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# A survey of elliptic Kato classes and some applications for partial differential equations

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**Abstract** In this paper, we give a survey of elliptic Kato classes. Next, we study existence results to some singular elliptic problems, adapted to such Kato classes. We end this paper by giving an asymptotic behavior for the solution of a singular elliptic Dirichlet problem on a bounded domain of $\mathbb{R}^n$ ($n \geqslant 2$).


## 1. Introduction

This paper is a survey on some Kato classes of functions and some existence results to singular elliptic problems. More precisely, in the first section, we give some properties of the Green function of the Laplace operator $\Delta$ in a regular domain $D$ in $\mathbb{R}^n$ ($n \geqslant 2$). In particular, we establish many 3G-inequalities (see Zhao, 1986; Kalton and Verbitsky, 1999; Selmi, 2000; Bachar et al., 2003). In Section 2, we introduce the classical Kato classes (see Aizenman and Simon, 1982; Zhao,

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1993) and the new ones (see Mâagli and Zribi, 2005; Zeddini, 2003; Bachar et al., 2002; Bachar et al., 2003; Bachar and Mâagli, 2005). Next, we give many examples of functions in Kato classes and some properties of these classes. In Section 3, these Kato classes of functions are adapted to study some existence results to singular elliptic problems (see Bachar et al., 2002; Bachar et al., 2003; Bachar and Mâagli, 2005; Ben Othman et al., 2009; Mâagli and Zribi, 2001; Mâagli and Zribi, 2005; Zeddini, 2003). In the last section, we are interested in the asymptotic behavior of the solution of a singular Dirichlet problem in a regular bounded domain in $\mathbb{R}^n$ (see Ben Othman et al., 2009; Crandall et al., 1977; Lazer and Mckenna, 1991; Zhang, 2005).

## 2. Green function

Let $(P_t)_{t>0}$ be the Gauss semigroup on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$:

$$P_tf(x) = g_t * f(x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right) f(y)dy$$

where $f \in \mathcal{B}^+(\mathbb{R}^n), x \in \mathbb{R}^n$ and $t > 0$.

Let $G(x,y) = \int_0^\infty g_t(x-y)dt = \int_0^\infty \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x-y|^2}{4t}\right)dt = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}} |x-y|^{2-n}$ if $n \geqslant 3$.

$G$ is called the Green function of Laplace operator $\Delta$ in $\mathbb{R}^n$. If $D$ is a bounded $C^{1,1}$-domain in $\mathbb{R}^n$, the Green function $G_D$ of $\Delta$ in $D$ satisfies $\Delta G_D(\cdot, y) = -\delta_y, G_D(x,y) = G_D(y,x), G_D(x,y) = 0$, if $y \in D,\ x \in \partial D$.

Let $f, g : S \rightarrow [0, \infty]$, we denote by $f \approx g$ if there exists a constant $c > 0$ such that $\frac{1}{c} f \leqslant g \leqslant cf$. For $x \in D$, $\delta(x) = d(x, \partial D)$ denotes the distance from $x$ to the boundary $\partial D$. Note that $\max(a,b) \approx a+b$ and $\min(a,b) \approx \frac{ab}{a+b}$ for $a, b > 0$.

**Theorem 2.1** (Zhao (1986)). *We have on $D \times D, G_D(x,y) \approx H(x,y)$, where*

$$H(x,y) = \begin{cases} \log\left(1 + \frac{\delta(x)\delta(y)}{|x-y|^2}\right) & \text{for } n = 2 \\ \frac{1}{|x-y|^{n-2}} \min\left(1, \frac{\delta(x)\delta(y)}{|x-y|^2}\right) & \text{for } n \geqslant 3 \end{cases}$$

**Example 2.1.** Let $B = B(0,1)$. The Green function $G_B$ is given by

$$G_B(x,y) = \begin{cases} \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}} \left(\frac{1}{|x-y|^{n-2}} - \frac{1}{[x,y]^{n-2}}\right); & n \geqslant 3, \\ \frac{1}{4\pi} Log\left(1 + \frac{(1-|x|^2)(1-|y|^2)}{|x-y|^2}\right); & n = 2, \end{cases}$$

where $[x,y]^2 = |x-y|^2 + (1-|x|^2)(1-|y|^2)$, for $x, y \in B(0,1)$.

Let $x \in D$ and $D_1 = \left\{y \in D : |x-y|^2 \leqslant \delta(x)\delta(y)\right\}$. Then we have the following.

**Lemma 2.2** (Mâagli and Selmi (2002)).

(1) *If* $|x-y|^2 \leqslant \delta(x)\delta(y)$, *then* $\frac{3-\sqrt{5}}{2}\delta(x) \leqslant \delta(y) \leqslant \frac{3+\sqrt{5}}{2}\delta(x)$ *and* $|x-y| \leqslant \frac{1+\sqrt{5}}{2}$ $\min(\delta(x),\delta(y))$

(2) *If* $\delta(x)\delta(y) \leqslant |x-y|^2$, *then* $\max(\delta(x),\delta(y)) \leqslant \frac{1+\sqrt{5}}{2}|x-y|$

*In particular, we have* $B\left(x, \frac{\sqrt{5}-1}{2}\delta(x)\right) \subset D_1 \subset B\left(x, \frac{\sqrt{5}+1}{2}\delta(x)\right)$.

**Remark 2.3.** Using this Lemma, we deduce the following properties:

$$\frac{\delta(y)}{\delta(x)} G_D(x,y) \approx \begin{cases} \min\left(1, \frac{\delta(y)^2}{|x-y|^2}\right) \frac{1}{|x-y|^{n-2}} \leqslant \frac{1}{|x-y|^{n-2}}, & n \geqslant 3 \\ \min\left(1, \frac{\delta(y)^2}{|x-y|^2}\right) \log(2 + \frac{\delta(x)\delta(y)}{|x-y|^2}) \leqslant \log \frac{2d}{|x-y|}, & n = 2 \end{cases}$$

If $D$ is an $C^{1,1}$-exterior domain in $\mathbb{R}^n$, then we have (see Bachar et al., 2003)

$$G_D(x,y) \approx \frac{1}{|x-y|^{n-2}} \min\left(1, \frac{\lambda(x)\lambda(y)}{|x-y|^2}\right), \quad n \geqslant 3$$

$$G_D(x,y) \approx \log\left(1 + \frac{\lambda(x)\lambda(y)}{|x-y|^2}\right), \quad n = 2$$

where $\lambda(x) = \delta(x)(\delta(x)+1)$.

**Example 2.2.** Let $B^* = \overline{B}(0,1)^c = \{x \in \mathbb{R}^n : |x| > 1\}$. Then we have

$$G_{B^*}(x,y) = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{[x,y]^{n-2}}\right)$$

where $[x,y]^2 = |x-y|^2 + (|x|^2-1)(|y|^2-1)$.

**Remark 2.4.** Let $\gamma : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$,

$$x \to \frac{x}{|x|^2}$$

then $G_{B^*}(x,y) = |x|^{2-n}|y|^{2-n} G_B(\gamma(x), \gamma(y))$.

Indeed, if $\gamma : \hat{\Omega} \to \Omega$ is a Möbius-transformation, then

$$G_{\widehat{\Omega}}(x,y) = |\gamma'(x)|^{\frac{n-2}{2}} |\gamma'(y)|^{\frac{n-2}{2}} G_\Omega(\gamma(x), \gamma(y)),$$

where $|\gamma'(x)| = |\det \gamma'(x)|, \ x \in \widehat{\Omega}$.

Let $\mathbb{R}^n_+$ be the half space: $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times ]0, +\infty[ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\}$, $n \geqslant 1$.

The density of the Gauss semigroup on $\mathbb{R}^n_+$ is given by

$$p(x,t,y) = \frac{1}{(4\pi t)^{\frac{n}{2}}}\left(e^{-\frac{|x-y|^2}{4t}} - e^{-\frac{|x-\overline{y}|^2}{4t}}\right)$$

where $x, y \in \mathbb{R}^n_+,\ t > 0,\ \overline{y} = (y', -y_n)$.

Then the Green function of $\Delta$ in $\mathbb{R}^n_+$ is given by

$$G_{\mathbb{R}^n_+}(x,y) = \int_0^\infty p(x,t,y)dt = \begin{cases} \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}\left(\frac{1}{|x-y|^{n-2}} - \frac{1}{|x-\overline{y}|^{n-2}}\right) & \text{if } n \geqslant 3 \\ \frac{1}{4\pi} Log\left(1 + \frac{4x_2y_2}{|x-y|^{n-2}}\right) & \text{if } n = 2 \\ \min(x,y) & \text{if } n = 1 \end{cases}$$

**3G-theorems:**

(a) The classical one (Zhao, 1986). Let $D$ be a $C^{1,1}$-domain in $\mathbb{R}^n$ $(n \geqslant 3)$

$$\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)} \leqslant C_0\left(\frac{1}{|x-z|^{n-2}} + \frac{1}{|y-z|^{n-2}}\right).$$

(b) A new 3G-theorem:

(i) Let $D$ be a bounded $C^{1,1}$-domain, then we have for $x, y, z \in D$,

$$\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)} \leqslant C_0\left(\frac{\delta(z)}{\delta(x)}G_D(x,z) + \frac{\delta(z)}{\delta(y)}G_D(y,z)\right)$$

(See Kalton and Verbitsky (1999) for $n \geqslant 3$ and Selmi (2000) for $n = 2$.)

(ii) Let $D$ be a $C^{1,1}$-exterior domain, then we have for $x, y, z \in D$ (see Bachar et al., 2003):

$$\frac{G_D(x,z)G_D(z,y)}{G_D(x,y)} \leqslant C_0\left(\frac{\min(1,\delta(z))}{\min(1,\delta(x))}G_D(x,z) + \frac{\min(1,\delta(z))}{\min(1,\delta(y))}G_D(y,z)\right)$$

(iii) Let $D = \mathbb{R}^n_+$, then we have for $x, y, z \in \mathbb{R}^n_+$ (see Bachar and Mâagli, 2005):

$$\frac{G_{\mathbb{R}^n_+}(x,z)G_{\mathbb{R}^n_+}(z,y)}{G_{\mathbb{R}^n_+}(x,y)} \leqslant C_0\left(\frac{z_n}{x_n}G_{\mathbb{R}^n_+}(x,z) + \frac{z_n}{y_n}G_{\mathbb{R}^n_+}(y,z)\right)$$

$C_0$ is a positive constant depending only on $D$.

## 3. Kato classes

### 3.1. Classical Kato class

Let $D$ be a domain in $\mathbb{R}^n$ $(n \geqslant 2)$. Aizenman and Simon (1982) introduced a Kato class $K_n(D)$ as follows: $q \in \mathcal{B}(D)$ is in the Kato class $K_n(D)$ if

$$\lim_{\alpha\to 0^+}\left(\sup_{x\in D}\int_{B(x,\alpha)\cap D}\frac{1}{|x-y|^{n-2}}|q(y)|dy\right)=0,\quad n\geqslant 3$$

$$\lim_{\alpha\to 0^+}\left(\sup_{x\in D}\int_{B(x,\alpha)\cap D}Log\frac{1}{|x-y|}|q(y)|dy\right)=0,\quad n\geqslant 2.$$

When $D$ is an exterior domain, Zhao (1993) introduced a Kato class $K_n^\infty(D)$ as follows: $q\in K_n^\infty(D)$ if $q\in K_n(D)$ and

$$\lim_{M\to+\infty}\left(\sup_{x\in D}\left(\int_{(|y|\geqslant M)}\frac{1}{|x-y|^{n-2}}|q(y)|dy\right)\right)=0,\quad n\geqslant 3$$

## 3.2. New Kato class

Let $D$ be a $C^{1,1}$-domain with $\partial D$ compact. Using Remark 2.3, we introduced (see Mâagli and Zribi, 2005; Zeddini, 2003) a new Kato class $K(D)$ as follows: $q\in\mathcal{B}(D)$ is in $K(D)$ if

$$\lim_{\alpha\to 0^+}\left(\sup_{x\in D}\int_{B(x,\alpha)\cap D}\frac{\rho(y)}{\rho(x)}G_D(x,y)|q(y)|dy\right)=0$$

where $\rho(x)=\min(1,\delta(x))$.

If $D$ is an exterior domain, the new Kato class $K^\infty(D)$ is defined (see Bachar et al., 2003) by $q\in K(D)$ and

$$\lim_{M\to+\infty}\left(\sup_{x\in D}\int_{(|y|\geqslant M)\cap D}\frac{\rho(y)}{\rho(x)}G_D(x,y)|q(y)|dy\right)=0$$

For $D=\mathbb{R}^n_+, q\in K^\infty(\mathbb{R}^n_+)$ iff $q\in\mathcal{B}(\mathbb{R}^n_+)$ and

$$\begin{cases}\lim_{\alpha\to 0^+}\left(\sup_{x\in\mathbb{R}^n_+}\int_{B(x,\alpha)\cap\mathbb{R}^n_+}\frac{y_n}{x_n}G_{\mathbb{R}^n_+}(x,y)|q(y)|dy\right)=0\\ \lim_{M\to+\infty}\left(\sup_{x\in\mathbb{R}^n_+}\int_{(|y|\geqslant M)\cap\mathbb{R}^n_+}\frac{y_n}{x_n}G_{\mathbb{R}^n_+}(x,y)|q(y)|dy\right)=0\end{cases}$$

(See Bachar et al., 2002; Bachar and Mâagli, 2005.)

**Example 3.1**

(1) Let $D$ be a $C^{1,1}$-bounded domain.
   (a) $q(x)=\frac{1}{\delta(x)^\lambda},\ q\in K(D)\iff\lambda<2$. If $1\leqslant\lambda<2$, then $q\notin K_n(D),\ n\geqslant 3$.
   (b) Let $p>\frac{n}{2}$, then for each $\lambda<2-\frac{n}{p}$, we have $\frac{1}{\delta(.)^\lambda}L^p(D)\subset K(D)$. Let $D=B(0,1)$ and $q$ be a Borel radial function in $D$ then $q\in K(D)\iff\int_0^1 r(1-r)|q(r)|dr<\infty$.

(2) Let $D$ be an $C^{1,1}$-exterior domain in $\mathbb{R}^n$ $(n \geqslant 3)$. The function $x \to \frac{1}{(|x|+1)^{\mu-\lambda}\delta(x)^{\lambda}} \in K^{\infty}(D) \iff \lambda < 2 < \mu$.

(3) The function $x \to \frac{1}{(|x|+1)^{\mu-\lambda}x_n^{\lambda}} \in K^{\infty}(\mathbb{R}^n_+) \iff \lambda < 2 < \mu$.

*3.3. Some properties*

Let $D$ be a $C^{1,1}$ domain in $\mathbb{R}^n$ $(n \geqslant 3)$ with $\partial D$ compact or $D = \mathbb{R}^n_+$ and $q \in K^{\infty}(D)$. Then we have

(1) $x \to q(x)\frac{\delta(x)}{(|x|+1)^{n-1}} \in L^1(D)$

(2) $x \to Vq(x) = \int_D G_D(x,y)q(y)dy \in C_0(D) = \{f \in C(D) : \lim_{x\to\partial^{\infty}D} f(x) = 0\}$

(3) Put for $q \in K^{\infty}(D)$, $\|q\|_D := \sup_{x\in D}\int_D \frac{\delta(y)}{\delta(x)} G_D(x,y)|q(y)|dy$

$$\alpha_q := \sup_{x,y\in D}\int_D \frac{G_D(x,z)G_D(z,y)}{G_D(x,y)}|q(y)|dy$$

Then we have $0 \leqslant \alpha_q \leqslant 2C_0\|q\|_D < +\infty$

(4) Let $q \in K^{\infty}(D)$. For any nonnegative superharmonic function $v$ in $D$, we have:

$$\int_D G_D(x,y)|q(y)|v(y)dy \leqslant \alpha_q v(x), \quad x \in D.$$

## 4. Applications

1. Consider the following nonlinear singular elliptic problem

$$(P)\begin{cases} -\Delta u = \varphi(\cdot,u) & \text{in } D \quad \text{(in the sense of distributions)} \\ u_{|\partial D} = 0 & \\ \lim_{|x|\to+\infty} u(x) = 0 & \text{(whenever } D \text{ is an } C^{1,1} - \text{exterior domain or} \\ & D = \mathbb{R}^n_+,\ n \geqslant 3) \\ u > 0 & \end{cases}$$

We aim to prove that (P) has a unique continuous solution in $D$.

**History of the problem (P)**

Let $\Omega$ be a $C^{1,1}$-bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$ and $0 \leqslant a \in C^{\alpha}_{loc}(\Omega)$, $g \in C^1(]0,\infty)$, $g' \leqslant 0$ and $g(0^+) = +\infty$.

The existence of a classical solution of the following problem

$$(1)\begin{cases} -\Delta u = a(x)g(u) & \text{in } D \\ u_{|\partial D} = 0 \\ u > 0 \end{cases}$$

was studied by many authors. For instance, Crandall et al. (1977) considered the case $a \equiv 1$, Lazer and Mckenna (1991) assume that $\frac{a}{\delta(x)^{\lambda}} \leqslant a(x) \leqslant \frac{b}{\delta(x)^{\lambda}};\ 0 < \lambda < 2$ and $g(u) = u^{-\sigma}(\sigma > 0)$.

Zhang (2005) considered the case $0 \leqslant a(x) \leqslant \frac{b}{\delta(x)^{\lambda}},\ \lambda < 2$.

These authors showed that (1) has a unique solution $u \in C^{2+\alpha}(\Omega) \cap C(\overline{\Omega})$.
In the next Theorem, we generalize their result.
We suppose that

- **(H$_1$)** $\varphi : D \times (0, +\infty) \to [0, +\infty)$ is a non-trivial measurable function, continuous and nonincreasing with respect to the second variable.
- **(H$_2$)** $\forall c > 0, \varphi(\cdot, c) \in K^{\infty}(D)$.

**Theorem 4.1.** *Under $(H_1)$ and $(H_2)$, problem (P) has a unique solution in $C_0(D)$.*

Idea of the proof: Let $\lambda > 0$ and consider the following problem

$$(P_\lambda)\begin{cases} -\Delta u = \varphi(\cdot, u) \quad \text{in } D & \text{(in the distributional sense)} \\ u_{|\partial D} = \lambda & \\ \lim\limits_{|x| \to +\infty} u(x) = \lambda & \text{(whenever } D \text{ is \textit{an} exterior domain or} \\ & D = \mathbb{R}^n_+, n \geqslant 3) \\ u > 0 & \end{cases}$$

Let $\beta = \lambda + \|V(\varphi(., \lambda))\|$ and $\Lambda = \left\{u \in B_b^+(D) :\ \lambda \leqslant u \leqslant \beta\right\}$.

We define the operator $Tu = \lambda + V(\varphi(\cdot, u))$, for $u \in \Lambda$. We show that $T : \Lambda \to \Lambda$ is compact. Hence by the Schäuder fixed point theorem, $(P_\lambda)$ has a unique solution $u_\lambda$ in $\Lambda$ satisfying

$$u_\lambda = \lambda + \int_D G_D(x, y)\varphi(y, u_\lambda)dy$$

Next we show that $u_\lambda \in C(D)$ and using the complete maximum principle, we obtain $0 \leqslant u_\mu - u_\lambda \leqslant \mu - \lambda$, for $0 < \lambda \leqslant \mu$. This implies the map $\lambda \to u_\lambda$ is nondecreasing on $(0, \infty)$.

Put $u = \inf_{\lambda > 0} u_\lambda = \sup_{\lambda > 0}(u_\lambda - \lambda)$, we deduce that $u$ is a continuous solution of (P). The uniqueness follows by the maximum principle.

2. Let $D$ be a bounded $C^{1,1}$ domain in $\mathbb{R}^n$ $(n \geqslant 3)$ and consider the following problem

$$(Q)\begin{cases} \Delta u + f(., u) = 0 \quad \textit{in } D \setminus \{0\} & \text{(in the sense of distributions)} \\ u_{|\partial D} = 0 & \\ u > 0 & \\ u(x) \sim_{x \to 0} \frac{c}{|x|^{n-2}} & (c \text{ small} > 0) \end{cases}$$

**History:** The existence of infinitely many singular positive solutions of the problem (Q) has been established, by Zhang and Zhao (1998) for the special nonlinearity $f(x,t) = p(x)t^{\mu}$ $(\mu > 1)$, where the function $x \to \frac{p(x)}{|x|^{(n-2)(\mu-1)}} \in K_n(D)$.

This problem was also studied by Kalton and Verbitsky (1999) by a different method.

Here we require the following hypotheses on $f$

- **(H$_1$)** $f$ is a Borel measurable function in $D \times (0,\infty)$, continuous with respect to the second variable.
- **(H$_2$)** $|f(x,t)| \leqslant tq(x,t)$, where $q \in \mathcal{B}^{+}(D \times (0,\infty))$ nondecreasing with respect to the second variable such that $\lim_{t\to 0^+} q(x,t) = 0$
- **(H$_3$)** The function $x \to q(x, G_D(x,0)) \in K(D)$.

Then we establish the following (see Mâagli and Zribi, 2005).

**Theorem 4.2.** *Assume (H$_1$)–(H$_3$). The problem (Q) has infinitely many solutions. More precisely, there exists $b_0 > 0$ such that for each $b \in (0, b_0]$, there exists a solution u of (Q) continuous in $D \setminus \{0\}$ and satisfying:*

$$u(x) \approx \frac{\delta(x)}{|x|^{n-2}} \quad \forall x \in D \setminus \{0\}$$

*and*

$$\lim_{|x|\to 0} u(x)|x|^{n-2} = bc_n$$

*where* $c_n = \frac{\Gamma(\frac{n}{2}-1)}{4\pi^{\frac{n}{2}}}$.

## 5. Asymptotic behavior

Let $D$ be a $C^{1,1}$-bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$. We return to problem

$$(P_0)\begin{cases} -\Delta u = a(x)g(u) & \text{in } D \\ u_{|\partial D} = 0 \\ u > 0 \end{cases}$$

where,

(i) $a \in C^{\alpha}_{loc}(D) \cap K^{+}(D)$
(ii) $g : (0,+\infty) \to [0,+\infty)C^1$, $g' \leqslant 0$ and $\lim_{t\to 0^+} \frac{g(\xi t)}{g(t)} = \xi^{-\gamma}$, $\gamma \geqslant 0$ and
(iii) Suppose that $\lim_{\delta(x)\to 0} \frac{a(x)}{h(\delta(x))} = c_0 > 0$, where $h : (0,\eta) \to (0,+\infty)$ is continuous, nonincreasing such that $h(0^+) = +\infty$. Let $p$ be the local solution of $\begin{cases} p''(t) = -h(t)g(p(t)) \\ p > 0, \ p(0) = 0, \ 0 < t < \eta \end{cases}$. Then we have the following

**Theorem 5.1** (Ben Othman et al. (2009)). *The solution u of* $(P_0)$ *is in* $C^{2+\alpha}(D) \cap C(\overline{D})$ *and satisfies*

$$\lim_{\delta(x)\to 0} \frac{u(x)}{p(\delta(x))} = c_0^{\frac{1}{\gamma+1}}.$$

This result improves those of Crandall et al. (1977) which they considered the case $a \equiv 1$ and showed that $u(x) \approx p(\delta(x))$. In particular they proved that $u(x) \approx \delta(x)^{\frac{2}{\gamma+1}}$ if $g(u) = u^{-\gamma},\ \gamma > 1$.

Lazer and Mckenna (1991) showed for $a(x) \approx \delta(x)^{\lambda},\ 0 < \lambda < 2$, that $c_1\delta(x)^{\frac{2}{\gamma+1}} \leqslant u(x) \leqslant c_2\delta(x)^{\frac{2-\lambda}{\delta+1}}$.

On the other hand, Zhang (2005) showed the same result in Theorem 5.1, when $g(0^+) = +\infty,\ 0 \leqslant a(x) \leqslant \frac{c}{\delta(x)^{\lambda}},\ 0 < \lambda < 2$ and $\lim_{t\to 0^+} \frac{h(\xi t)}{h(t)} = \xi^{-\theta},\ 0 < \theta < 2$

**Example 5.1.** Let $D$ be a $C^{1,1}$-bounded domain in $\mathbb{R}^n$ $(n \geqslant 2)$. Consider the following problem

$$(*) \quad \begin{cases} \Delta u = -a(x)u^{-\sigma} & \text{in } D \\ u_{|\partial D} = 0 \\ u > 0 \end{cases}$$

where $0 \leqslant \sigma < 1,\ a \in C^{\alpha}_{loc}(D)$ and $a(x) = \delta(x)^{\lambda-1}\left(\log\frac{2d}{\delta(x)}\right)^{\lambda}$ with $0 < \lambda < 1$.

The solution of $\begin{cases} p''(t) = -t^{\lambda-1}(\log\frac{1}{t})^{\lambda}(p(t))^{-\sigma} \\ p > 0,\ p(0) = p(1) = 0. \end{cases}$ is given by $p(t) = t\log\frac{1}{t}$.

Then, the solution of $(*)$ satisfies:

$$\lim_{\delta(x)\to 0} \frac{u(x)}{\delta(x)\log\frac{1}{\delta(x)}} = 1 \quad \text{and} \quad u(x) \approx \delta(x)\log\frac{2}{\delta(x)}$$

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