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STOCHASTIC COMPARISONS OF RESIDUAL LIFETIMES

IBRAHIM A. AHMAD

ABSTRACT. The random residual lifetime is a random variable representing the remaining life of a system of age $t \geq 0$. Its expected value, the mean residual lifetime, is well known and extensively studied for both testing and estimation. In the present investigation we concentrate on comparing two independent residual lifetimes of systems of same age, uniformly in time also for a specified age. In addition, we estimate the probability that a residual lifetime is smaller than another of the same age. We also discuss comparing a residual lifetime with another new life both uniformly in age and for a specified age. Estimation of the probability that a residual lifetime at a certain age is less than a new independent life is also presented. Whenever possible the asymptotic relative efficiencies of the procedures discussed here are compared to those of the usual Mann-Whitney-Wilcoxon statistics which compares two complete lives and is a special case of our procedures for life distributions.

1. INTRODUCTION

A life is a nonnegative random variable (rv) X having distribution function (df)F and a survival function (sf) $\bar{F} = 1 - F$. For a system whose life is X and has survived up to the age $t \geq 0$, we define the "residual lifetime" (RL) X_t as the random variable whose sf is given by:

$$\bar{F}_t(x) = \bar{F}(x+t)/\bar{F}(t) \quad \text{for all } x, t \geq 0. \tag{1.1}$$

Thus the df of X_t is $[F(x+t) - F(t)]/\bar{F}(t)$, for all $x, t \geq 0$.

The definition (1.1) is not new since its expected value $E(X_t) = \mu(t)$ has for long been known to researchers in lifetesting and reliability as "the mean residual lifetime" (MRL) and is formally given by:

$$(1.2) \quad \mu(t) = E(X - t|X > t) = E(X_t) = \int_0^\infty \bar{F}(x+t)dx/\bar{F}(t).$$

When $\mu(t)$ decreases it defines an important and practical class of life distributions, the decreasing mean residual lifetime distributions, (DMRL). It was introduced by Bryson and Siddiqi (1969) and independently by Marshall and Proschan (1972). Testing that $\mu(t)$ is constant (the survival function is exponential) against that it is decreasing was discussed by several authors among them Bryson (1974), Blakema and DeHaan (1974), Hollander and Proschan (1975), Langenberger and Srinivasan (1979), Hall and Wellner (1981) and Ahmad (1992). While estimating $\mu(t)$ nonparametrically and as a stochastic process was discussed by Yang (1978), Hall and Wellner (1981), and Ahmad (1983a,b), among others, for both large and small samples. For a survey of properties, uses, and studies about $\mu(t)$ we refer the reader to Guess and Proschan (1985). The corresponding study of the df of X_t did not receive attention as did $\mu(t)$. We intend to remedy this in the current work. Let Y be another independent life with *df* G and *sf* $\bar{G}$. Define the RL $Y_t, t \geq 0$ as above.

On the other hand, stochastic comparison between two independent (not necessarily life) random variables X and Y is one of the oldest and most important problems in nonparametric statistics. We say that X is stochastically smaller than Y (written $X \overset{st}{\leq} Y$) if for all $x, F(x) \geq G(x)$ (i.e., $\bar{F}(x) \leq \bar{G}(x)$). Testing $H_0 : X = Y$ versus $H_1 : X \overset{st}{\leq} Y$ and not identical as well as estimating $\theta = P(X \leq Y)$ are among the most celebrated problems in nonparametric statistics today. Wilcoxon (1945) and Mann and Whitney (1947) gave the now famous Mann-Whitney-Wilcoxon statistics for testing H_0 against H_1. This was followed by a large literature, cf. Randles and Wolfe (1979). While estimating θ was initiated by Birnbaum (1956) and furthered by Birnbaum and McCarty (1958) and followed by many authors, cf. Pratt and Gibbons (1981) and Mee (1990) for further details and references.

In the current investigation we discuss several problems in comparing X_t with either Y_t or Y itself. Specifically we address the following problems:

(i) *Comparing two RL's X_t and Y_t:*

Problem Number One: We want to test $H_0^{(1)} : X_t = Y_t$ for all $t \geq 0$, versus $H_1^{(1)} : X_t \overset{st}{\leq} Y_t$ for all $t \geq 0$, i.e. $H_0^{(1)} : F = G$ versus $H_1^{(1)}$: $\frac{\bar{F}(x+t)}{\bar{F}(t)} \leq \frac{\bar{G}(x+t)}{\bar{G}(t)}$ for all $x, t \geq 0$.

Problem Number Two: We will test $H_0^{(2)} : X_{t_0} = Y_{t_0}$ for a specified value t_0 against $H_1^{(2)} : X_{t_0} \overset{st}{\leq} Y_{t_0}$, i.e. $H_0^{(2)} : \frac{\bar{F}(x+t_0)}{\bar{F}(t_0)} = \frac{\bar{G}(x+t_0)}{\bar{G}(t_0)}$ for all $x \geq 0$ versus $H_1^{(2)} : \frac{\bar{F}(x+t_0)}{\bar{F}(t_0)} \leq \frac{\bar{G}(x+t_0)}{\bar{G}(t_0)}$ for all $x \geq 0$.

Problem Number Three: We want to estimate $\mathcal{P}_{t_0} = P(X_{t_0} \leq Y_{t_0})$ for some $t_0 \geq 0$ and offer a confidence interval for $\mathcal{P}_{t_0}$.

(ii) *Comparing on RL, X_t with another Complete Life Y:*

Problem Number Four: We want to test $H_0^{(3)} : X_t = Y$ for all $t \geq 0$, versus $H_1^{(3)} : X_t \overset{st}{\leq} Y$ for all $t \geq 0$, i.e. $H_0^{(3)} : F = G$ versus $H_1^{(3)}$: $\frac{\bar{F}(x+t)}{\bar{F}(t)} \leq \bar{G}(x)$ for all $x, t \geq 0$.

Problem Number Five: We want to test $H_0^{(4)} : X_{t_0} = Y$ for a specific $t_0 \geq 0$ against $H_1^{(4)} : X_{t_0} \overset{st}{\leq} Y$, for a $t_0 \geq 0$, i.e. $H_0^{(4)} : \frac{\bar{F}(x+t_0)}{\bar{F}(t_0)} = \bar{G}(x)$ against $H_1^{(4)} : \frac{\bar{F}(x+t_0)}{\bar{F}(t_0)} \leq G(x)$ for a specified t_0 and all $x \geq 0$.

Problem Number Six: We want to estimate $Q_{t_0} = P(X_{t_0} \leq Y)$ for some $t_0 \geq 0$ and offer a confidence interval for Q_{t_0}.

Some comments on the above problems are in order. Problem number one is well-known since it is equivalent to testing $H_0^{(1)} : h_X(t) = h_Y(t)$, for all $t \geq 0$, where $h_X(t)$ is the hazard rate of X against $H_1^{(1)} : h_X(t) \leq h_Y(t)$

for all $t \geq 0$. The testing of $H_0^{(1)}$ using rank statistics was discussed by Chikkagouder and Shuster (1974); while Kochar (1979) and (1981) proposed statistics of the U-type in (1979) and log ranks in (1981). Our test statistic is of the U-statistic type and is simpler than that of Kochar (1979) but is equivalent to it in the sense of Pitman's asymptotic relative efficiency. Hence no need for the complicated calculation of Kochar's (1979) statistic or the use of the log ranks as in Kochar (1981) (since its relative efficiency is not always better).

The second problem is new and should be used to compare two systems of the same age to see which one has longer residual life time. This would be helpful in buying used items (cars, appliances,..., etc.) of the same age but different brands. Note here that the data we need to conduct this comparison are the complete lives of the systems (may be put to test in accelerated fashion).

Problem number three can be used for marketing strategies. Showing that if the probability that the residual life of a product of age t_0 being longer than that of a competitor of same age is quite high, then this means that this is a superior product.

Problem number four is also new. Here we are comparing a system of age t_0 to a new system. This problem has many applications in studies of whether to buy a used system of age t_0 or a new one (perhaps of lower quality but same price). Same thing may be said when comparing X_{t_0} to a different new system with life Y. Finally problem number six may be used to advocate whether to buy new (may be of lesser overall quality) to used of age t_0.

The practical part in all of the above problems, is that testing and estimation are based on samples of complete lives $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ from F and G respectively.

2. COMPARING TWO INDEPENDENT RL'S

2.1. UNIFORM STOCHASTIC COMPARISON

Here we want to test $H_0^{(1)} : F = G$ against $H_1^{(1)} : \bar{F}(x+t)\bar{G}(t) \leq \bar{G}(x+t)\bar{F}(t)$ for all $x, t \geq 0$, based on two independent samples $X_1, \cdots, X_m$ from F and $Y_1, \cdots, Y_n$ from G. Following reasoning similar to that used in the Mann-Whitney-Wilcoxon setting, we see that under $H_1^{(1)}$ we have

$$\int_0^\infty \int_0^\infty \bar{F}(x+t)\bar{G}(x) d_x \bar{G}(x+t) d\bar{F}(t) \leq \int_0^\infty \int_0^\infty \bar{G}(x+t)\bar{F}(t) d_x \bar{G}(x+t) d\bar{F}(t). \tag{2.1}$$

Note also that the

$$\text{RHS of (2.1)} = \frac{1}{2}\int_0^\infty \bar{G}^2(t)\bar{F}(t)dF(t), \tag{2.2}$$

while the

$$\text{LHS of (2.1)} = \int_0^\infty \int_t^\infty \bar{F}(u)\bar{G}(t)dG(u)dF(t). \tag{2.3}$$

Under $H_0^{(1)}$, both sides of (2.1) are equal to 1/8. Thus as a measure of departure from $H_0^{(1)}$ in favor of $H_1^{(1)}$ we take:

$$\Delta^{(1)}(F,G) = \int_0^\infty \bar{G}^2(t)\bar{F}(t)dF(t) - 2\int_0^\infty \int_t^\infty \bar{F}(u)\bar{G}(t)dF(t)dG(u). \tag{2.4}$$

Note that $\Delta^{(1)}(F,G) = 0$ under $H_0^{(1)}$ and is strictly positive under $H_1^{(1)}$. Thus a test statistic can be based on an empirical estimate of $\Delta^{(1)}(F,G)$, namely,

$$\hat{\Delta}^{(1)}(F_m,G_n) = \int_0^\infty \bar{G}_n^2(t)\bar{F}_m(t)dF_m(t) - 2\int_0^\infty \int_t^\infty \bar{F}_m(u)\bar{G}_n(t)dG_n(u)dF_m(t) \tag{2.5}$$

It is easy to see that

$$\hat{\Delta}^{(1)}(F_m,G_n) = (m^2n^2)^{-1} \sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\sum_{j_1=1}^{n}\sum_{j_2=1}^{n} \varphi(X_{i_1}, X_{i_2}, Y_{j_1}, Y_{j_2}), \tag{2.6}$$

where

$$\varphi(X_1, X_2, Y_1, Y_2) = I(Y_1 > X_1)I(Y_2 > X_1) \{I(X_2 > X_1) - 2I(X_2 > Y_1)\}. \tag{2.7}$$

We now state and prove the following

Theorem 2.1. *If* $\frac{m}{m+n} \to \lambda\epsilon(0,1)$ *as* $\min(m,n) \to \infty$, *then* $\sqrt{m+n}$ $(\hat{\Delta}^{(1)}(F_m, G_n) - \Delta^{(1)}(F,G))$ *is asymptotically normal with mean 0 and variance* $\sigma^2 = \sigma_1^2/\lambda + \sigma_2^2/(1-\lambda)$, *where*

$$\sigma_1^2 = Var\{\bar{G}^2(X_1)\bar{F}(X_1) - 2\bar{G}(X_1)\int_{X_1}^{\infty}[\bar{G}(X_1) - \bar{G}(x)]dF(x) + \int_0^{X_1}\bar{G}^2(x)dF(x) - 2\int_0^{X_1}\bar{G}(x)[\bar{G}(x) - \bar{G}(X_1)]dF(x)\}, \tag{2.8}$$

and

$$\sigma_2^2 = Var\{2\int_0^{Y_1}\bar{G}(x)\bar{F}(x)dF(x) - 2\bar{F}(Y_1)\int_0^{Y_1}\bar{G}(x)dF(x) -2[\int_0^{Y_1}\bar{F}(x)F(x)dG(x) + F(Y_1)\int_{Y_1}^{\infty}\bar{F}(x)dG(x)]\}. \tag{2.9}$$

Under $H_0, \sigma_0^2 = 2/105\lambda(1-\lambda)$. *Thus the procedure is distribution free.*

Proof. Using Theorem 3.4.13 of Randles and Wolfe (1979) we need only evaluate the variance. Now,

$$E[\varphi(X_1, X_2, Y_1, Y_2)|X_1] = \bar{G}^2(X_1)\bar{F}(X_1) -2\bar{G}(X_1)\int_{X_1}^{\infty}(\bar{G}(X_1) -\bar{G}(u))dF(u), \tag{2.10}$$

and

$$E[\varphi(X_1, X_2, Y_1, Y_2)|X_2] = \int_0^{X_2}\bar{G}^2(x)dF(x) -2\int_0^{X_2}\bar{G}(x)[\bar{G}(x) - \bar{G}(X_2)]dF(x). \tag{2.11}$$

But since

$$\sigma_1^2 = Var\{E[\varphi(X_1, X_2, Y_1, Y_2)|X_1] + E[\varphi(X_2, X_1, Y_1, Y_2) + X_1]\},$$

(2.8) now follows directly. Deriving (2.9) is entirely similar. Now, under $H_0^{(1)} : F = G$, $\sigma_{10}^2 = Var\{\frac{2}{3}\bar{F}^3(X_1) - \bar{F}(X_1) + \frac{1}{3}\}$ and $\sigma_{10}^2 = \sigma_{20}^2$. Direct calculations give that $\sigma_{10}^2 = \sigma_{20}^2 = \frac{2}{105}$. This concludes the proof.

In order to compare the above procedure with others such as the Mann-Whitney-Wilcoxon statistic we evaluate the Pitman's asymptotic efficacy of our procedure and compare it to others. Note that the asymptotic efficacy of one procedure is equal to: $\{\frac{d}{d\theta}\Delta_\theta^{(1)}\}^2/\sigma_0^2$, when $\theta \to \theta_0$, where $\Delta_\theta^{(1)} = \Delta^{(1)}(F, F_\theta)$. Thus in the case of location shift $G_\theta(x) = F(x - \theta)$ we get that the efficacy is:

$$(2.12) \qquad eff(\hat{\Delta}_0^{(1)}) = \{2f(0) + \int_0^\infty \bar{F}^2(t) f^2(t) dt + 2\int_0^\infty \int_t^\infty \bar{F}(u)\bar{F}(t) f'(u) f(t) du dt\}^2/\sigma_0^2.$$

Direct calculations for the exponential distribution gives the value $eff(\hat{\Delta}) = 14.21$ similar value for the Mann-Whitney-Wilcoxon is equal to 3 cf. Randles and Wolfe (19879), Section 5:4. Thus, the relative efficiency of our test to the Mann-Whitney-Wilcoxon is 4.73. Next, let us consider the shape alternative $G_\theta(x) = F(x^\theta)$. Thus the efficacy is given by

$$(2.13) \qquad \begin{aligned} eff(\hat{\Delta}^{(1)}) = \Big\{ & \frac{1}{4} - 2\int_0^\infty x \ln x \bar{F}^2(x) f^2(x) dx - 2\int_0^\infty x \ln x \bar{F}^2(x) f(x) dx \\ & -2\int_0^\infty \int_x^\infty \bar{F}(x) f(x) \ln u \bar{F}(u) f(u) du dx \\ & -2\int_0^\infty \int_x^\infty \bar{F}(x) f(x) u \ln u \bar{F}(u) f(u) du dx \Big\}^2 / \sigma_0^2. \end{aligned}$$

In the case of uniform (0,1) distribution, direct calculation gives the efficacy of 9.75 for $\hat{\Delta}_1^{(1)}$ while that of the Wilcoxon statistic is equal to 0.75. Hence the asymptotic relative efficacy of 13.0.

For the case $\bar{G}_\theta(x) = (\bar{F}(x))^{1+\theta}$ and some others, see Kochar (1979 and 1981) for efficiencies of $\hat{\Delta}_1^{(1)}$ relative to other statistics.

2.2. POINTWISE STOCHASTIC COMPARISON

Here we fix t_0 and want to test $H_0^{(2)} : \bar{F}(x+t_0)\bar{G}(t_0) = \bar{G}(x+t_0)\bar{F}(t_0)\ \forall\ x \geq 0$ against $H_1^{(2)} : \bar{F}(x+t_0)\bar{G}(t_0) \leq \bar{F}(t_0)\bar{G}(x+t_0)$ for all $x \geq 0$, based on two independent random samples $X_1, \cdots, X_m$ from F and $Y_1, \cdots, Y_n$ from G. Following a reasoning similar to that used in Section 2.1 above, we see that under $H_1^{(2)}$ we have

$$\text{(2.14)} \quad \bar{G}(t_0)\int_0^\infty \bar{F}(x+t_0)d_x\bar{G}(x+t_0) \leq \bar{F}(t_0)\int_0^\infty \bar{G}(x+t_0)d\bar{G}_x(x+t_0).$$

Now, the RHS of (2.16) is equal to $\frac{-1}{2}\bar{F}(t_0)\bar{G}^2(t_0)$ while the LHS of (2.16) is equal to $-\bar{G}(t_0)\int_{t_0}^\infty \bar{F}(u)dG(u)$. Thus we may take as a measure of departure from $H_0^{(2)}$,

$$\text{(2.15)} \qquad \Delta_{t_0}^{(2)}(F,G) = \bar{F}(t_0)\bar{G}(t_0) - 2\int_{t_0}^\infty \bar{F}(u)dG(u).$$

Note that under $H_0^{(2)}, \Delta_{t_0}^{(2)}(F,G) = 0$ while it is positive under $H_1^{(2)}$. Hence a test statistic of $H_0^{(2)}$ based on $\Delta_{t_0}^{(2)}(F,G)$ is given by its empirical counter-part, viz.,

$$\text{(2.16)} \qquad \hat{\Delta}_{t_0}^{(2)}(F_m,G_n) = \bar{F}_m(t_0)\bar{G}_n(t_0) - 2\int_{t_0}^\infty \bar{F}_m(u)dG_n(u).$$

It is easy to see that

$$\text{(2.17)} \qquad \hat{\Delta}_{t_0}^{(2)}(F_m,G_n) = (mn)^{-1}\sum_{i=1}^m\sum_{j=1}^n \psi_{t_0}(X_i,Y_j),$$

where $\psi_{t_0}(x,y) = I(\min(x,y) > t_0) - 2I(x > y > t_0)$. Now, clearly, $E(\psi_{t_0}(X_1,Y_1)|X_1] = I(X_1 > t_0)\{\bar{G}(X_1) - \bar{G}(t_0)\}$ and $E[\psi_{t_0}(X_1,Y_1)|Y_1] = I(Y_1 > t_0)\{\bar{F}(t_0) - 2\bar{F}(Y_1)\}$. Hence the asymptotic variance of $\sqrt{m+n}(\hat{\Delta}_{t_0}^{(2)}(F_m,G_n) - \Delta_{t_0}^{(2)}(F,G))$ is given by:

$$\text{(2.18)} \qquad \begin{aligned}\sigma^2 &= Var\{I(X_1 > t_0)[2\bar{G}(X_1) - \bar{G}(t_0)]\}/\lambda \\ &\quad + Var\{I(Y_1 > t_0)[\bar{F}(t_0) - 2\bar{F}(Y_1)]\}/(1-\lambda),\end{aligned}$$

and by direct calculations we see that under $H_0^{(2)}$ we have

$$(2.19) \qquad \sigma_0^2 = \frac{\bar{F}(t_0)\bar{G}(t_0)}{3}\left\{\frac{\bar{F}(t_0)}{\lambda} + \frac{\bar{G}(t_0)}{1-\lambda}\right\}.$$

We have thus proved the following theorem.

Theorem 2.2. *If* $\frac{m}{m+n} \to \lambda\epsilon(0,1)$ *as* $\min(m,n) \to \infty$, *then* $\sqrt{m+n}$ $(\hat{\Delta}_{t_0}^{(2)}(F_m, G_n) - \Delta_{t_0}^{(2)}(F,G))$ *is asymptotically normal with mean 0 and variance* σ^2 *given in (2.18) while under* $H_0^{(2)}, \sigma^2$ *reduces to (2.19).*

2.3. ESTIMATION OF $\mathcal{P}_{t_0} = P(X_{t_0} < Y_{t_0})$

For a fixed point t_0, one might want to estimate function of the type $\mathcal{P}_{t_0} = P(X_{t_0} < Y_{t_0})$. First we note that

$$(2.20)\, \mathcal{P}_{t_0} = P(t_0 < X < Y)/\bar{F}(t_0)\bar{G}(t_0) = \int_{t_0}^{\infty} \bar{G}(x)dF(x)/\bar{F}(t_0)\bar{G}(t_0).$$

Using two independent samples; $X_1, \ldots, X_m$ from F and $Y_1, \ldots, Y_n$ from G, we can estimate $\mathcal{P}_{t_0}$ easily as follows:

$$(2.21) \qquad \hat{\mathcal{P}}_{t_0} = \int_{t_0}^{\infty} \bar{G}_n(x)dF_m(x)/\bar{F}_m(t_0)\bar{G}_n(t_0).$$

It is not difficult to see that $(\hat{\mathcal{P}}_{t_0} - \mathcal{P}_{t_0})$ has the same limiting distribution as:

$$(2.22) \qquad \begin{aligned} &(\bar{F}(t_0)\bar{G}(t_0))^{-1}\{m^{-1}\sum_{i=1}^{m}\bar{G}(X_i)I(X_i > t_0) - \mathcal{P}_{t_0}\} \\ &-\frac{\mathcal{P}_{t_0}}{\bar{F}(t_0)}\{\bar{F}_m(t_0) - \bar{F}(t_0)\} \\ &+(\bar{F}(t_0)\bar{G}(t_0))^{-1}\{n^{-1}\sum_{i=1}^{n}\bar{F}(t_0) - \bar{F}(Y_j)) - \mathcal{P}_{t_0}\} \\ &-\frac{\mathcal{P}_{t_0}}{\bar{F}(t_0)}\{\bar{G}_n(t_0) - \bar{G}(t_0)\}, \end{aligned}$$

which is normal with mean 0 and variance

$$\begin{aligned}(2.23) \quad \sigma^2 \;=\; &(\bar{F}^2(t_0)\bar{G}^2(t_0))^{-2}\{\frac{1}{\lambda}\int_{t_0}^{\infty}(\bar{G}(u) - \mathcal{P}_{t_0}\bar{G}(t_0))^2 dF(u) \\ &+\frac{1}{1-\lambda}\int_{t_0}^{\infty}(\bar{F}(t_0) - \bar{F}(u) - \mathcal{P}_{t_0}\bar{F}(t_0))^2 dG(u)\}.\end{aligned}$$

If we are interested in establishing an asymptotically distribution-free normal confidence interval for $\mathcal{P}_{t_0}$, we simply estimate σ^2 by plugging empirical estimates all across and use the confidence interval $\hat{\mathcal{P}}_{t_0} \pm Z_{\alpha/2}\frac{\hat{\sigma}}{\sqrt{m+n}}$.

3. COMPARING A RESIDUAL LIFE WITH A COMPLETE ONE

3.1. UNIFORM STOCHASTIC COMPARISON

Here we want to test $H_0^{(3)}: F = G$ against $H^{(3)}: \bar{F}(x+t) \leq \bar{F}(t)\bar{G}(x)$ for all $x, t \geq 0$. In this case, a measure of departure from $H_0^{(3)}$ is:

$$\Delta^{(3)}(F, G) = \frac{1}{4} - \int_0^\infty \int_0^\infty \bar{F}(x+t)dF(t)dG(x). \tag{3.1}$$

Based on two independent samples $X_1, \cdots, X_m$ from F and $Y_1, \cdots, Y_n$ from G we have the test statistic:

$$\hat{\Delta}^{(3)}(F_m, G_n) = \frac{1}{4} - \int_0^\infty \int_0^\infty \bar{F}_m(x+t)dF_m(t)dG_n(x). \tag{3.2}$$

Note that with $\xi(X_1, X_2, Y_1) = I(X_1 > X_2 + Y_1)$ we easily see that:

$$\hat{\Delta}^{(3)}(F_m, G_n) = \frac{1}{4} - m^{-2}n^{-1}\sum_{i_1=1}^{m}\sum_{i_2=1}^{m}\sum_{j=1}^{n}\xi(X_{i_1}, X_{i_2}, Y_j). \tag{3.3}$$

As above, we can easily see that the asymptotic variance of $\sqrt{m+n}$ $(\hat{\Delta}^{(3)}(F_m, G_n) - \Delta^{(3)}(F, G))$ is given by

$$\begin{aligned} \sigma^2 &= \frac{1}{\lambda}Var\{F * G(X_1) + \int_0^\infty \bar{F}(X_1+u)dG(u)\} \\ &\quad + \frac{1}{1-\lambda}Var\{\int_0^\infty \bar{F}(x+Y_1)F(x)\}. \end{aligned} \tag{3.4}$$

Under $H_0^{(3)}: F = G, \sigma^2$ reduces to:

$$\sigma_0^2 = \frac{1}{54\lambda} + \frac{1}{48(1-\lambda)} = (8+\lambda)/432\lambda(1-\lambda). \tag{3.5}$$

Collecting all of the above we now state the following theorem.

Theorem 3.1. $\dfrac{m}{m+n} \rightarrow \lambda\epsilon(0,1)$ *as* $\min(m,n) \rightarrow \infty$, *then* $\sqrt{m+n}$ $(\hat{\Delta}^{(3)}(F_m, G_n) - \Delta^{(3)}(F,G))$ *has asymptotically normal distribution with mean 0 and variance* σ^2 *as given in (3.4). Under* $H_0^{(3)}, \sigma^2$ *reduces to (3.5).*

To study the relative efficiency of the test procedure of this section to that of Section 2.1 as well as to that of the Mann-Whitney-Wilcoxon, we observe that the efficacy of $\hat{\Delta}^{(3)}(F_m, G_n)$ in the location shift case, $\bar{G}_\theta(x) = \bar{F}(x-\theta)$, we get

$$eff_L(\hat{\Delta}^{(3)}) = \{f(0)/2 - \int_0^\infty \int_0^\infty \bar{F}(x+t) f(x) f'(t) dx dt\}^2 / \sigma_0^2. \tag{3.6}$$

When F is exponential the efficacy is equal to $27\lambda(1-\lambda)/(8+\lambda)$ while the efficacy of the Mann- Whitney-Wilcoxon statistic is equal to $3\lambda(1-\lambda)$ giving an asymptotic relative efficiency equal to $9/(8+\lambda)$. Hence $1 < ARE(\hat{\Delta}^{(3)}, W) < 1.125$, since $0 < \lambda < 1$. For the shape alternative we see that the efficacy is equal to:

$$\begin{aligned} eff_s(\hat{\Delta}^{(3)}) = & \left\{ \int_0^\infty \int_0^\infty \bar{F}(x+t) dF(t) \right. \\ & + \int_0^\infty \int_0^\infty \bar{F}(x+t) t \ln t dF(x) dF(t) \\ & \left. + \int_0^\infty \int_0^\infty \bar{F}(x+t) \ln t dF(x) dF(t) \right\}^2 / \sigma_0^2 \end{aligned} \tag{3.7}$$

In the case of uniform (0.1) distribution, the asymptotic relative efficiency of $\hat{\Delta}^{(3)}$ with respect to the Wilcoxon statistic is between (4.457, 5.012).

3.2. POINTWISE STOCHASTIC COMPARISON

In this section we want to test $H_0^{(4)} : \bar{F}(x+t_0) = \bar{F}(t_0)\bar{G}(x)$ for all $x \geq 0$ and a fixed value $t_0 \geq 0$ against $H_1^{(4)} : \bar{F}(x+t_0) \leq \bar{F}(t_0)\bar{G}(x)$ for all $x \geq 0$. As in the sections above we use the measure of departure for $H_0^{(4)}$,

$$\Delta_{t_0}^{(4)}(F,G) = \bar{F}(t_0) - 2\int_0^\infty \bar{F}(x+t_0) dG(x), \tag{3.8}$$

which can be estimated by

$$(3.9) \qquad \Delta_{t_0}^{(4)}(F_m, G_n) = \bar{F}_m(t_0) - 2\int_0^\infty \bar{F}_m(x+t_0)dG_n(x),$$

If we set $\xi_{t_0}(X_1, Y_1) = I(X_1 > t_0) - 2I(X_1 > Y_1 + t_0)$, then we can write

$$\Delta_{t_0}^{(4)}(F_m, G_n) = (mn)^{-1}\sum_{i=1}^{m}\sum_{j=1}^{n}\xi_{t_0}(X_i, Y_j).$$

Now we easily see that $E[\xi_{t_0}(X_1, Y_1)|X_1] = I(X_1 > t_0) - 2I(X_1 > t_0) - 2I(X_1 > t_0)G(X_1 - t_0)$ and $E[\xi_{t_0}(X_1, Y_1)|Y_1] = \bar{F}(t_0) - 2\bar{F}(Y_1 + t_0)$. Thus the asymptotic variance of $\sqrt{m+n}(\hat{\Delta}_{t_0}^{(4)}(F_m, G_n) - \Delta_{t_0}^{(4)}(F, G))$ equals:

$$(3.10) \qquad \begin{aligned} \sigma^2 \;=\; & Var\{I(X_1 > t_0) - 2I(X_1 > t_0)G(X_1 - t_0)\}/\lambda \\ & + Var\{\bar{F}(t_0) - 2\bar{F}(Y_1 + t_0)\}/(1-\lambda). \end{aligned}$$

Under $H_0^{(4)}, \sigma^2$ reduces to

$$(3.11) \qquad \sigma_0^2 = \frac{2}{3}\bar{F}^2(t_0).$$

We can now state the following result,

Theorem 3.2. $\frac{m}{m+n} \to \lambda\epsilon(0,1)$ *as* $\min(m,n) \to \infty$, *then* $\sqrt{m+n}$ $(\hat{\Delta}_{t_0}^{(4)}(F_m, G_n) - \Delta_{t_0}^{(4)}(F, G))$ *is asymptotically normal with mean 0 and variance* σ^2 *given in (3.10). Under* $H_0^{(4)}$, *the null variance is* σ_0^2 *given in (3.11).*

3.3. ESTIMATION OF $Q_{t_0} = P(Y_{t_0} < Y)$

In this case

$$(3.12) \qquad Q_{t_0} = 1 - \frac{1}{\bar{F}(t_0)}\int_0^\infty \bar{F}(t_0 + u)dG(u),$$

which can be estimated by its empirical version $\hat{Q}_{t_0}$ by plugging $\bar{F}_m$ and G_n in place of $\bar{F}$ and G. After some algebra one can show that $\hat{Q}_{t_0} - Q_{t_0}$

has the same limiting distribution as

$$\frac{1}{\bar{F}(t_0)}\left\{m^{-1}\sum_{i=1}^{m}[I(X_i > t_0)\bar{G}(X_i - t_0) + (1 - Q_{t_0})I(X_i > t_0)]\right.$$

$$(3.13) \quad \left. - \left[\int_0^\infty \bar{F}(t_0 + x)dG(x) + (1 - Q_{t_0})\bar{F}(t_0)\right]\right\}$$

$$+\frac{1}{\bar{F}(t_0)}\{n^{-1}\sum_{j=1}^{n} F(Y_j + t_0) - \int_0^\infty F(t_0 + x)dG(x)\}.$$

Hence $\sqrt{m+n}(\hat{Q}_{t_0} - Q_{t_0})$ is asymptotically normal with mean 0 and variance

$$(3.14) \quad \begin{aligned} \sigma_{t_0}^2 &= \bar{F}^2(t_0)\{\int_{t_0}^\infty [G(u - t_0) + 1 - Q_{t_0}]^2 dF(u)/\lambda \\ &\quad + \int_0^\infty [F(u + t_0) \\ &\quad - EF(Y_1 + t_0)]^2 dG(u)/(1 - \lambda)\}. \end{aligned}$$

APPENDIX

(i) *Main Steps in deriving (2.12):*

$$\begin{aligned} \Delta^{(1)}(F, G) &= \int_0^\infty \bar{G}^2(t)\bar{F}(t)dF(t) \\ &\quad -2\int_0^\infty \int_0^\infty \bar{F}(u)\bar{G}(x)dF(t)dG(u) = I_1 - I_2. \end{aligned}$$

In the location case, $\bar{G}_\theta(u) = \bar{F}(u - \theta)$ when $u \geq \theta$ and $\bar{G}_\theta(u) = 1$ if $u < \theta$. Thus

$$I_1 = \int_0^\theta \bar{F}(t)dF(t) + \int_\theta^\infty \bar{F}^2(t - \theta)\bar{F}(t)dF(t).$$

Hence

$$\frac{\partial I_1}{\partial \theta}\bigg|_{\theta \to 0} = 2\int_0^\infty \bar{F}^2(t)f^2(t)dt,$$

and

$$\begin{aligned} I_2 &= 2\int_0^\infty \int_{\max(t,\theta)}^\infty f(t)\bar{F}(u)\bar{F}(t-\theta)f(u-\theta)dudt \\ &= 2F(\theta)\int_0^\infty f(u-\theta)\bar{F}(u)du \\ &\quad +2\int_0^\infty \int_t^\infty \bar{F}(t-\theta)f(u-\theta)\bar{F}(u)f(t)dudt \\ &= J_1 + J_2, \quad \text{say.} \end{aligned}$$

Now,

$$\frac{\partial J_1}{\partial\theta}\,|_{\theta\to 0} = -f(0),$$

and

$$\begin{aligned} \frac{\partial J_2}{\partial\theta}\,|_{\theta\to 0} &= 2\left\{\frac{-f(0)}{2} + \frac{1}{2}\int_0^\infty \bar{F}^2(t)f^2(t)dt \right. \\ &\quad \left. -\int_0^\infty \int_t^\infty \bar{F}(t)\bar{F}(u)f'(u)f(t)dudt\right\}. \end{aligned}$$

(ii) *Main Steps in deriving (2.13):*

In this case, $\bar{G}_\theta(x) = \bar{F}(x^\theta)$ and using this in $\Delta^{(1)}(F,G)$ we easily see that

$$\frac{\partial I_1}{\partial\theta}\,|_{\theta\to 1} = -2\int_0^\infty \bar{F}^2(x)f^2(x)x\ln x dx,$$

and

$$\begin{aligned} \frac{\partial I_2}{\partial\theta}\,|_{\theta\to 1} &= \frac{1}{4} - 2\int_0^\infty \int_x^\infty x\ln x f(x)\bar{F}(u)f(u)dudx \\ &\quad -2\int_0^\infty \int_x^\infty \bar{F}(x)f(x)u\ln u f(u)\bar{F}(u)dudx \\ &\quad -2\int_0^\infty \int_x^\infty \bar{F}(x)f(x)\ln u f(u)\bar{F}(u)dudx. \end{aligned}$$

(iii) *Main Steps in deriving (3.5) and (3.6):*

$$\begin{aligned} \Delta^{(3)}(F,G) &= \int_0^\infty \int_0^\infty \bar{F}(x+t)dF(t)dG(x) \\ &= \int_0^\infty \int_\theta^\infty \bar{F}(x+t)f(t)f(x-\theta)dxdt, \end{aligned}$$

in the location case. Hence

$$\frac{\partial \Delta^{(3)}}{\partial \theta}|_{\theta\to 0} = -f(0)\int_0^\infty \bar{F}(t)f(t)dt - \int_0^\infty\int_0^\infty \bar{F}(x+t)f(t)f'(x)dtdx.$$

While in the shape case $\Delta^{(3)}(F,G) = \int_0^\infty\int_0^\infty \theta\bar{F}(x+t)f(x)f(t^\theta)t^{\theta-1}dxdt$. Thus differentiating with respect to θ and letting $\theta \to 1$ gives the result.

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Division of Statistics, Northern Illinois University, DeKalb, Illinois 60115, U.S.A.

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