{g}_{i-r}\} p_i = 3 - \rho.$$

2) Dropping the N-policy assumption (take $N = r$), we are left with the r-quorum queueing system with random server capacity and impatient customers. Let $P^* = (p_i^*; i \in N)$ and $P^*(z) = \sum_{i=0}^{\infty} p_i^* z^i$ denote the steady state vector of probability and the probability generating function of the embedded Markov chain, respectively. From (3.8) we get

$$P^*(z) = \frac{\sum_{i=0}^{M} \{z^r A_i^*(z) - z^i F(z)\} p_i^*}{z^r - F(z)}, \tag{3.15}$$

where

$$A_i^*(z) = \begin{cases} \sum_{k=1}^{r} f_{r-k}(z)F_k(z)\gamma_k & \text{if } i < r \\ A_i(z) & \text{if } i \geq r \end{cases} \tag{3.16}$$

Combining (3.8) and (3.15) we get the decomposition property:

$$P(z) = P^*(z)\frac{\sum_{i=0}^{M}\{z^r A_i(z) - z^i F(z)\}p_i}{\sum_{i=0}^{M}\{z^r A_i^*(z) - z^i F(z)\}p_i^*}. \tag{3.17}$$

By the stochastic decomposition result of Fuhrmann and Cooper (1985), the second term on the right-hand side of expression (3.17) is the probability generating function of the number of customers at a departure epoch, given that the server is idle.

3) Similarly, we may drop the random server capacity assumption (take $\gamma_r = 1, \gamma_j = 0$ for $j = 1, ..., r-1$), the r-quorum assumption (take $r = 1$), the scenario 1 (take $f_{r-j}^{(r-j)} = 1$ and $f_s^{(r-j)} = 0$ for $s = 0, ..., r-j-1$), scenario 2 (take $g_{i-r}^{(i-r)} = 1$ and $g_s^{(i-r)} = 0$ for $s = 0, ..., i-r-1$), or both scenarios of impatience or any combination of these features. In each case, the probability generating function of the system size at a departure epoch of the corresponding queueing system may be derived from (3.8).

3.3 System Characteristics

We derive now some characteristics of the queueing system.

i) Let $\beta = (\beta_i; i = 0, 1, ...)$ and $\rho = (\rho_i; i = 0, 1, ...)$ such that $\beta_i = E[T_{n+1} - T_n | Q_n = i]$ and $\rho_i = \lambda\beta_i$. Then

$$\beta_i = E[T_1 | Q_1 = i] = \begin{cases} \frac{N-i}{\lambda} + \hat{b} & \text{if } i < r \\ b & \text{if } i \geq r \end{cases} \tag{3.18}$$

and

$$\rho_i = \begin{cases} N - i + \lambda\hat{b} & \text{if } i < r \\ \rho & \text{if } i \geq r \end{cases} \tag{3.19}$$

Therefore,

$$P\beta = \sum_{i=0}^{r-1}(\frac{N-i}{\lambda} + \hat{b} - b)p_i + b. \tag{3.20}$$

(3.21) $$P\rho = \sum_{i=0}^{r-1}(N - i + \lambda\hat{b} - \rho)p_i + \rho.$$

$P\beta$ is called the stationary mean service cycle.

ii) Let l denote the stationary mean server load. Then

(3.22) $$l = \sum_{i=0}^{\infty} \bar{\Gamma}_i p_i,$$

where $\bar{\Gamma}_i = E[\Gamma_n(Q_n)|Q_n = i]$ and the server capacity Γ_n is given by:

(3.23) $$\Gamma_n(Q_n) = \begin{cases} c_n(Q_n) & \text{if } Q_n < r \\ r & \text{if } Q_n \geq r \end{cases}$$

Therefore,

(3.24) $$l = r - \epsilon \sum_{i=0}^{r-1} p_i,$$

where

(3.25) $$\epsilon = r - \bar{c},$$

and

(3.26) $$\bar{c} = E[c_1].$$

iii) Let d denote the mean number of disappointed customers. Then

(3.27) $$d = d_1 + d_2,$$

where d_1 and d_2 represent the mean number of impatient customers in scenario 1 and scenario 2 respectively. It is easy to check, by probability arguments that:

(3.28) $$d_1 = (N - \bar{c} - \hat{f}) \sum_{i=0}^{r-1} p_i,$$

where $\hat{f}$ is defined in (3.12) and

(3.29) $$d_2 = \sum_{i=r}^{M}(i - r - \bar{g}_{i-r})p_i,$$

where $\bar{g}_{i-r}$ is defined in (2.6). Therefore

$$d = (N - \bar{c} - \hat{f}) \sum_{i=0}^{r-1} p_i + \sum_{i=r}^{M} (i - r - \bar{g}_{i-r}) p_i. \tag{3.30}$$

iv) Let $\mathcal{I}$ denote the system intensity. Then

$$\begin{aligned} \mathcal{I} &= P\rho - d, && (3.31)\\ &= \sum_{i=0}^{r-1} (\bar{c} + \hat{f} - i + \lambda(\hat{b} - b)) p_i - \sum_{i=r}^{M} (i - r - \bar{g}_{i-r}) p_i + \rho. && (3.32) \end{aligned}$$

A remarkable property of the system in equilibrium is that the stationary mean server load l and the the system intensity $\mathcal{I}$ are equal:

$$l = \mathcal{I}. \tag{3.33}$$

This follows from the definitions of l and $\mathcal{I}$ and relation (3.10).

4. THE PROCESS $\{Q(t)\}$

4.1 Probability generating function of the system

The process $\{Q(t)\}$ is a semi-regenerative process with conditional regenerations at points $T_n, n \geq 0$ and $\{(Qn, Tn)\}$ is the associated Markov renewal process. $\{Q(t)\}$ is ergodic if and only if (3.7) is satisfied (see for example Dshalalow and Tadj [3]). Our objective now is to derive, for the semi-regenerative process $\{Q(t)\}$, the steady-state vector of probabilities $\pi = (\pi_i, i \geq 0)$ or, equivalently, its probability generating function $\pi(z) = \sum_{i=0}^{\infty} \pi_i z^i$. Let $K(t) = \{K_{ij}(t); i, j \geq 0\}$ be the semi-regenerative kernel, i.e., $K_{ij}(t) = P\{Q(t) = j, T_1 > t | Q(0) = i\}$. Then we have:

$$K_{ij}(t) = e^{-\lambda t} \frac{(\lambda t)^{j-i}}{(j-i)!}, 0 \leq i \leq j < r \leq N \text{ or } 0 \leq i < r \leq j \leq N,$$

$$\begin{aligned} K_{ij}(t) &= \int_0^t \lambda e^{-\lambda(t-u)} \frac{[\lambda(t-u)]^{N-i-1}}{(N-i-1)!} e^{-\lambda u} \sum_{l=1}^{r} \sum_{s=0}^{N-l} \frac{(\lambda u)^{j-l-s}}{(j-l-s)!} \\ &\quad \times [1 - B_l(u)] f_s^{(N-l)} \gamma_l du, 0 \leq i < r \leq N \leq j, \end{aligned}$$

$$K_{ij}(t) = \sum_{s=0}^{i-r} e^{-\lambda t}\frac{(\lambda t)^{j-r-s}}{(j-r-s)!}[1-B(t)]g_s^{(i-r)}, r \le i \le j,$$

$$K_{ij}(t) = e^{-\lambda t}\frac{(\lambda t)^{j-i}}{(j-i)!}[1-B(t)], M < i \le j,$$

$$K_{ij}(t) = 0, 0 \le j < i.$$

The following is the main result of this section.

Proposition 4. *Given the equilibrium condition* (3.7), *the probability generating function* $\pi(z)$ *is given by:*

$$\begin{aligned}\lambda(1-z)P\beta\pi(z) &= (1-z^r)P(z) \\ &+ \sum_{i=0}^{r-1}[\sum_{l=1}^{r}\sum_{s=0}^{N-l} f_s^{(N-l)}\gamma_l u_{N-l-s}^{(l)}(z) - z^N + z^r A(z)]p_i \\ &+ \sum_{i=r}^{M}[\sum_{s=0}^{i-r} g_s^{(i-r)} u_{i-r-s}(z) - 1 + z^r g_{i-r}(z)F(z)]p_i,\end{aligned} \tag{4.1}$$

where $P(z)$ *is the generating function of* $P, P\beta$ *is determined by* (3.20), *and*

$$u_s^{(i)} = \sum_{m=0}^{s}(z^{-m}-1)(-\lambda)^{(s-m)}\frac{\beta_i^{(s-m)}(\lambda)}{(s-m)!} + 1 - z^{-s}F_i(z), \tag{4.2}$$

and the index i *may be dropped for all* i *exceeding* r.

Proof. We apply the main convergence theorem for semi-regenerative processes in the form

$$\pi = \frac{PH}{P\beta}. \tag{4.3}$$

We need to find the integrated semi-regenerative kernel $H = \int_0^\infty K(t)dt$, which is done by routine calculus, and then the generating functions $h_i(z)$ of all rows of H. We show that

$$h_i(z) = \frac{z^N}{\lambda(1-z)}[(z^{i-N}-1) + \sum_{l=1}^{r}\sum_{s=0}^{N-l} f_s^{(N-l)}\gamma_l u_{N-l-s}^{(l)}(z)], 0 \le i < r,$$

$$h_i(z) = \frac{z^i}{\lambda(1-z)}\sum_{s=0}^{i-r} g_s^{(i-r)} u_{i-r-s}(z), r \le i \le M,$$

$$h_i(z) = \frac{z^i}{\lambda} \frac{1 - F(z)}{1 - z}, i \geq M.$$

Now, from relation (4.3),

$$P\beta\pi(z) = (\sum_{i=0}^{r-1} + \sum_{i=r}^{M} + \sum_{i=M+1}^{\infty})h_i(z)p_i.$$

Therefore,

$$\begin{aligned}\lambda(1 - z)P\beta\pi(z) &= (\sum_{i=0}^{r-1}[z^i - z^N + \sum_{l=1}^{r}\sum_{s=0}^{N-l} f_s^{(N-l)}\gamma_l u_{N-l-s}^{(l)}(z)])p_i \\ &+ \sum_{i=r}^{M}\sum_{s=0}^{i-r} g_s^{(i-r)}u_{i-r-s}(z)p_i z^i + [1 - F(z)] \sum_{i=M+1}^{\infty} p_i z^i.\end{aligned}$$

We add and subtract $[1 - F(z)] \sum_{i=0}^{M} p_i z^i$ to the equation above. Then, using from relation (3.8) the fact that $F(z) \sum_{i=0}^{M} p_i z^i = z^r \sum_{i=0}^{M} A_i(z)p_i - [z^r - F(z)]P(z)$, we get relation (4.1).

4.2 Examples

1) Dropping the N-policy assumption (take $N = r$), we are left with the r-quorum queueing system with random server capacity and impatient customers. The decomposition property of Fuhrmann and Cooper (1985) may be used to get the probability generating function of the number of customers at an arbitrary point of time given that the server is idle.

2) As in the discrete case, we may drop any feature(s) of the model (random server capacity assumption, the r-quorum assumption, the scenario 1, scenario 2, or both scenarios of impatience) to obtain from (4.1) the probability generating function of the system size at an arbitrary point of time of the corresponding queueing system.

4.3 System characteristics

i) Let $\bar{I}$ and $\bar{B}$ denote the expected length of an idle and busy period of the server in the steady-state. The server starts an idle period every time the queue length i is smaller than r at a service completion. He then has to wait $N-i$ exponential phases. Therefore the probability that he idles in the steady-state is:

$$(4.4) \qquad \frac{\sum_{i=0}^{r-1} p_i e_{\lambda,N-i}(t)}{\sum_{i=0}^{r-1} p_i},$$

where $e_{\lambda,k}(t)$ is the k-Erlang probability density function with parameter λ, which yields:

$$(4.5) \qquad \bar{I} = \frac{\sum_{i=0}^{r-1} p_i \frac{N-i}{\lambda}}{\sum_{i=0}^{r-1} p_i}.$$

On the other hand, the probability that the server is idle is:

$$(4.6) \qquad \sum_{i=0}^{r-1} \pi_i = \frac{\bar{I}}{\bar{I}+\bar{B}}.$$

This allows us to derive the mean busy period in the steady-state $\bar{B}$:

$$(4.7) \qquad \bar{B} = \frac{\sum_{i=r}^{\infty} \pi_i}{\sum_{i=0}^{r-1} \pi_i} \bar{I},$$

where $\bar{I}$ is given by (4.5).

ii) The mean output rate, $\mathcal{O} = \lim_{t\to\infty} \frac{E[S(t)|Q(0)=i]}{t}$, where $S(t)$ is the number of customers completely processed during the time interval $[0,t]$, can be shown, as in Dshalalow [2], to be:

$$(4.8) \qquad \mathcal{O} = \frac{l}{P\beta}.$$

iii) The (actual) mean input rate κ of any random process is: (see Dshalalow [2]):

$$(4.9) \qquad \kappa = \frac{\mathcal{I}}{P\beta}.$$

Similarly, the (total) mean input rate can be shown to be:

$$(4.10) \qquad \bar{\lambda} = \frac{P\rho}{P\beta}.$$

It follows from (3.31) that the mean impatience rate $\delta = \bar{\lambda} - \kappa$ is given by:

$$\delta = \frac{d}{P\beta}. \tag{4.11}$$

Obviously, the mean impatience rate δ_i of scenario i $(i = 1, 2)$ is

$$\delta_i = \frac{d_i}{P\beta}. \tag{4.12}$$

Note that given the equilibrium condition $\rho < r$, the (actual) mean input rate κ and the mean output rate $\mathcal{O}$ are equal, that is $\kappa = \mathcal{O}$.

To summarize, we considered in this paper a queueing model under N-policy characterized by a delayed service, a random server capacity, and two scenarios of impatient customers. An ergodicity condition and steady-state probabilities are derived. System characteristics are computed.

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