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On the wreath product of groups containing M_{24}

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ABSTRACT. In this paper, we show the structure of some groups containing the Mathieu group M_{24}. The structure of the groups founded is determined in terms of wreath product. Some related cases are also included. We also show that S_{24k+1} and A_{24k+1} can be generated using the wreath product $M_{24}\, wr\, C_k$ and an element of order 4 in S_{24k+1} and element of order 3 in $_{24k+1}$ for all integers $k \geq 2$.

Keywords. The Mathieu group M_{24}, The wreath product.
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1. Introduction

Al-Amri and Easssa [1] have shown that the group generated by the two 5-cycles $(k,\ 2k,\ 3k,\ 7k,\ 6k)(4k,\ 8k,\ 5k,\ 9k,\ 10k)$, the permutation $(1,\ 2, \ldots,\ 11k)$ $(11k+1,\ 11k+2, \ldots, 12k)$ and the involution $(k,\ 12k)(2k,\ 11k)(3k,\ 6k)(4k,\ 8k)$ $(5k,\ 9k)(7k,\ 10k)$ is the wreath product of M_{12} by C_k. They also gave a generating sets of $M_{12}\, wr\, S_k$, $M_{12}\, wr\, A_k$ and $M_{12}\, wr\, (S_m\, wr\, C_a)$.

In [2], Al-Amri and Al-Shehri have shown that the group S_{9k+1} and A_{9k+1} can be generated using the wreath product $M_9\, wr\, C_k$ and an element of order 4 in S_{9k+1} and element of order 5 in A_{9k+1} for all odd integers $k > 2$.

The Mathieu group M_{24} of order 244823040 is one of the well known simple groups. In [4], M_{24} is fully described. In a matter of fact, M_{24} can be generated using three permutations, the first is of order 23, the second is of order 5 and

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the third is an involution as follows;

$$M_{24} = \Big\langle (1,2,...,23)(3,17,10,7,9)(4,13,14,19,5)(8,18,11,12,23)(15,20,22,21,16),\ (1,24)(2,23)(3,12)(4,16)(5,18)(6,10)(7,20)(8,14)(9,21)(11,17)(13,22)(15,19)\Big\rangle.$$

M_{24} can be also faintly presented in different ways. One of the well known presentation of M_{24} is:

$$M_{24} = \Big\langle X,Y,T \,|\, X^{23} = Y^5 = T^2 = (XY)^5 = (XT)^3 = (YT)^{10} = 1,\ \langle X,Y\rangle = M_{23}\Big\rangle$$

Another presentation for M_{24} is found as follows:

$$M_{24} = \Big\langle X,Y \,|\, X^{23} = Y^3 = (XY)^{12} = [X,Y]^3 = \big[X^2Y\big]^{21} = 1,\ \langle X,[X,Y]\big[X^2Y\big]\rangle = M_{23}\Big\rangle$$

In this paper, we will show the structure of the group generated by three permutations, the first is of order $23k$, the second is of order 5 and the third is an involution. We will show that the group obtained is the wreath product of $M_{24}\, wr\, C_k$. Some related cases are also included. We also show that S_{24k+1} and A_{24k+1} can be generated using the wreath product $M_{24}\, wr\, C_k$ and an element of order 4 in S_{24k+1} and an element of order 3 in A_{24k+1}.

2. PRELIMINARY RESULTS

Definition 2.1. Let A and B be groups of permutations on non empty sets Ω_1 and Ω_2 respectively, where $\Omega_1 \cap \Omega_2 = \phi$. The wreath product of A and B is denote by $A\, wr\, B$ and defined as $A\, wr\, B = A^{\Omega_2} \times_\theta B$, i.e., the direct product of $|\Omega_2|$ copies of A and mapping θ where $\theta : B \rightarrow Aut\big(A^{\Omega_2}\big)$, is defined by $\theta_y(x) = x^y$, for all $x \in A^{\Omega_2}$. It follows that $|A\, wr\, B| = (|A|)^{|\Omega_2|}\,|B|$.

Theorem 2.1. [3] *Let G be the group generated by the n-cycle $(1,2,...,n)$ and the 2-cycle (n,a). If $1 < a < n$ is an integer with $n = a\,m$, then $G \cong S_m\, wr\, C_a$.*

Theorem 2.2. [3] *Let* $1 \leq a_1 \neq a_2 \neq a_3 < n$. *Let* $n \geq 7$ *be an odd integer. Let* G *be the group generated by the* n*-cycle* $(1, 2, ..., n)$ *and the* 4*-cycle* (n, a_1, a_2, a_3). *If* $hcf(n, a_1, a_2, a_3) = 1$, *then* $G \cong S_m$.

Theorem 2.3. [3] *Let* $1 \leq a \neq b < n$ *be any integers. Let* n *be an odd integer and let* G *be the group generated by the* n*-cycle* $(1, 2, ..., n)$ *and the* 3*-cycle* (n, a, b). *If* $hcf(n, a, b) = 1$, *then* $G = A_n$. *While if* n *an even then* $G \cong S_n$.

3. The RESULTS

Theorem 3.1. *Let* G *be the group generated by the permutation* $(1,\ 2, ..., 23k)$ $(23k+1, 23k+2, ..., 24k)$, *the four* 5*-cycles* $(3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k)(15k, 20k, 22k, 21k, 16k)$ *and the involution* $(k, 24k)(2k, 23k)(3k, 12k)(4k, 16k)(5k, 18k)(6k, 10k)(7k, 20k)(8k, 14k)(9k, 21k)$ $(11k, 17k)(13k, 22k)(15k, 19k)$. *if* $k > 1$ *is an integer, then* $G \cong M_{24}\, wr\, C_k$ *of order* $\left(|M_{24}|\right)^k \times k$, *where* M_{24} *is the Mathieu group of order* 244823040.

Proof. Let $\sigma = (1,\ 2, ..., 23k)(23k{+}1, 23k{+}2, ..., 24k)$, $\tau = (3k, 17k, 10k, 7k, 9k)$ $(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k)(15k, 20k, 22k, 21k, 16k)$ and $\gamma = (k, 24k)(2k, 23k)(3k, 12k)(4k, 16k)(5k, 18k)(6k, 10k)(7k, 20k)(8k, 14k)(9k, 21k)$ $(11k, 17k)(13k, 22k)(15k, 19k)$. Let $\alpha = \prod_{i=0}^{6} \tau^{\sigma^{ik}}$, we get an element $\delta = \alpha^{10} = (k, 2k, 3k, ..., 23k)$. Let $G_i = \langle \delta^{\sigma^i}, \tau^{\sigma^i}, \gamma^{\sigma^i} \rangle$ for all $1 \leq i \leq k$, be the groups of act on the sets $\Gamma_i = \left\{ \left(jk^{\sigma^i}\right) : 1 \leq j \leq 24 \right\}$, for all $i = 1, 2, ..., k$ respectively, and since $\bigcap_{i=1}^{k} \Gamma_i = \phi$, then we get the direct product $G_1 \times G_2 \times ... \times G_k$, where $G_i \cong M_{24}$. Let $\beta = \delta^{-1}\sigma = (1,\ 2, ..., k)(k + 1, k + 2, ..., 2k)(22k + 1, 22k + 2, ..., 23k)(23k + 1, 23k + 2, ..., 24k)$. Let $H = \langle \beta \rangle \cong C_k$. H conjugates G_1 into G_2, G_2 into G_3,... and G_k into G_1. Hence we get the wreath product $M_{24}\, wr\, C_k \subseteq G$. On the other hand, since $\delta\beta = (1,\ 2, ..., k, k{+}1, k{+}2, ..., 2k, 22k{+}1, 22k{+}2, ..., 23k, 23k{+}1, 23k{+}2, ..., 24k) = \sigma$ then $\sigma \in M_{24}\, wr\, C_k$. Hence $G = \langle \sigma, \tau, \gamma \rangle \cong M_{24}\, wr\, C_k$. □

Theorem 3.2. *Let* G *be the group generated by the permutation* $(1,\ 2, ..., 23k)$ $(23k+1, 23k+2, ..., 24k)$, *the four* 5*-cycles* $(3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k)(15k, 20k, 22k, 21k, 16k)$, *the involution* $(k, 24k)$ $(2k, 23k)(3k, 12k)(4k, 16k)(5k, 18k)(6k, 10k)(7k, 20k)(8k, 14k)(9k, 21k)(11k, 17k)(13k, 22k)(15k, 19k)$ *and the involution* $(1, 2)(k + 1, k + 2)(2k + 1, 2k +$

2)...(23k + 1, 23k + 2). *if* $k > 2$ *is an integer, then* $G \cong M_{24}\, wr\, S_k$ *of order* $\left(|M_{24}|\right)^k \times k!$.

Proof. Let $\sigma = (1, 2, ..., 23k)(23k{+}1, 23k{+}2, ..., 24k)$, $\tau = (3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k)(15k, 20k, 22k, 21k, 16k)$ and $\gamma = (k, 24k)(2k, 23k)(3k, 12k)(4k, 16k)(5k, 18k)(6k, 10k)(7k, 20k)(8k, 14k)(9k, 21k)(11k, 17k)(13k, 22k)(15k, 19k)$ and $\mu = (1, 2)(k{+}1, k{+}2)(2k{+}1, 2k{+}2)...(23k{+}1, 23k{+}2)$. From Theorem 2.1, $\langle\sigma, \tau\rangle \cong M_{24}\, wr\, C_k$. Since $C_k = \big\langle(1, 2, ..., k)(k{+}1, k{+}2, ..., 2k)(22k{+}1, 22k{+}2, ..., 23k)(23k{+}1, 23k{+}2, ..., 24k)\big\rangle$, then $\big\langle(1, 2, ..., k)(k{+}1, k{+}2, ..., 2k)(22k{+}1, 22k{+}2, ..., 23k)(23k{+}1, 23k{+}2, ..., 24k), \mu\big\rangle \cong S_k$. Hence $G = \langle\sigma, \tau, \mu\rangle \cong M_{24}\, wr\, S_k$. □

Theorem 3.3. *Let G be the group generated by the permutation* $(1, 2, ..., 23k)(23k+1, 23k+2, ..., 24k)$, *the four* 5*-cycles* $(3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k)(15k, 20k, 22k, 21k, 16k)$, *the involution* $(k, 24k)(2k, 23k)(3k, 12k)(4k, 16k)(5k, 18k)(6k, 10k)(7k, 20k)(8k, 14k)(9k, 21k)(11k, 17k)(13k, 22k)(15k, 19k)$ *and the product of the* 3*-cycles* $(1, 2, 3)(k{+}1, k{+}2, k{+}3)(2k{+}1, 2k{+}2, 2k{+}3)...(23k{+}1, 23k{+}2, 23k{+}3)$. *if* $k > 3$ *is an odd integer, then* $G \cong M_{24}\, wr\, A_k$ *of order* $\left(|M_{24}|\right)^k \times \dfrac{k!}{2}$.

Proof. The proof is similar to the proof of the previous result. □

Theorem 3.4. *Let* $k = am$ *be an integer,* $1 < a < k$. *Let G be the group generated by the permutation* $(1, 2, ..., 23k)(23k + 1, 23k + 2, ..., 24k)$, *the four* 5*-cycles* $(3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k)(15k, 20k, 22k, 21k, 16k)$, *the involution* $(k, 24k)(2k, 23k)(3k, 12k)(4k, 16k)(5k, 18k)(6k, 10k)(7k, 20k)(8k, 14k)(9k, 21k)(11k, 17k)(13k, 22k)(15k, 19k)$ *and the involution* $(k, a)(2k, k{+}a)(3k, 2k{+}a)...(24k, 23k{+}a)$, *then* $G \cong M_{24}\, wr\, (S_k\, wr\, C_a)$ *of order* $\left(|M_{24}|\right)^k \times (m!)^a \times a$.

Proof. Let $\sigma = (1, 2, ..., 23k)(23k{+}1, 23k{+}2, ..., 24k)$, $\tau = (3k, 17k, 10k, 7k, 9k)(4k, 13k, 14k, 19k, 5k)(8k, 18k, 11k, 12k, 23k)(15k, 20k, 22k, 21k, 16k)$ and $\gamma = (k, 24k)(2k, 23k)(3k, 12k)(4k, 16k)(5k, 18k)(6k, 10k)(7k, 20k)(8k, 14k)(9k, 21k)(11k, 17k)(13k, 22k)(15k, 19k)$ and $\mu = (1, 2)(k{+}1, k{+}2)(2k{+}1, 2k{+}2)...(23k{+}1, 23k{+}2)$. From Theorem 2.1, $\langle\sigma, \tau\rangle \cong M_{24}\, wr\, C_k$. Since $C_k = \big\langle(1, 2, ..., k)(k{+}1, k{+}2, ..., 2k)(22k{+}1, 22k{+}2, ..., 23k)(23k{+}1, 23k{+}2, ..., 24k)\big\rangle$, then $\big\langle(1, 2, ...,$

$k)(k+1, k+2, ..., 2k)(22k+1, 22k+2, ..., 23k)(23k+1, 23k+2, ..., 24k), \mu\rangle \cong S_k$.
Hence $G = \langle\sigma, \tau, \mu\rangle \cong M_{24}\, wr\, S_k$. □

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